

CAMBIO DI VARIABILI IN GENERALE

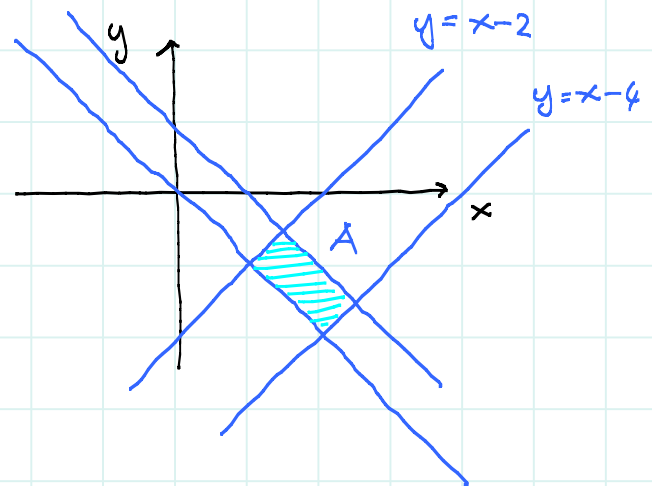
(In 2 variabili, ma in n è completamente uguale)

$$\iint_A f(x,y) dx dy = \iint_B \underbrace{g(u,v)}_{\text{funzione } f \text{ scritta nelle variabili } u \text{ e } v} \underbrace{J(u,v)}_{\text{pagamento "Determinante JACOBIANA"}}$$

$\uparrow$ insieme A descritto usando u e v

Esempio $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x+y \leq 1, 2 \leq x-y \leq 4\}$

$$\iint_A (x^2 - y^2) dx dy$$



Come insieme normale sono almeno 3 pezzi ☹️

Pongo

$$u := x+y$$

$$v := x-y$$

L'insieme diventa $0 \leq u \leq 1, 2 \leq v \leq 4$ ~ rettangolo ☺️

Funzione diventa $(x+y)(x-y) = uv$ ~ ☺️

$$\iint_A (x^2 - y^2) dx dy = \int_0^1 du \int_2^4 dv uv J(u,v)$$

Come calcolo $J(u,v)$?

Procedura in 3 passi:

① Ricavare (x, y) in funzione di u e v

② Matrice Jacobiana $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$

③ Determinante $J(u,v) = |\text{Det matrice Jacobiana}|$

$$\textcircled{1} \quad \begin{aligned} u &= x+y \\ v &= x-y \end{aligned} \quad x = \frac{u+v}{2} \quad y = \frac{u-v}{2}$$

$$\textcircled{2} \quad \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\textcircled{3} \quad J(u, v) = |\text{Det}| = \frac{1}{2}$$

Alla fine $\iint_A \dots = \frac{1}{2} \int_0^1 u \, du \int_{-2}^0 v \, dv = \text{banale.}$

Esempio $x = \rho \cos \theta \quad y = \rho \sin \theta$

$\textcircled{1}$ Ricavare ... già fatto

$$\textcircled{2} \quad \begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{pmatrix}$$

$$\textcircled{3} \quad J(\rho, \theta) = |\text{Det}| = |\rho \cos^2 \theta + \rho \sin^2 \theta| = \rho$$

Casi speciali : $\rightarrow$ se la trasformazione è affine, $J = \text{costante}$
 $\rightarrow$ se la trasformazione è una isometria, allora
 $J = 1$ (vale in particolare per tutte le traslazioni)

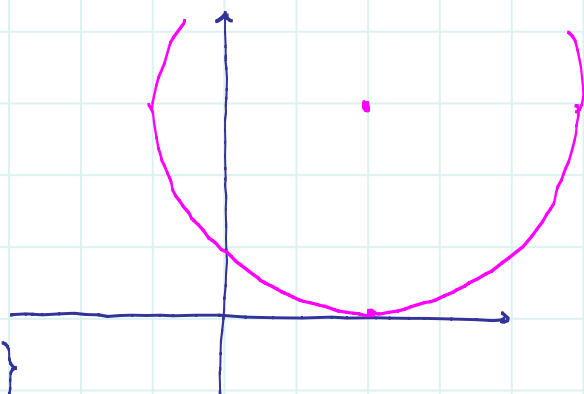
Esempio $A = \text{cerchio con centro in } (2, 3) \text{ e raggio } 3.$

Calcolare

$$\iint_A x^2 \, dx \, dy$$

Scrivo A in cartesiana

$$A = \{ (x, y) \in \mathbb{R}^2 : \underbrace{(x-2)^2} + \underbrace{(y-3)^2} \leq 9 \}$$



Normale asse x :

$$A = \{ (x,y) \in \mathbb{R}^2 : -1 \leq x \leq 5, 3 - \sqrt{9 - (x-2)^2} \leq y \leq 3 + \sqrt{9 - (x-2)^2} \} \quad \text{☹}$$

2° modo Uso la traslazione

$$\iint_A x^2 dx dy = \iint_{u^2+v^2 \leq 9} \underbrace{(u+2)^2}_{x^2} \cdot \underbrace{1}_{J(u,v)} du dv$$

$$= \iint_{u^2+v^2 \leq 9} (u^2 + 4u + 4) du dv$$

$$= \iint_{u^2+v^2 \leq 9} u^2 du dv + 36\pi$$

Cosa ho usato?

$$\iint_{u^2+v^2 \leq 9} 4 du dv = 4 \cdot \text{area cerchio} = 4 \cdot 9\pi$$

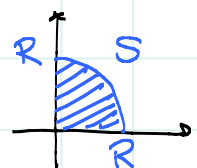
Fatto generale $\iint_A 1 dx dy = \text{Area}(A) =$ volume di un "cilindro" con base A e altezza 1

$\int_a^b 1 dx = \text{lungh. interv.} =$ area rett. con base ... e altezza 1.

Ho poi usato che

$$\iint_{u^2+v^2 \leq 9} u du dv = 0 \quad (\text{per simmetria i pezzi + semplificano i pezzi -})$$

Fatto generale $\iint_S x dx dy = \iint_S y dx dy$



e tanti altri uguali per simmetria

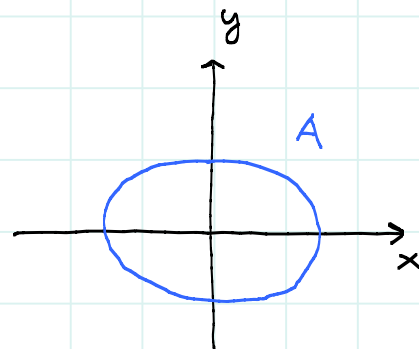
$$\iint_{u^2+v^2 \leq 9} u^2 du dv = \int_0^3 dp \int_0^{2\pi} d\theta \, p^2 \cos^2 \theta \cdot p$$

$$= \underbrace{\int_0^3 p^3 dp}_{\frac{81}{4}} \underbrace{\int_0^{2\pi} \cos^2 \theta d\theta}_{\pi} = \frac{81}{4} \pi$$

— 0 — 0 —

Esempio $A = \{ (x, y) \in \mathbb{R}^2 : x^2 + 2y^2 \leq 3 \}$

$$\iint_A |y| dx dy$$



$$u = x$$

$$v = \sqrt{2}y$$

$$\leadsto u^2 + v^2 \leq 3 \quad \ddot{\cup}$$

① $x = u$
 $y = \frac{1}{\sqrt{2}} v$

② $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

③ $J(u, v) = \frac{1}{\sqrt{2}}$

Quindi $\iint_A |y| dx dy = \iint_{\underbrace{u^2+v^2 \leq 3}_{\substack{\text{nuovo} \\ \text{insieme}}}} \underbrace{\frac{1}{\sqrt{2}} |v|}_{\substack{\uparrow \\ |y|}} \cdot \underbrace{\frac{1}{\sqrt{2}}}_{\substack{\uparrow \\ J}}$

$$= \frac{1}{2} \iint_{\substack{u^2+v^2 \leq 3 \\ \uparrow \\ \text{per simmetria}}} |v| du dv = \iint_{\substack{u^2+v^2 \leq 3 \\ v \geq 0}} v du dv$$

$$= \int_0^{\sqrt{3}} dp \int_0^{\pi} d\theta \, p \sin \theta \, p = \text{banale.}$$

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