



Integrali tripli in coord. cilindriche e sferiche

Coord. cilindriche

ρ, θ, z

esattamente come coord. polari

raggio = ρ

Coord. sferiche

ρ, θ, φ

raggio = $\rho^2 \cos \varphi$

$\rho^2 \sin \varphi$ per i
fisici

$$x = \rho \cos \varphi \cos \theta$$

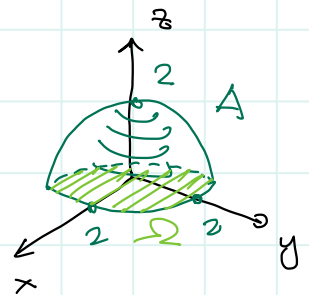
$$y = \rho \cos \varphi \sin \theta$$

$$z = \rho \sin \varphi$$

Oss. Ricordarsi che il raggio è sempre ≥ 0 e infatti $\cos \varphi \geq 0$ per $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Esempio 1 $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, z \geq 0\}$

$$\iiint_A z^2 dx dy dz$$



lo modo

Per colonne

$$A = \{(x, y, z) : (x, y) \in \Omega, 0 \leq z \leq \sqrt{4 - x^2 - y^2}\}$$

$\uparrow$
 cerchio base $x^2 + y^2 \leq 4$

$$\iiint_A z^2 dx dy dz = \iint_{\Omega} dx dy \int_0^{\sqrt{4-x^2-y^2}} dz \quad z^2 = \frac{1}{3} \iint_{\Omega} dx dy [z^3]_{z=0}^{z=\sqrt{4-x^2-y^2}}$$

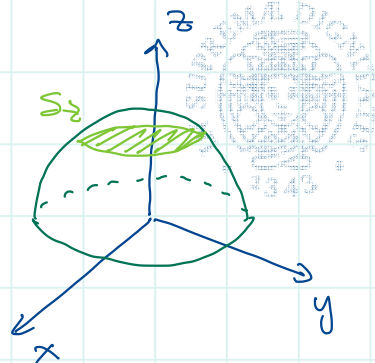
$$= \frac{1}{3} \iint_{\Omega} (4 - x^2 - y^2)^{3/2} dx dy \quad \text{si fa in coord. polari}$$

$$= \frac{1}{3} \int_0^2 \rho d\rho \int_0^{2\pi} d\theta (4 - \rho^2)^{3/2} \quad \rho \uparrow \text{rag.} = \frac{2\pi}{3} \int_0^2 \rho (4 - \rho^2)^{3/2} d\rho$$

$$= \frac{2\pi}{3} \left[- (4 - \rho^2)^{5/2} \cdot \frac{2}{5} \cdot \frac{1}{2} \right]_{\rho=0}^{\rho=2} = \frac{2\pi}{15} 4^{5/2} = \boxed{\frac{64\pi}{15}}$$

2° modo Per sezioni

$$A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 2, \underbrace{x^2 + y^2 \leq 4 - z^2}_{S_z = \text{cerchio di raggio } \sqrt{4 - z^2}}\}$$



$$\begin{aligned} \iiint_A z^2 dx dy dz &= \int_0^2 dz \iint_{S_z} z^2 dx dy = \int_0^2 z^2 \cdot \text{Area}(S_z) dz \\ &= \int_0^2 z^2 \pi (4 - z^2) dz = \pi \int_0^2 (4z^2 - z^4) dz = \pi \left[\frac{4}{3} z^3 - \frac{1}{5} z^5 \right]_{z=0}^{z=2} \\ &= \pi \left(\frac{32}{3} - \frac{32}{5} \right) = 32\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \boxed{\frac{64}{15} \pi} \quad \text{😊} \end{aligned}$$

3° modo Coord. sferiche Come si descrive A in tali coordinate?

$$\boxed{0 \leq \rho \leq 2 \quad 0 \leq \theta \leq 2\pi \quad 0 \leq \varphi \leq \frac{\pi}{2}}$$

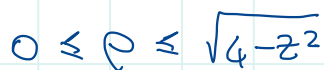
↑↑ semisfera

$$\iiint_A z^2 dx dy dz = \int_0^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \quad \boxed{\rho^2 \sin^2 \varphi} \quad \boxed{\rho^2 \cos \varphi}$$

descrizione di A in coord. sferiche z^2 pagamento

$$\begin{aligned} &= \int_0^2 \rho^4 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^2 \varphi \cos \varphi d\varphi \\ &= \int_0^2 \rho^4 d\rho \int_0^{2\pi} d\theta \left[\frac{1}{3} \sin^3 \varphi \right]_{\varphi=0}^{\varphi=\pi/2} \\ &= \frac{1}{3} \int_0^2 \rho^4 d\rho \int_0^{2\pi} d\theta = \frac{2\pi}{3} \int_0^2 \rho^4 d\rho \\ &= \frac{2\pi}{3} \left[\frac{1}{5} \rho^5 \right]_{\rho=0}^{\rho=2} = \boxed{\frac{64}{15} \pi} \end{aligned}$$

(un po' buffo nella sfera :))


$$\iiint_A z \, dx \, dy \, dz = \int_0^2 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{4-z^2}} \rho \, d\rho \quad z^2 \boxed{\rho} \uparrow \text{proj.}$$

$$= 2\pi \int_0^2 z^2 \int_0^{\sqrt{4-z^2}} \rho \, d\rho \quad = \pi \int_0^2 z^2 [\rho^2]_{\rho=0}^{\rho=\sqrt{4-z^2}}$$

$$= \pi \int_0^2 z^2 (4-z^2) \, dz$$

= da qui se poi (ma si risulta anche prima)
coincide con il metodo per sezioni)
— 0 — 0 —

$$\iiint_S (x+y)^2 dx dy dz = \frac{256}{15} \pi$$
$$\iiint_S (x^2 + y^2 + 2xy) \, dx \, dy \, dz$$

$$\iiint_S xy \, dx \, dy \, dz = 0 \quad \text{per simmetria (è dispari in } x \text{ e } y)$$

$$\iiint_S x^2 d\tau = \iiint_S y^2 d\tau = \iiint_S z^2 d\tau = 2 \iiint_A z^2 dx dy dz$$

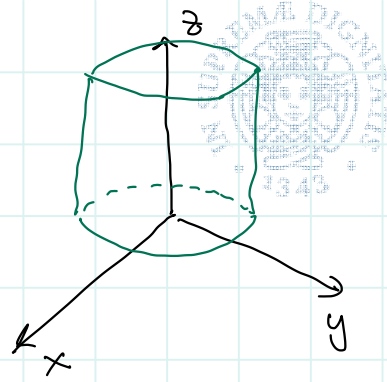
$\uparrow$ per simmetria (se permuto le variabili, la sfera rimane la stessa)

$\uparrow$ semisfera di prima

Esempio 3 C = cilindro come in figura

Raggio base: 1

Altezza: $[0, 3]$



$$\iiint_C x^2 (x^2 + y^2 + z^2) dx dy dz$$

1° modo Coord. cilindriche $0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3$

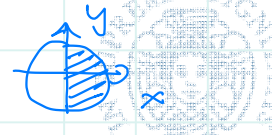
$$\begin{aligned} \iiint_C \dots &= \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^3 dz \underbrace{\rho^2 \cos^2 \theta (\rho^2 + z^2)}_{\text{funz. in coord. cilindriche}} \cdot \underbrace{\rho}_{\text{raggiamento}} \\ &= \int_0^1 \rho^3 d\rho \underbrace{\int_0^{2\pi} \cos^2 \theta d\theta}_{\pi} \int_0^3 (\rho^2 + z^2) dz \\ &= \pi \int_0^1 \rho^3 d\rho \left[\rho^2 z + \frac{1}{3} z^3 \right]_{z=0}^{z=3} = \pi \int_0^1 \rho^3 (3\rho^2 + 9) d\rho \\ &= \pi \int_0^1 (3\rho^5 + 9\rho^3) d\rho = \pi \left(\frac{1}{2} + \frac{9}{4} \right) = \boxed{\frac{11}{4} \pi} \end{aligned}$$

2° modo Per colonne

$$\begin{aligned} \iiint_C \dots &= \iint_{\Omega} dx dy \int_0^3 dz x^2 (x^2 + y^2 + z^2) \\ &\quad \uparrow \text{base} \\ &= \iint_{\Omega} dx dy x^2 \left[zx^2 + zy^2 + \frac{1}{3} z^3 \right]_{z=0}^{z=3} \\ &= \iint_{\Omega} x^2 (3x^2 + 3y^2 + 9) \\ &= \int_0^1 d\rho \int_0^{2\pi} d\theta \rho^2 \cos^2 \theta (3\rho^2 + 9) \rho \end{aligned}$$

e da qui è lo stesso di prima

Esempi di descrizioni in coord. sferiche



$$x^2 + y^2 + z^2 \leq 2, \quad x \geq 0, \quad z \leq 0 \quad 0 \leq \rho \leq \sqrt{2}, \quad -\frac{\pi}{2} \leq \varphi \leq 0, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

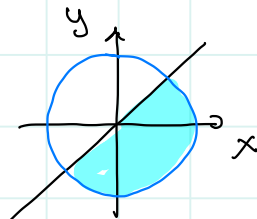
$$1 \leq x^2 + y^2 + z^2 \leq 3$$

$$1 \leq \rho \leq \sqrt{3}, \quad 0 \leq \theta \leq 2\pi, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

(corona sferica)

$$x^2 + y^2 + z^2 \leq 4, \quad x \geq y$$

$$0 \leq \rho \leq 2, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$



$$-\frac{3\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

$$\frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4}$$

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 5, \quad x \leq 0, \quad |z| \leq 3\}$$

In coord. cilindriche diventa: $0 \leq \rho \leq \sqrt{5}, \quad -3 \leq z \leq 3, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

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