

INTEGRALI DOPPI IN COORDINATE POLARI

$$\iint_A f(x,y) dx dy = \iint_B f(\rho \cos \theta, \rho \sin \theta) \cdot \rho d\rho d\theta$$

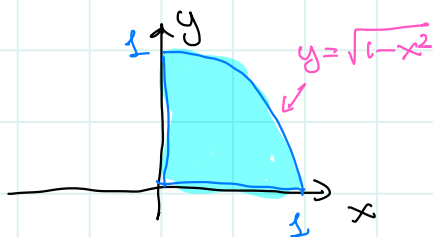
insieme A descritto
in coord. polari

funzione f in
coord. polari

PAGAMENTO

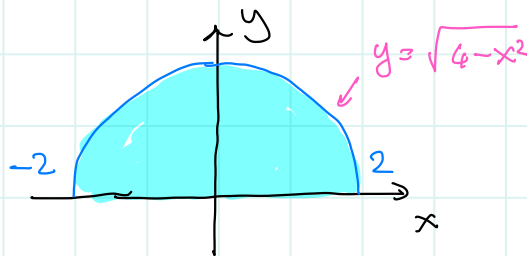
Descrizione di insiemi in coord. polari

Gli insiemi che si descrivono bene sono quelli a simmetria radiale



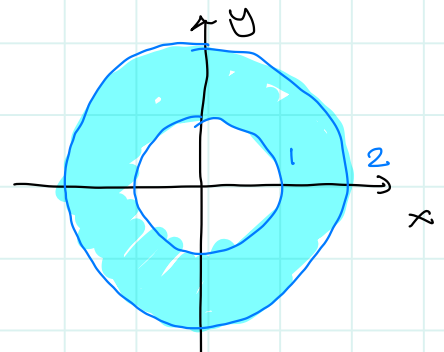
$$0 \leq \rho \leq 1$$

$$0 \leq \theta \leq \frac{\pi}{2}$$



$$0 \leq \rho \leq 2$$

$$0 \leq \theta \leq \pi$$



$$1 \leq \rho \leq 2$$

$$0 \leq \theta \leq 2\pi$$

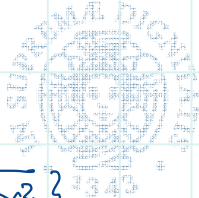
Esempio 1 $A =$ insieme di sinistra $f(x,y) = x$

$$\iint_A x dx dy = \int_0^1 d\rho \int_0^{\frac{\pi}{2}} d\theta \quad \boxed{\rho \cos \theta} \quad \boxed{\rho}$$

descrizione di A in coord. polari x in coord. polari pagamento

$$= \int_0^1 \rho^2 d\rho \underbrace{\int_0^{\frac{\pi}{2}} \cos \theta d\theta}_1 = \int_0^1 \rho^2 d\rho = \boxed{\frac{1}{3}}$$

Domanda: potevamo usare gli insiemi normali? SI!!



Normale asse x : $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$

$$\iint_A x \, dx \, dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \, x = \int_0^1 x \, dx \underbrace{\int_0^{\sqrt{1-x^2}} dy}_{\substack{\text{integrale di 1 su un interv.} \\ = \text{lunghezza interv.}}}$$

$$= \int_0^1 x \sqrt{1-x^2} \, dx = \left[-\frac{1}{3} (1-x^2)^{\frac{3}{2}} \right]_{x=0}^{x=1} = \boxed{\frac{1}{3}} \quad \text{☺}$$

↑
verificare la primitiva
facendo la derivata

Normale asse y : $B = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2}\}$

$$\begin{aligned} \iint_A x \, dx \, dy &= \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx \, x = \frac{1}{2} \int_0^1 dy [x^2]_{x=0}^{x=\sqrt{1-y^2}} \\ &= \frac{1}{2} \int_0^1 (1-y^2) \, dy = \frac{1}{2} \left[y - \frac{1}{3} y^3 \right]_{y=0}^{y=1} \\ &= \frac{1}{2} \cdot \frac{2}{3} = \boxed{\frac{1}{3}} \quad \text{☺} \end{aligned}$$

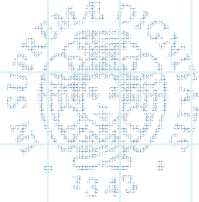
Esempio 2 $A =$ insieme al centro $f(x, y) = y^2$

$$\iint_A y^2 \, dx \, dy = \boxed{\int_0^2 \rho \, d\rho \int_0^\pi d\theta} \quad \boxed{\rho^2 \sin^2 \theta} \quad \boxed{\rho}$$

insieme in coord.
polari y^2

$$\begin{aligned} &= \int_0^2 \rho^3 \, d\rho \underbrace{\int_0^\pi \sin^2 \theta \, d\theta}_{=\frac{\pi}{2}} = \frac{\pi}{2} \int_0^2 \rho^3 \, d\rho \\ &= \frac{\pi}{2} \left[\frac{1}{4} \rho^4 \right]_{\rho=0}^{\rho=2} = \boxed{2\pi} \end{aligned}$$

Fatto generale di Analisi 1



$$\int_a^b \cos^2 \theta d\theta = \int_a^b \sin^2 \theta d\theta = \frac{b-a}{2}$$

SE gli estremi a e b sono multipli interi di $\frac{\pi}{2}$

Idea fondamentale: $\int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta$ perché le due funzioni sono
D'una simmetrica dell'altra
 $\cos \theta = \sin(\frac{\pi}{2} - \theta)$

[Lo stesso vale qui per ogni potenza]

Detto I il valore comune di questi 2 integrali, allora

$$2I = \int_0^{\frac{\pi}{2}} (\cos^2 \theta + \sin^2 \theta) d\theta = \int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

[Qui è fondamentale il quadrato!!]

Lo stesso discorso vale su tutti gli intervalli $[k\frac{\pi}{2}, (k+1)\frac{\pi}{2}]$

Potremmo fare l'esempio 2 come insieme normale? Sì, quasi

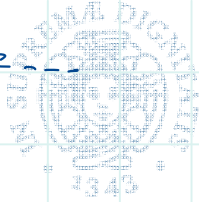
Normale axe x : $A = \{(x, y) \in \mathbb{R}^2: -2 \leq x \leq 2, 0 \leq y \leq \sqrt{4-x^2}\}$

$$\iint_A y^2 dx dy = \int_{-2}^2 dx \int_0^{\sqrt{4-x^2}} y^2 dy = \frac{1}{3} \int_{-2}^2 dx [y^3]_{y=0}^{y=\sqrt{4-x^2}}$$

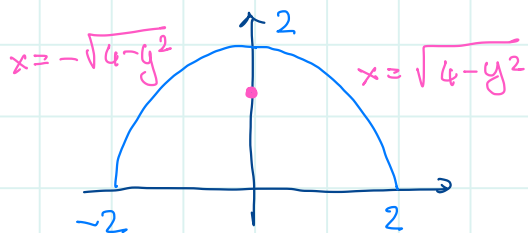
$$= \frac{1}{3} \int_{-2}^2 (4-x^2) \sqrt{4-x^2} dx = \text{ad analisi 1 si fa, con un po' di fatica}$$

Pongo $x = 2 \sin t$ $dx = 2 \cos t dt$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cos^2 t \cdot 2 \cos t \cdot 2 \cos t \, dt = \frac{16}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t \, dt = \dots$$



Normale asse y : $A = \{ (x,y) \in \mathbb{R}^2 : 0 \leq y \leq 2, -\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2} \}$



$$\begin{aligned} \iint_A y^2 \, dx \, dy &= \int_0^2 dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \, y^2 = \int_0^2 y^2 dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \\ &= 2 \int_0^2 y^2 \sqrt{4-y^2} \, dy = \text{con fatica si fa.} \end{aligned}$$

Esempio 3 $f(x,y) = x+y$

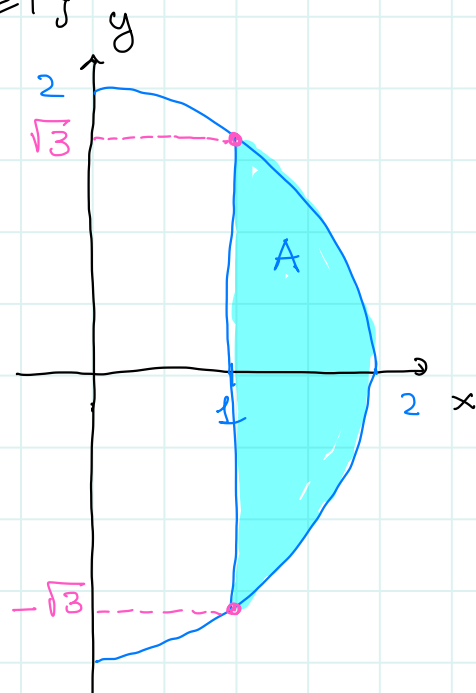
$$A = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, x \geq 1 \}$$

1° modo A è normale risp. asse x

$$A = \{ (x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} \}$$

2° modo A è normale risp. asse y

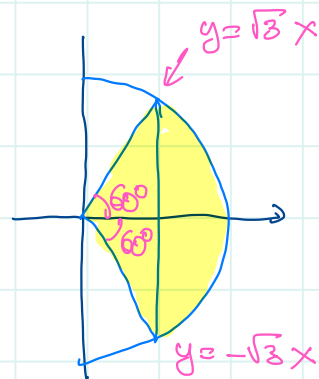
$$A = \{ (x,y) \in \mathbb{R}^2 : -\sqrt{3} \leq y \leq \sqrt{3}, 1 \leq x \leq \sqrt{4-y^2} \}$$



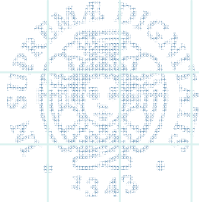
3° modo Scrivo A come differenza tra un settore circolare S e un triangolo T

$$S \text{ in coord. polari è } 0 \leq \rho \leq 2, -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$$

$$T \text{ è } \{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, -\sqrt{3}x \leq y \leq \sqrt{3}x \}$$



Seguendo questa idea otteniamo



$$\begin{aligned} \iint_A (x+iy) dx dy &= \iint_S (x+iy) dx dy - \iint_T (x+iy) dx dy \\ &= \int_0^2 d\rho \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta [e^{i\theta} \cos\theta + e^{i\theta} \sin\theta] \rho \end{aligned}$$

S in coord. polari

↑ pagamento

$$- \int_0^1 dx \int_{-\sqrt{3}x}^{\sqrt{3}x} dy (x+iy)$$

Entrambi gli integrali si calcolano senza problemi.

4° modo Polari dirette. Come scrivo A?

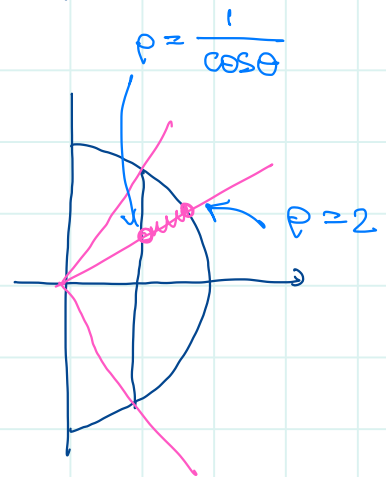
$$-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$$

$$x \geq 1$$

$$\rho \cos\theta \geq 1$$

$$\rho \geq \frac{1}{\cos\theta}$$

$$\frac{1}{\cos\theta} \leq \rho \leq 2$$



In questo modo

$$\iint_A (x+iy) dx dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_{\frac{1}{\cos\theta}}^2 d\rho [e^{i\theta} \cos\theta + e^{i\theta} \sin\theta] \rho$$

Descrizione di A in coord. polari

[Esercizio: controllare che coi 4 modi venga lo stesso numero].

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