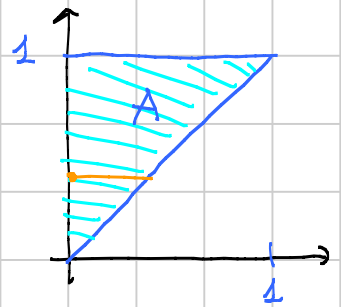


Integrali doppi su insiemi normali (Lezioni 23/24 di AM2-14)Esempio 1 $\iint_A (x^2+y) dx dy$ 1° modo Normale asse x

$$\int_0^1 dx \int_x^1 dy (x^2+y) = \int_0^1 dx \left[x^2 y + \frac{1}{2} y^2 \right]_{y=x}^{y=1}$$

$$= \int_0^1 dx \left\{ x^2 + \frac{1}{2} - x^3 - \frac{x^2}{2} \right\} = \int_0^1 \left\{ \frac{1}{2} + \frac{1}{2} x^2 - x^3 \right\} dx$$

$$= \left[\frac{1}{2} x + \frac{1}{6} x^3 - \frac{1}{4} x^4 \right]_{x=0}^{x=1} = \frac{1}{2} + \frac{1}{6} - \frac{1}{4} = \frac{6+2-3}{12} = \boxed{\frac{5}{12}}$$

2° modo Normale asse y

$$\int_0^1 dy \int_0^y dx (x^2+y) = \int_0^1 dy \left[\frac{1}{3} x^3 + yx \right]_{x=0}^{x=y} = \int_0^1 \left\{ \frac{1}{3} y^3 + y^2 \right\} dy$$

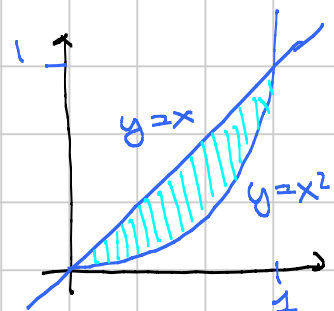
$$= \left[\frac{1}{12} y^4 + \frac{1}{3} y^3 \right]_{y=0}^{y=1} = \frac{1}{12} + \frac{1}{3} = \boxed{\frac{5}{12}} \quad \text{☺}$$

Oss. Svolgere i calcoli in 2 modi è ottimo per verificare la correttezza. In questo caso "normale asse y" era più comodo, perché c'era estremo $y=0$ nell'integrale interno.

Esempio 2 $\iint_A (x^2+xy^2) dx dy$ $\iint_A y dx dy$

$$\textcircled{1} = \int_0^1 dx \int_{x^2}^x dy (x^2+xy^2)$$

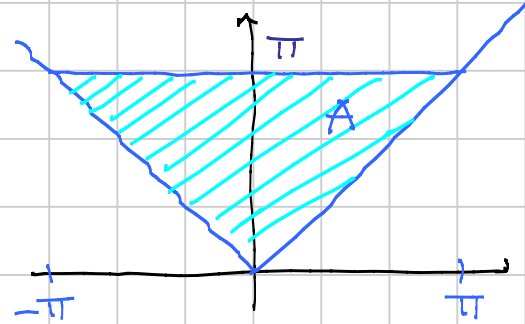
$$= \int_0^1 dx \left[x^2 y + \frac{1}{3} x y^3 \right]_{y=x^2}^{y=x} = \int_0^1 dx \left\{ x^3 + \frac{1}{3} x^4 - x^4 - \frac{1}{3} x^7 \right\} = \text{si fa}$$



$$\begin{aligned} \textcircled{2} &= \int_0^1 dx \int_{x^2}^x dy \cdot y = \int_0^1 dx \left[\frac{1}{2} y^2 \right]_{y=x^2}^{y=x} = \frac{1}{2} \int_0^1 (x^2 - x^4) dx \\ &= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15} \end{aligned}$$

Esempio 3 $\iint_A x^2 \sin y \, dx dy = \textcircled{1}$

$$\iint_A y^2 \sin x \, dx dy = \textcircled{2}$$



Osserviamo subito che $\textcircled{2} = 0$

- $f(-x, y) = -f(x, y)$ **DISPARI** in x
- insieme simmetrico rispetto asse y

Osserviamo anche che $\textcircled{1} > 0$ in quanto $f(x, y) \geq 0$ in A e non identicamente nulla.

Inoltre per simmetria posso integrare sul solo pezzo di destra e moltiplicare per 2 (la $f(x, y)$ è **PARI** in x).

$$\textcircled{1} = 2 \int_0^\pi dx \int_x^\pi dy \, x^2 \sin y = 2 \int_0^\pi x^2 dx \int_x^\pi \sin y \, dy$$

$$= 2 \int_0^\pi x^2 dx \left[-\cos y \right]_{y=x}^{y=\pi} = 2 \int_0^\pi x^2 (1 + \cos x) dx$$

$$= 2 \int_0^\pi x^2 dx + 2 \int_0^\pi x^2 \cos x \, dx$$

"si fa con 2 passaggi per parti".

Si poteva impostare come insieme normale risp. asse y :

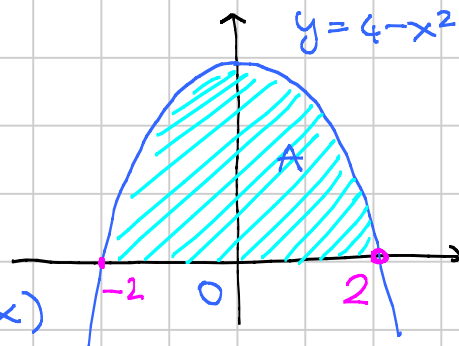
$$\textcircled{1} = 2 \int_0^\pi dy \int_0^y dx \, x^2 \sin y = 2 \int_0^\pi \sin y \, dy \int_0^y x^2 dx$$

$$= \frac{2}{3} \int_0^\pi y^3 \sin y \, dy$$

si fa con 3 passaggi per parti

Esempio 4 $\iint_A x^2 y \, dx \, dy = \textcircled{1}$

$\iint_A x y^2 \, dx \, dy = \textcircled{2}$



Osserviamo che $\textcircled{2} = 0$ ($f(x,y)$ dispari in x)
e che $\textcircled{1} > 0$ ed è uguale al doppio
dell'integrale nella parte destra (o sinistra) [$f(x,y)$ pari in x]

$$\begin{aligned} \textcircled{1} &= \int_0^2 dx \int_0^{4-x^2} dy \, x^2 y = \int_0^2 x^2 dx \left[\frac{1}{2} y^2 \right]_{y=0}^{y=4-x^2} \\ &= \frac{1}{2} \int_0^2 x^2 (4-x^2)^2 dx = \text{si fa facile} \end{aligned}$$

Esempio 5 $A = \{(x,y) \in \mathbb{R}^2 : x^8 \leq y \leq x^9 \leq e^9\}$

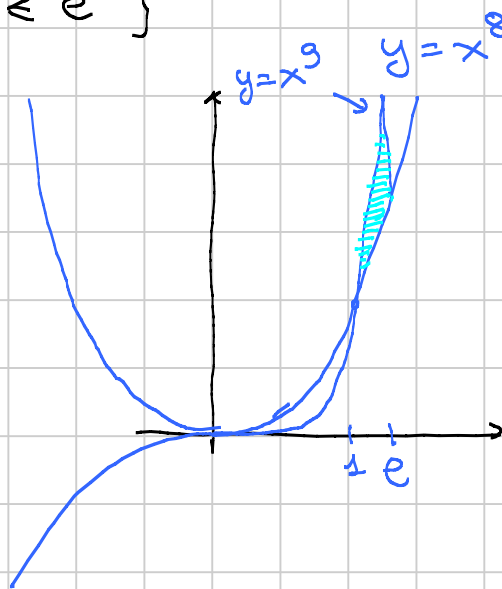
Come è fatto $\mathcal{Q}$ insieme? Vedi figura

$A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq e, x^8 \leq y \leq x^9\}$

↑
normale axe x

$$\iint_A \frac{1}{xy} \, dx \, dy = \int_1^e dx \int_{x^8}^{x^9} dy \, \frac{1}{xy}$$

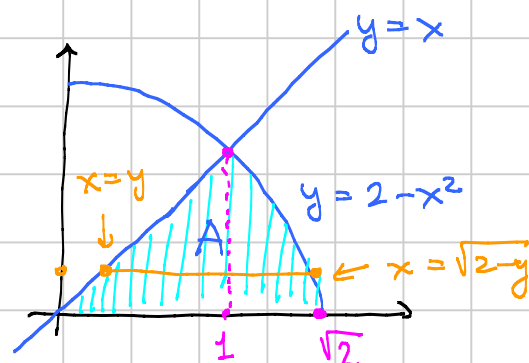
$$\begin{aligned} &= \int_1^e \frac{1}{x} dx \int_{x^8}^{x^9} \frac{1}{y} dy = \int_1^e \frac{1}{x} dx [\log y]_{y=x^8}^{y=x^9} \quad (\text{Precorso!}) \\ &= \int_1^e \frac{\log x}{x} dx = \frac{1}{2} [\log^2 x]_{x=1}^{x=e} = \frac{1}{2} \end{aligned}$$



Esempio 6 $A = \{(x,y) \in \mathbb{R}^2 : y \geq 0, y \leq x, y \leq 2-x^2\}$

$\iint_A x \, dx \, dy$

$\iint_A y \, dx \, dy$



$$\begin{aligned}
 \iint_A x \, dx \, dy &= \int_0^1 dx \int_0^x dy \, x + \int_1^{\sqrt{2}} dx \int_0^{2-x^2} dy \, x \\
 &= \int_0^1 x \, dx \int_0^x 1 \, dy + \int_1^{\sqrt{2}} x \, dx \int_0^{2-x^2} dy \\
 &= \int_0^1 x^2 \, dx + \int_1^{\sqrt{2}} x(2-x^2) \, dx = \text{si fa.}
 \end{aligned}$$

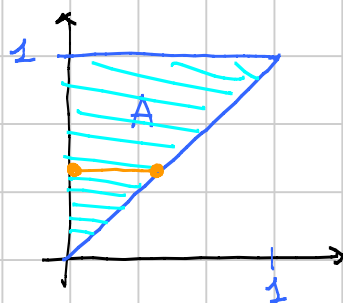
Se lo facevo normale rispetto all'asse y diventava

$$\begin{aligned}
 \iint_A x \, dx \, dy &= \int_0^1 dy \int_y^{\sqrt{2-y}} dx \, x = \int_0^1 dy \left[\frac{1}{2} x^2 \right]_{x=y}^{x=\sqrt{2-y}} \\
 &= \frac{1}{2} \int_0^1 (2-y-\frac{y^2}{2}) dy = \text{banale in un pezzo solo} \text{ 😊} \\
 &\quad \text{[Estremo di integrazione corretto dopo video]}
 \end{aligned}$$

Esempio 7 $\iint_A x e^{y^3} \, dx \, dy$

Normale risp. asse x :

$$\int_0^1 dx \int_x^1 dy \, x e^{y^3} = \int_0^1 x \, dx \int_x^1 e^{y^3} \, dy$$



↑ Non so fare la primitiva 😞

Normale risp. asse y :

$$\begin{aligned}
 \int_0^1 dy \int_0^y dx \, x e^{y^3} &= \int_0^1 e^{y^3} dy \int_0^y x \, dx = \frac{1}{2} \int_0^1 y^2 e^{y^3} dy \\
 &= \frac{1}{2} \frac{1}{3} [e^{y^3}]_{y=0}^{y=1} = \frac{e-1}{6}
 \end{aligned}$$

Oss (molto hard) Se uno non si arrende, veniva anche con il 2° approccio!

Non so fare la primitiva di e^{y^3} . La chiamo $F(y)$ (esiste per analisi 1).

$$\int_0^1 x dx \int_x^1 e^{y^3} dy = \int_0^1 x dx [F(y)]_{y=x}^{y=1} =$$

$$= \int_0^1 \{x F(1) - x F(x)\} dx$$

$$= F(1) \left[\frac{1}{2} x^2 \right]_{x=0}^{x=1} - \int_0^1 x F(x) dx$$

$$= \frac{1}{2} F(1) - \left[\frac{1}{2} x^2 F(x) - \int \frac{x^2}{2} F'(x) dx \right]_0^1$$

$$= \cancel{\frac{1}{2} F(1)} - \cancel{\frac{1}{2} F(1)} + \int_0^1 \frac{x^2}{2} e^{x^3} dx = \frac{1}{2} \int_0^1 x^2 e^{x^3} dx$$

e questo si fa come sopra.

— 0 — 0 —