

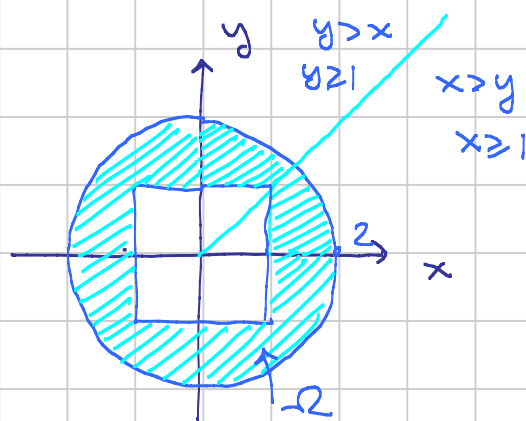
Esempio 1 $A = \{(x, y, z) \in \mathbb{R}^3 : \max\{|x|, |y|\} \geq 1, |z| \leq 1, x^2 + y^2 \leq 4\}$

$$\iiint_A |z + z^3| dx dy dz$$

Come è fatto A ? È una specie di "cilindro" con altezza $|z| \leq 1$ e base descritta di $x^2 + y^2 \leq 4$, $\max\{|x|, |y|\} \geq 1$

Osserviamo che la base è simm.

rispetto agli assi, quindi basta copiare il 1° quadrante



$$\iiint_A |z + z^3| \dots = 2 \iiint_{A^+} z + z^3 dx dy dz$$

$\uparrow$
 $z > 0 \Leftrightarrow z > 0$
 $z(1+z^2)$

$\uparrow$
 p.ti di A con $z > 0$

$$= 2 \int_0^1 dz (z + z^3) \iint_{S_z} 1 dx dy = 2 \int_0^1 (z + z^3) \text{Area}(R) dz$$

$\uparrow$
 S_z
 $\uparrow$
 R

$$= 8(\pi - 1) \int_0^1 z + z^3 dz = \text{si fa.}$$

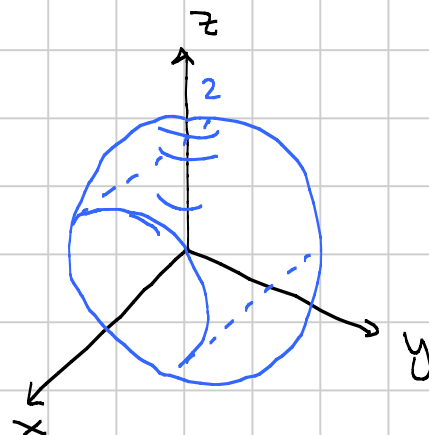
$\uparrow$
 $4\pi - 4$

Si poteva fare per colonne con la stessa difficoltà.

Esempio 2 $A = \{(x, y, z) \in \mathbb{R}^3 : \underbrace{x^2 + y^2 + z^2 \leq 4}_{\text{Dentro la sfera}}, \underbrace{y^2 + z^2 \geq 1}_{\text{fuori dal cilindro}}\}$

$$\iiint_A z^2 dx dy dz$$

Moralmente: stiamo facendo un buco nella sfera.



1° modo Per sezioni lungo l'asse x . Dove varia x ?

$$x^2 \leq 4 - y^2 - z^2 \leq 4 - 1 = 3 \leadsto x^2 \leq 3 \leadsto -\sqrt{3} \leq x \leq \sqrt{3}$$

Fissato x , dove variano y e z ?

$$1 \leq y^2 + z^2 \leq 4 - x^2$$

$$\iiint_A z^2 dx dy dz = \int_{-\sqrt{3}}^{\sqrt{3}} dx \iint_{S_x} z^2 dy dz$$

$$\begin{aligned} y &= \rho \cos \theta \\ z &= \rho \sin \theta \end{aligned}$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} dx \int_1^{\sqrt{4-x^2}} \rho d\rho \int_0^{2\pi} d\theta \underbrace{\rho^2 \sin^2 \theta}_z \cdot \rho$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} dx \underbrace{\int_1^{\sqrt{4-x^2}} \rho^3 d\rho}_{\frac{1}{4}[(4-x^2)^2 - 1]} \underbrace{\int_0^{2\pi} \sin^2 \theta d\theta}_{\pi}$$

$$= \frac{\pi}{4} \int_{-\sqrt{3}}^{\sqrt{3}} \{(4-x^2)^2 - 1\} dx$$

= si fa

2° modo Per colonne con base nel piano yz .

Quando mettiamo $x=0$ troviamo $1 \leq y^2 + z^2 \leq 4$

↑
cil.
↑
sfera

Fissati $(y, z) \in \text{Base}$, dove varia x ? $x^2 \leq 4 - y^2 - z^2 \leadsto -\sqrt{\dots} \leq x \leq \sqrt{\dots}$

$$\iiint_A z^2 dx dy dz = \iint_{\text{Base}} dy dz \cdot z^2 \int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} dx$$

$$= 2 \iint_{\text{Base}} dy dz \cdot z^2 \sqrt{4-y^2-z^2} = 2 \int_1^2 \rho d\rho \int_0^{2\pi} d\theta \rho^2 \sin^2 \theta \sqrt{4-\rho^2} \cdot \rho$$

$$= 2 \int_1^2 \rho^3 \sqrt{4-\rho^2} d\rho \underbrace{\int_0^{2\pi} \sin^2 \theta d\theta}_{\pi} = 2\pi \int_1^2 \rho^3 \sqrt{4-\rho^2} d\rho$$

Pongo $4 - \rho^2 = w \leadsto -2\rho d\rho = dw$

$$= \pi \int_0^3 (4-w) \sqrt{w} dw$$

= si fa

3° modo Si può impostare per differenza

$$\iiint_A z^2 \dots = \iiint_{\text{sfera}} z^2 - \iiint_{\text{Buco}} z^2$$

$\uparrow$ $x^2+y^2+z^2 \leq 4$ $\uparrow$ $x^2+y^2+z^2 \leq 4$
 $y^2+z^2 \leq 1$


$$\iiint_{\text{Buco}} z^2 dx dy dz = \iint_{y^2+z^2 \leq 1} z^2 dy dz \int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}} dx = \text{più o meno come prima.}$$

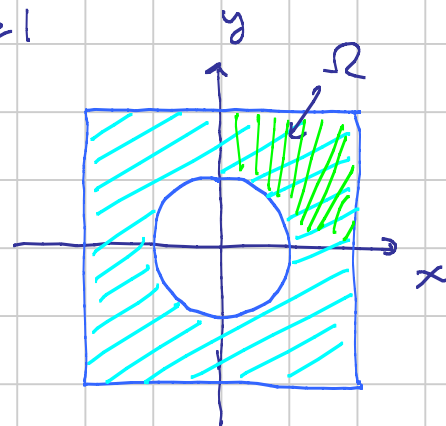
Esempio 4 $A = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 2, |y| \leq 2, |z| \leq 2, x^2 + y^2 \geq 1\}$

$$\iiint_A |x| dx dy dz$$

A è una specie di cilindro con altezza $|z| \leq 2$ e base descritta da $|x| \leq 2, |y| \leq 2, x^2 + y^2 \geq 1$

$$= \iint_{\text{Base}} |x| dx dy \int_{-2}^2 1 dz$$

$$= 4 \iint_{\text{Base}} |x| dx dy = 16 \iint_{\text{Base} \cap 1^{\circ} \text{quadr}} x dx dy$$



$$= 16 \left\{ \iint_{\text{Quadr.}} x dx dy - \iint_{\text{Settore circ.}} x dx dy \right\}$$

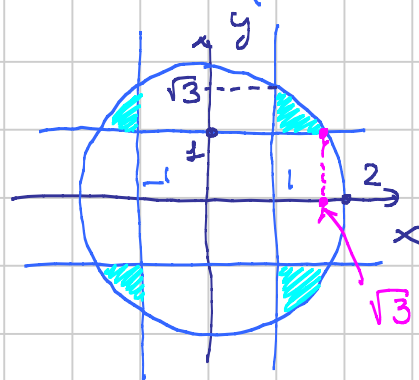
$$= 16 \left\{ \int_0^2 x dx \int_0^2 dy - \int_0^1 \rho d\rho \int_0^{\pi/2} d\theta \rho \cos \theta \cdot \rho \right\}$$

$$= 16 \left\{ 2 \int_0^2 x dx - \int_{\frac{1}{3}}^1 \rho^2 d\rho \int_0^{\pi/2} \underbrace{\cos \theta}_{1} d\theta \right\} = 16 \left\{ 4 - \frac{1}{3} \right\}.$$

Esempio 5 $A = \{(x, y, z) \in \mathbb{R}^3 : |x| \geq 1, |y| \geq 1, \underbrace{x^2 + y^2 + z^2 \leq 4}_{\text{Dentro la sfera}}\}$

$$\iiint_A |x| \, dx \, dy \, dz$$

$$\stackrel{\substack{\uparrow \\ \text{colonne}}}{=} \iint_{\text{Base}} |x| \, dx \, dy \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 1 \, dz$$



$$= 2 \iint_{\text{Base}} |x| \sqrt{4-x^2-y^2} \, dx \, dy$$

$$= 8 \int_1^{\sqrt{3}} dx \int_1^{\sqrt{4-x^2}} x \sqrt{4-x^2-y^2} \, dy \quad \dots \text{ forse era meglio normale asse } y$$

$$= 8 \int_1^{\sqrt{3}} dy \int_1^{\sqrt{4-y^2}} x \sqrt{4-x^2-y^2} \, dx$$

$$= 8 \int_1^{\sqrt{3}} dy \left[\frac{x}{3} (4-x^2-y^2)^{\frac{3}{2}} \left(-\frac{1}{2}\right) \right]_{x=1}^{x=\sqrt{4-y^2}}$$

$$= \frac{1}{24} \int_1^{\sqrt{3}} dy (3-y^2)^{\frac{3}{2}} = \text{si fa con la sostituzione } y = \sqrt{3} \sin w \quad dy = \sqrt{3} \cos w$$

= ci si riduce a integrare $\cos^4 w$.

Esempio 6 $A = [0, 1] \times [0, 1] \times [0, 1] \quad \iiint_A |2x-1| \, dx \, dy \, dz$

Analizziamo per sezioni lungo asse x

$$= \int_0^1 dx |2x-1| \iint_{S_x} 1 \, dy \, dz = \int_0^1 |2x-1| \text{Area}(S_x) \, dx$$

$\uparrow$
quadrato

$$= \int_0^1 |2x-1| \, dx = \frac{1}{2} \quad (\text{verificare spezzando il valore assoluto!})$$

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