

Integrali tripli con cambi di variabile

Esempio 1 $A = \{ (x, y, z) \in \mathbb{R}^3 : \underbrace{(x+2)}_u^2 + \underbrace{(y-3)}_v^2 + \underbrace{(z+1)}_w^2 \leq 4 \}$

$$\iiint_A x \, dx \, dy \, dz$$

$$= \iiint_{\text{nuovo } A} (u-2) \, du \, dv \, dw = -2 \text{ Volume (sfera)}$$

$$\underbrace{u^2+v^2+w^2 \leq 4}_{\substack{\downarrow \\ \text{per simmetria}}} = -2 \frac{4}{3} \pi \cdot 8 = -\frac{64}{3} \pi$$

Non abbiamo pagato (cioè $J=1$) perché il cambio di variabili è una traslazione

Oss. Il volume della sfera è l'integrale di 1 sulla sfera stessa

$$\begin{aligned} \iiint_S 1 \, dx \, dy \, dz &= \int_0^R \rho \, d\rho \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \, \rho^2 \cos \varphi \\ &= \underbrace{\int_0^R \rho^2 \, d\rho}_{\frac{R^3}{3}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \, d\varphi}_2 = \frac{4}{3} \pi R^3 \end{aligned}$$

Esempio 2 $A = \{ (x, y, z) \in \mathbb{R}^3 : 2x^2 + 3y^2 + 5z^2 \leq 6 \}$

$$\underbrace{(\sqrt{2}x)}_u^2 + \underbrace{(\sqrt{3}y)}_v^2 + \underbrace{(\sqrt{5}z)}_w^2 \leq 6$$

$$\iiint_A |x| \, dx \, dy \, dz$$

$$= \iiint_{u^2+v^2+w^2 \leq 6} \frac{|u|}{\sqrt{2}} \cdot \frac{1}{\sqrt{30}} \, du \, dv \, dw = \frac{1}{\sqrt{60}} \iiint_{u^2+v^2+w^2 \leq 6} |u| \, du \, dv \, dw$$

Devo calcolare J seguendo la procedura

① Ricavare: $x = \frac{u}{\sqrt{2}} \quad y = \frac{v}{\sqrt{3}} \quad z = \frac{w}{\sqrt{5}}$

② Matrice jacobiana $\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$

③ $J = |\text{Det}| = \frac{1}{\sqrt{30}}$

Calcolo dell'integrale

$$\iiint_{\dots} |u| \, du \, dv \, dw = \iiint_{\dots} |w| \, du \, dv \, dw$$

$$= 2 \int_0^{\sqrt{6}} dp \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} dp \, p \sin \varphi \cdot p^2 \cos \varphi$$

$$= 2 \int_0^{\sqrt{6}} p^3 dp \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi$$

$$= 2 \cdot \frac{36}{4} \cdot 2\pi \left[\frac{1}{2} \sin \varphi \right]_{\varphi=0}^{\varphi=\frac{\pi}{2}} = 18\pi$$

Esempio 3 $A = \{ (x, y, z) \in \mathbb{R}^3 : 2x^2 + 3y^2 + 4z^4 \leq 5 \}$

$$\underbrace{(\sqrt{2}x)^2}_{u} + \underbrace{(\sqrt{3}y)^2}_{v} + \underbrace{(\sqrt{2}z)^4}_{w} \leq 5$$

$$\iiint_A (x^2 y^2) \, dx \, dy \, dz$$

$$= \iiint_{u^2 + v^2 + w^4 \leq 5} \left(\frac{u^2}{2} + \frac{v^2}{3} \right) \underbrace{\frac{1}{2\sqrt{3}}}_{J} \, du \, dv \, dw$$

$$= \frac{1}{2\sqrt{3}} \frac{5}{6} \iiint_{u^2 + v^2 + w^4 \leq 5} u^2 \, du \, dv \, dw$$

↑
simmetria rispetto a u e v

(e insieme resta lo stesso se scambio u e v)

L'integrale finale si fa per sezioni

$$\int_{-\sqrt[4]{6}}^{\sqrt[4]{6}} dw \iint_{S_w} u^2 du dv = \int_{-\sqrt[4]{6}}^{\sqrt[4]{6}} dw \int_0^{\sqrt{6-w^4}} dp \int_0^{2\pi} p^2 \cos^2 \theta \cdot p d\theta$$

↑
sezioni $u^2+v^2 \leq 6-w^4$

$$= \int_{-\sqrt[4]{6}}^{\sqrt[4]{6}} dw \int_0^{\sqrt{6-w^4}} p^3 dp \underbrace{\int_0^{2\pi} \cos^2 \theta d\theta}_{\pi} = \frac{\pi}{4} \int_{-\sqrt[4]{6}}^{\sqrt[4]{6}} (6-w^4)^2 dw$$

= si fa.

Altro modo: provare a farlo per colonne!!!

Esempio 4 $A = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^4 \leq 1 \}$

$$\iiint_A z^2 dx dy dz$$

1° modo: per sezioni $= \int_{-1}^1 dz z^2 \text{Area}(S_z) = \pi \int_{-1}^1 z^2 (1-z^4) dz$

↑
 $x^2+y^2 \leq 1-z^4$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{7} \right) = \frac{8}{21} \pi$$

2° modo: per colonne $= \iint_{x^2+y^2 \leq 1} dx dy \int_{-\sqrt[4]{1-x^2-y^2}}^{\sqrt[4]{1-x^2-y^2}} z^2 dz$

$$= \frac{2}{3} \iint_{x^2+y^2 \leq 1} dx dy (1-x^2-y^2)^{\frac{3}{4}} = \frac{2}{3} \int_0^1 dp \int_0^{2\pi} d\theta (1-p^2)^{\frac{3}{4}} p$$

$$\left[\frac{2}{3} \cdot 2\pi \cdot \frac{4}{7} (1-p^2)^{\frac{7}{4}} \left(-\frac{1}{2} \right) \right]_{p=0}^{p=1} = \frac{8}{21} \pi \quad \text{😊}$$

3° modo Cambio di variabili creativo

$$x^2 + y^2 + (z^2)^2 \leq 1$$

$x = u \quad y = v \quad z^2 = w$

$$= 2 \iiint_{\substack{u^2+v^2+w^2 \leq 1 \\ w \geq 0}} w \cdot \frac{1}{2\sqrt{w}} du dv dw$$

Vediamo il calcolo di J ④ Ricavare: $x=u$, $y=v$, $z=\sqrt{w}$

② Matrice jacobiana $\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{w}} \end{pmatrix}$

$$= \iiint_{\substack{u^2+v^2+w^2 \leq 1 \\ w \geq 0}} \sqrt{w} du dv dw = \int_0^1 dp \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \sqrt{p \sin \varphi} \cdot p^2 \cos \varphi d\varphi$$

$$= \underbrace{\int_0^1 p^{\frac{5}{2}} dp}_{\frac{2}{7}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \int_0^{\frac{\pi}{2}} \sqrt{\sin \varphi} \cdot \cos \varphi d\varphi = \frac{4\pi}{7} \left[\frac{2}{3} (\sin \varphi)^{\frac{3}{2}} \right]_{\varphi=0}^{\varphi=\frac{\pi}{2}} = \frac{4\pi}{7} \cdot \frac{2}{3} = \boxed{\frac{8\pi}{21}} \text{ 😊}$$

Esempio 5 $A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 + xy + yz + zx \leq 18\}$

$$\iiint_A y^2 dx dy dz$$

$$2x^2+2y^2+2z^2+2xy+2yz+2zx \leq 36$$

$$\underbrace{(x+y)}_u^2 + \underbrace{(y+z)}_v^2 + \underbrace{(z+x)}_w^2 \leq 36$$

$$= \iiint_{u^2+v^2+w^2 \leq 36} \left(\frac{u+v-w}{2} \right)^2 \cdot \frac{1}{2} du dv dw$$

$$x = \frac{u+w-v}{2}$$

$$y = \frac{u+v-w}{2}$$

$$z = \frac{-u+v+w}{2}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\text{Det} = \frac{1}{8} (\cancel{1} \cdot \cancel{1} + 1 + 1 + 1 + 1) = \frac{1}{2}$$

[Jacobiano corretto dopo video]

$$= \frac{1}{8} \iiint_{\dots} (u+v-w)^2 du dv dw$$

$$= \frac{1}{8} \iiint_{\dots} (u^2 + v^2 + w^2 + \underbrace{2uv}_{0} - \underbrace{2uw}_{0} - \underbrace{2vw}_{0}) du dv dw$$

per simmetria

$$= \frac{1}{16} \int_0^6 dp \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \, p^2 \cdot p^2 \cos\varphi = \text{si fa facile :}$$

Esempio 6 $A = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z+y)^2 \leq 4, y \geq 0 \}$

$$\iiint_A z \, dx \, dy \, dz$$

$$x = u$$

$$y = v$$

$$(z+y) = w$$

Che cosa diventa A ? $\{ u^2 + v^2 + w^2 \leq 4, v \geq 0 \} = A_{u,w}$

$$= \iiint_{A_{u,w}} (w-v) \, du \, dv \, dw = - \int_0^2 dp \int_0^{\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p \cos\varphi \sin\theta \, p^2 \cos\varphi \, d\varphi$$

per simmetria

$$x = u$$

$$y = v$$

$$z = w - v$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rightsquigarrow J = 1$$

$$= - \underbrace{\int_0^2 p^3 dp}_4 \underbrace{\int_0^{\pi} \sin\theta d\theta}_2 \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi d\varphi}_{\frac{\pi}{2}} = -4\pi$$

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