

Integrali tripli in coordinate sferiche

$$(p, \theta, \varphi) \quad p \geq 0 \quad \theta \in [0, 2\pi] \quad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$x = \overbrace{p \cos \varphi}^{\text{OQ}} \cos \theta$$

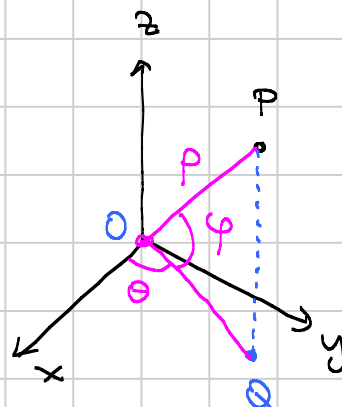
$$y = p \cos \varphi \sin \theta$$

$$z = p \sin \varphi$$

} coordinate sferiche
dei geografi

φ = latitudine

θ = longitudine



Achtung! Le coordinate sferiche di fisica / ingegneri (anche i matematici si stanno adeguando) sono diverse, nel senso che

→ φ può essere diverso

→ talvolta φ e θ sono scambiati!

Usando le coord. dei geografi si ha

$$J = p^2 \cos \varphi$$

fisici si ha

$$J = p^2 \sin \varphi \sin \theta$$

Descrizione di insiemi in coord. sferiche

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\} \quad 0 \leq p \leq 1, 0 \leq \theta \leq 2\pi, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

$$C = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 2\} \quad 1 \leq p \leq \sqrt{2}, 0 \leq \theta \leq 2\pi, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

$$A = \{(x, y, z) \in S, z \leq 0\} \quad 0 \leq p \leq 1, 0 \leq \theta \leq 2\pi, -\frac{\pi}{2} \leq \varphi \leq 0$$

semisfera "bassa" sotto piano xy

$$B = \{(x, y, z) \in S, z \geq 0, y \leq 0\} \quad 0 \leq p \leq 1, \pi \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{2}$$

Quanto di sfera

Esempio 1 $\iiint_S y \, dx \, dy \, dz = 0$ per simmetria

$$\iiint_C y \, dx \, dy \, dz = \iiint_A y \, dx \, dy \, dz$$

$\iiint_B y \, dx \, dy \, dz =$

(da subito sappiamo che viene < 0)

$$= \int_0^1 \rho \, d\rho \int_{\pi}^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \underbrace{\rho \cos\varphi \sin\theta}_y \underbrace{\frac{\rho^2 \cos\varphi}{J}}_J$$
$$= \underbrace{\int_0^1 \rho^3 \, d\rho}_{\frac{1}{4}} \underbrace{\int_{\pi}^{2\pi} \sin\theta \, d\theta}_{-2} \underbrace{\int_0^{\frac{\pi}{2}} \cos^2\varphi \, d\varphi}_{\frac{\pi}{4}} = -\frac{\pi}{8}$$

Esempio 2 $\iiint_S |y| \, dx \, dy \, dz = \frac{\pi}{2}$ (è 4 volte il precedente cambiato di segno)

[Fare il conto esplicito per avere conferma]

Esempio 3 $\iiint_S |z| \, dx \, dy \, dz = 2 \iiint_{S_+} z \, dx \, dy \, dz = \frac{\pi}{2}$

dove $S_+ := \{(x, y, z) \in S : z \geq 0\}$ $\rho \in [0, 1], \theta \in [0, 2\pi]$
 $\varphi \in [0, \frac{\pi}{2}]$

$$\iiint_{S_+} z \, dx \, dy \, dz = \int_0^1 \rho \, d\rho \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \underbrace{\rho \sin\varphi}_z \cdot \underbrace{\frac{\rho^2 \cos\varphi}{J}}_J$$
$$= \underbrace{\int_0^1 \rho^3 \, d\rho}_{\frac{1}{4}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\frac{\pi}{2}} \sin\varphi \cdot \cos\varphi \, d\varphi}_{\left[\frac{1}{2} \sin^2\varphi\right]_0^{\pi/2} = \frac{1}{2}} = \frac{\pi}{4}$$

Oss. importante Era ovvio a priori che

$$\iiint_S |z| \, dx \, dy \, dz = \iiint_S |y| \, dx \, dy \, dz = \iiint_S |x| \, dx \, dy \, dz$$

sempre per ragioni di simmetria.

Esempio 4 $\iiint_S x^2 dx dy dz$

1° modo Conto completo in coord. così come sono

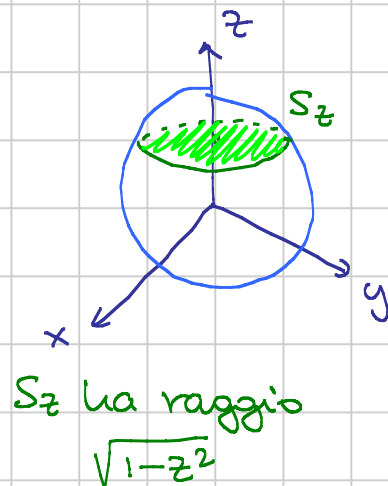
$$\begin{aligned}
 &= \int_0^1 dp \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \underbrace{\rho^2 \cos^2 \varphi \cos^2 \theta}_{x^2} \cdot \underbrace{\rho^2 \cos \varphi}_J \\
 &= \underbrace{\int_0^1 \rho^4 d\rho}_{\frac{1}{5}} \underbrace{\int_0^{2\pi} \cos^2 \theta d\theta}_{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 \varphi d\varphi = \frac{\pi}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos \varphi}_{\text{"cos } \varphi \cdot \cos^2 \varphi} d\varphi \\
 &= \frac{\pi}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos \varphi (1 - \sin^2 \varphi)}_{\cos \varphi - \sin^2 \varphi \cos \varphi} d\varphi = \frac{\pi}{5} \left[+\sin \varphi - \frac{1}{3} \sin^3 \varphi \right]_{\varphi = -\frac{\pi}{2}}^{\varphi = \frac{\pi}{2}} \\
 &= \frac{2\pi}{5} \left[1 - \frac{1}{3} \right] = \boxed{\frac{4\pi}{15}}
 \end{aligned}$$

2° modo Usiamo la simmetria per integrare z^2 , che è più semplice ☺

$$\begin{aligned}
 \iiint_S x^2 dx dy dz &= \iiint_S z^2 dx dy dz = \int_0^1 dp \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \underbrace{\rho^2 \sin^2 \varphi}_{z^2} \cdot \underbrace{\rho^2 \cos \varphi}_J \\
 &= \underbrace{\int_0^1 \rho^4 d\rho}_{\frac{1}{5}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi \cos \varphi d\varphi = \frac{2\pi}{5} \left[\frac{1}{3} \sin^3 \varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= \frac{2\pi}{5} \cdot \frac{2}{3} = \boxed{\frac{4\pi}{15}}
 \end{aligned}$$

3° modo Integro z^2 per sezioni!

$$\begin{aligned}
 \iiint_S z^2 dx dy dz &= \int_{-1}^1 dz z^2 \iint_{S_z} dx dy \\
 &= \int_{-1}^1 z^2 \text{Area}(S_z) dz = \pi \int_{-1}^1 z^2 (1 - z^2) dz = \pi \left[\frac{1}{3} - \frac{1}{5} \right] \\
 &= \boxed{\frac{4\pi}{15}}
 \end{aligned}$$



Esempio 5 $\iiint_A y^4 dx dy dz$

$A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x \geq 0\} = \text{mezza sfera}$

$= \frac{1}{2} \iiint_S y^4 dx dy dz = \frac{1}{2} \iiint_S z^4 dx dy dz$

↑
tutta la sfera
di raggio 2

↑
più comodo!

$= \frac{1}{2} \int_0^2 \rho^6 d\rho \int_0^{2\pi} d\sigma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \rho^4 \sin^4 \varphi \cdot \rho^2 \cos \varphi$

$= \frac{1}{2} \underbrace{\int_0^2 \rho^6 d\rho}_{\frac{2^7}{7}} \underbrace{\int_0^{2\pi} d\sigma}_{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 \varphi \cdot \cos \varphi d\varphi = \frac{2^7}{7} \pi \left[\frac{1}{5} \sin^5 \varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$
 $= \frac{256}{35} \pi$

2° modo: per sezioni

cerchio di raggio $\sqrt{4-z^2}$

$\iiint_S z^4 dx dy dz = \int_{-2}^2 dz \cdot z^4 \cdot \text{Area}(S_z) = \pi \int_{-2}^2 z^4 (4-z^2) dz$

$= 2\pi \int_0^2 (4z^4 - z^6) dz = 2\pi \left[\frac{4}{5} z^5 - \frac{1}{7} z^7 \right]_0^2 = 2\pi \cdot 2^7 \left(\frac{1}{5} - \frac{1}{7} \right)$
 $= 2\pi \cdot 2^7 \cdot \frac{2}{35}$
 $= \frac{\pi \cdot 2^9}{35}$ 😊

3° modo: per colonne!

$\sqrt{4-x^2-y^2}$

$\iiint_S z^4 dx dy dz = \iint_{\text{Base}} dx dy \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} z^4 dz$

$= \frac{2}{5} \iint_{\text{Base}} dx dy \left[\sqrt{4-x^2-y^2} \right]^5 = \frac{2}{5} \int_0^2 \rho^6 d\rho \int_0^{2\pi} d\sigma (4-\rho^2) \cdot \rho$

$= \frac{4\pi}{5} \int_0^2 \rho (4-\rho^2)^{\frac{5}{2}} d\rho = \left[\frac{4\pi}{5} \cdot \frac{2}{7} (4-\rho^2)^{\frac{7}{2}} \left(-\frac{1}{2} \right) \right]_{\rho=0}^{\rho=2}$

$$= \frac{4\pi}{35} 4^{\frac{7}{2}} = \frac{4\pi}{35} \cdot 2^7 = \frac{\pi \cdot 2^9}{35}$$

Quando divido per 2 ritrovo la risposta del 1° modo.

4° modo Provare ad integrare direttamente x^4 in coord. sferiche !!!

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