

Integrali tripli Prerequisiti: LEZIONI 25/26 di AM2-14

Formula di spezzamento: $\rightarrow$ sui parallelepipedi $3 = 1+1+1$
 $\rightarrow$ per colonne $3 = 2+1$
 $\rightarrow$ per sezioni $3 = 1+2$

Esempio 1 $A = [0,1] \times [2,3] \times [4,5]$ $\iiint_A (xy+z) dx dy dz$

$$= \int_0^1 dx \int_2^3 dy \int_4^5 dz (xy+z) = \frac{5}{4} + \frac{9}{2}$$

$$\int_0^1 dx \int_2^3 dy \int_4^5 xy dz = \int_0^1 x dx \int_2^3 y dy \underbrace{\int_4^5 1 dz}_1 = \int_0^1 x dx \left[\frac{1}{2} y^2 \right]_{y=2}^{y=3}$$

$$= \frac{5}{2} \left[\frac{1}{2} x^2 \right]_{x=0}^{x=1} = \frac{5}{4}$$

$$\int_0^1 dx \int_2^3 dy \int_4^5 dz z = \int_0^1 dx \int_2^3 dy \left[\frac{1}{2} z^2 \right]_{z=4}^{z=5} = \frac{9}{2} \underbrace{\int_0^1 dx \int_2^3 dy}_{\text{area di } [0,1] \times [2,3] = 1} = \frac{9}{2}$$

Oss. Posso scegliere a piacere in che ordine gestire le 3 variabili

Ad esempio il secondo lo potevo gestire come

$$\int_4^5 dz \int_2^3 dy \int_0^1 dx z = \int_4^5 dz \underbrace{\int_2^3 dy \int_0^1 dx \cdot 1}_{\text{Area}=1} = \int_4^5 z dz = \dots$$

Esempio 2 $A = [0,\pi] \times [0,\pi] \times [0,2]$ $f(x,y,z) = \sin(x+y) \cdot z$

$$\int_0^\pi dx \int_0^\pi dy \int_0^2 dz z \cdot \sin(x+y) = \int_0^\pi dx \int_0^\pi dy \sin(x+y) \left[\frac{1}{2} z^2 \right]_{z=0}^{z=2}$$

$$2 \int_0^\pi dx \int_0^\pi dy \sin(x+y) = 2 \int_0^\pi dx \left[-\cos(x+y) \right]_{y=0}^{y=\pi} = \dots \text{ si fa}$$

Esempio 3 $A = \{(x, y, z) \in \mathbb{R}^3 : \underbrace{0 \leq x \leq 1, 0 \leq y \leq 1}_{\text{base}}, \underbrace{0 \leq z \leq x+y}_{\text{colonna}}\}$

$\iiint_A z \, dx \, dy \, dz$ insieme normale rispetto al piano xy

Uso la formula di riduzione per colonne

$$= \int_0^1 dx \int_0^1 dy \int_0^{x+y} dz \cdot z = \frac{1}{2} \int_0^1 dx \int_0^1 dy [z^2]_{z=0}^{z=x+y}$$

↑ integrale sulla colonna

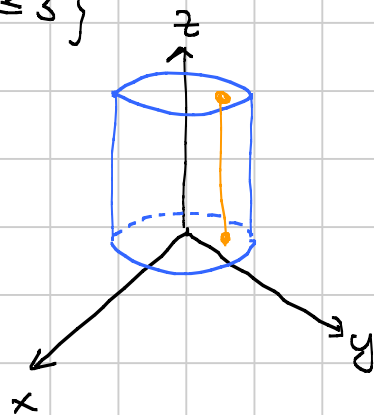
$$= \frac{1}{2} \int_0^1 dx \int_0^1 dy (x+y)^2 = \text{si fa facilmente}$$

Esempio 4 $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2, 0 \leq z \leq 3\}$

$$\iiint_C z \, dx \, dy \, dz$$

$$\iiint_C (x^2 + y^2 + z^2) \, dx \, dy \, dz$$

Possiamo pensare C come insieme normale risp. al piano xy e integrare per colonne



$$\iiint_C z \, dx \, dy \, dz = \iint_{x^2+y^2 \leq 2} dx \, dy \int_0^3 dz \cdot z$$

$$= \frac{1}{2} \iint_{x^2+y^2 \leq 2} dx \, dy [z^2]_{z=0}^{z=3} = \frac{9}{2} \iint_{x^2+y^2 \leq 2} 1 \cdot dx \, dy = 9\pi$$

area cerchio

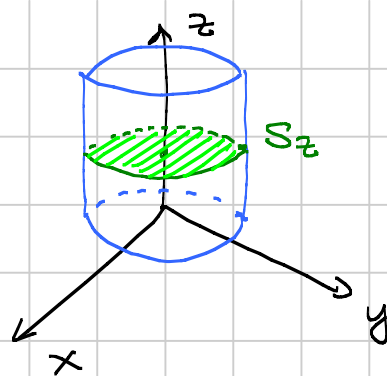
$$\iint_C (x^2 + y^2 + z^2) \, dx \, dy \, dz = \iint_{x^2+y^2 \leq 2} dx \, dy \int_0^3 dz (x^2 + y^2 + z^2)$$

$$= \iint_{x^2+y^2 \leq 2} dx \, dy \left[(x^2 + y^2)z + \frac{1}{3}z^3 \right]_{z=0}^{z=3} = \iint_{x^2+y^2 \leq 2} dx \, dy [3(x^2 + y^2) + 9]$$

$$= 9 \text{ Area} + 3 \int_0^2 \rho \, d\rho \int_0^{2\pi} d\varphi \, \rho^2 \cdot \underset{\substack{\uparrow \\ \text{rag.}}}{\rho} = 18\pi + 2\pi \left[\frac{3}{4} \rho^4 \right]_{\rho=0}^{\rho=2} = 18\pi + 24\pi = 42\pi$$

In alternativa potevo ragionare per sezioni

$$\begin{aligned}\iiint_C z \, dx \, dy \, dz &= \int_0^3 dz \iint_{S_z} z \, dx \, dy \\ &= \int_0^3 z \, dz \underbrace{\iint_{S_z} 1 \, dx \, dy}_{\text{area}(S_z) = 2\pi} \\ &= 2\pi \int_0^3 z \, dz = \pi [z^2]_{z=0}^{z=3} = 9\pi \quad \text{😊}\end{aligned}$$



Calcolare l'altro per sezioni poteva essere fortidioso, quindi conveniva spezzare

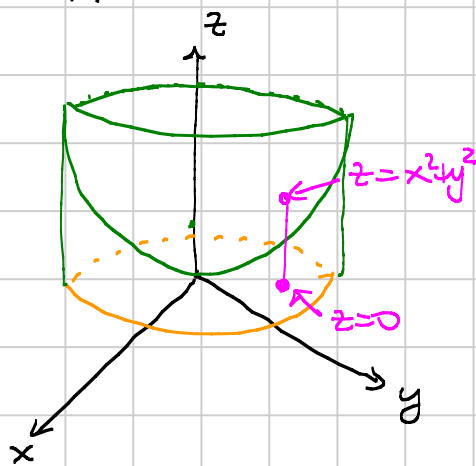
$$\iiint_C z^2 \, dx \, dy \, dz = \int_0^3 z^2 \, dz \underbrace{\iint_{S_z} 1 \, dx \, dy}_{\text{sezioni}} = 2\pi \int_0^3 z^2 \, dz = 18\pi$$

$$\iiint_C (x^2 + y^2) \, dx \, dy \, dz = \iint_{x^2 + y^2 \leq 1} (x^2 + y^2) \, dx \, dy \underbrace{\int_0^3 1 \cdot dz}_{= 3} = 3 \iint_{x^2 + y^2 \leq 1} (x^2 + y^2) \, dx \, dy = \text{si fa.}$$

Esempio 5 $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq x^2 + y^2 \leq 1\} = A$

$$\iiint_A z^2 \, dx \, dy \, dz$$

↑ sono i punti che stanno nel cilindro, ma sotto il paraboloide



Descrizione come insieme normale

$$A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \leftarrow \text{base}, 0 \leq z \leq x^2 + y^2 \leftarrow \text{cilindro}\}$$

per cui come

$$\downarrow = \iint_{x^2 + y^2 \leq 1} dx \, dy \int_0^{x^2 + y^2} dz \, z^2 = \frac{1}{3} \iint_{x^2 + y^2 \leq 1} dx \, dy [z^3]_{z=0}^{z=x^2 + y^2}$$

$$= \frac{1}{3} \iint_{x^2 + y^2 \leq 1} (x^2 + y^2)^3 = \frac{1}{3} \int_0^1 \rho \, d\rho \int_0^{2\pi} d\theta \, \rho^6 \cdot \rho = \frac{2\pi}{3} \left[\frac{1}{8} \rho^8 \right]_0^1 = \frac{\pi}{12}$$

Si poteva fare per sezioni? Sì, a patto di capire come sono fatte!

Fissato $z \in [0, 1]$, le sezioni S_z sono $\{(x, y) \in \mathbb{R}^2: z \leq x^2 + y^2 \leq 1\}$
 corona circolare con raggio interno $\sqrt{z}$ e raggio est. 1

$$\begin{aligned} \iiint_A z^2 dx dy dz &= \int_0^1 dz \underset{\text{sezioni}}{z^2 \iint_{S_z} 1 dx dy} = \int_0^1 z^2 \cdot \text{Area}(S_z) dz \\ &= \int_0^1 z^2 \pi (1-z) dz = \pi \int_0^1 (z^2 - z^3) dz = \pi \left[\frac{1}{3} z^3 - \frac{1}{4} z^4 \right]_{z=0}^{z=1} \\ &= \frac{\pi}{12} \quad \text{☺} \end{aligned}$$

Oss. Il conto nei 2 casi è completamente diverso, quindi è utilissimo farlo in 2 modi diversi per VERIFICA.

Esempio 6 $A = \{(x, y, z) \in \mathbb{R}^3: \underbrace{x^2 + y^2 \leq 4}_{\text{cilindro}}, z \in [0, 2], \underbrace{x \geq 0}_{\text{metà cilindro}}\}$

$$\iiint_A (x+y+z) dx dy dz$$

$$\iiint_A y dx dy dz = 0 \quad \text{per simmetria } (x, y, z) \rightarrow (x, -y, z)$$

- La funzione è dispari
- L'insieme A viene mandato in sé.

$$\begin{aligned} \iiint_A x dx dy dz &= \iint_{x^2+y^2 \leq 4} x dx dy \underbrace{\int_0^2 dz}_{=2} = 2 \iint_{x^2+y^2 \leq 4, x \geq 0} x dx dy \\ &= 2 \int_0^2 dp \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \underbrace{p \cos \theta p}_{=2} = 2 \int_0^2 p^2 dp \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta = 4 \left[\frac{1}{3} p^3 \right]_{p=0}^{p=2} = \frac{32}{3} \end{aligned}$$

$$\iiint_A z dx dy dz = \int_0^2 dz \underset{\text{sezioni}}{z \cdot \text{Area}(S_z)} = 2\pi \int_0^2 z dz = 4\pi.$$