

Utilizzo delle coordinate polari negli integrali doppi (lez. 27 AM2-14)

Esempio 1 $\iint_{S_{NE}} x \, dx \, dy$

Descrivo S_{NE} in coord. polari

$$0 \leq \rho \leq 1 \quad 0 \leq \theta \leq \frac{\pi}{2}$$

descrizione di tipo rettangolo

L' integrale diventa

$$\int_0^1 \rho \, d\rho \int_0^{\frac{\pi}{2}} d\theta$$

descrizione insieme
in coord. polari

$$\rho \cos \theta$$

x in coord.
polari

$$\rho$$

↑ pagamento

$$= \underbrace{\int_0^1 \rho^2 \, d\rho}_{\frac{1}{3}} \underbrace{\int_0^{\frac{\pi}{2}} \cos \theta \, d\theta}_1 = \frac{1}{3}$$

Oss. Potrei calcolare l' integrale pensando S_{NE} come insieme normale risp. asse x

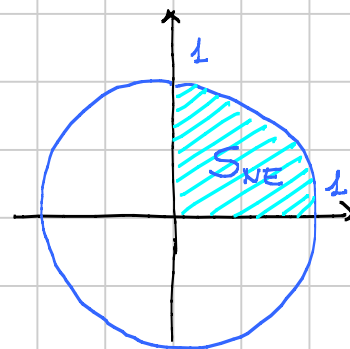
$$\iint_{S_{NE}} x \, dx \, dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} x \, dy = \int_0^1 x \, dx \int_0^{\sqrt{1-x^2}} 1 \, dy$$

$$= \int_0^1 x \sqrt{1-x^2} \, dx = \left[-\frac{1}{3} (1-x^2)^{\frac{3}{2}} \right]_{x=0}^{x=1} = \frac{1}{3} \quad \text{☺}$$

↑
VERIFICA!!!

$$\iint_{S_{NE}} x \, dx \, dy = \int_0^1 dy \int_0^{\sqrt{1-y^2}} x \, dx = \int_0^1 dy \left[\frac{1}{2} x^2 \right]_{x=0}^{x=\sqrt{1-y^2}}$$

$$= \frac{1}{2} \int_0^1 (1-y^2) \, dy = \frac{1}{2} \left[y - \frac{1}{3} y^3 \right]_0^1 = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \quad \text{☺}$$



Esempio 2 $\iint_{SE} x^2 dx dy$

In coord. polari diventa

$$\int_0^1 dp \int_0^{\frac{\pi}{2}} d\theta \underbrace{p^2 \cos^2 \theta}_{x^2} \overset{\substack{\uparrow \\ \text{rag.}}}{p} = \int_0^1 p^3 dp \underbrace{\int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta}_{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} \int_0^1 p^3 dp = \frac{\pi}{16}$$

Fatto generale (analisi 1): l'integrale di $\cos^2 \theta$ e di $\sin^2 \theta$ su intervalli i cui estremi sono multipli interi di $\frac{\pi}{2}$ è sempre uguale alla metà della lunghezza dell'intervallo.

[Dim: l'integrale di $\cos^2 \theta$ e di $\sin^2 \theta$ sono uguali per ragioni di simmetria, e la loro somma è l'integrale di 1, cioè la lunghezza]

Se lo volevo fare come insieme normale

$$\iint_{S_{NE}} x^2 dx dy = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy x^2 = \int_0^1 x^2 \sqrt{1-x^2} dx \quad \text{☹}$$

↑
asse x

(serve sostituz. trig o altro)

$$= \int_0^1 dy \int_0^{\sqrt{1-y^2}} x^2 dx = \frac{1}{3} \int_0^1 (1-y^2)^{\frac{3}{2}} dy \quad \text{☹ ☹}$$

Esempio 3 $\iint_{S_{NE}} y dx dy = \frac{1}{3}$ $\iint_{S_{NE}} y^2 dx dy = \frac{\pi}{16}$

Sono uguali ai precedenti per ragioni di simmetria!

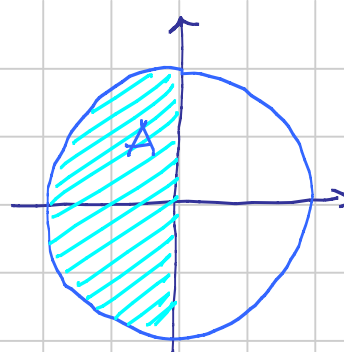
$$\iint_{S_{SW}} y dx dy = \iint_{S_{SE}} y dx dy = -\frac{1}{3} \quad (\text{simmetria!})$$

$$\iint_{S_{NW}} x^2 dx dy = \iint_{S_{SW}} y^2 dx dy = \frac{\pi}{16} \quad (\quad " \quad)$$

Esempio 4

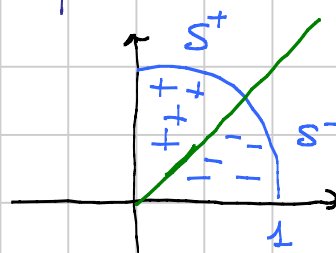
$$\iint_A |y| \, dx \, dy = \frac{2}{3}$$

$$\iint_A y^2 \, dx \, dy = \frac{\pi}{8}$$



Esempio 5

$$\iint_{S_{NE}} |y-x| \, dx \, dy$$



$$= \iint_{S^+} (y-x) \, dx \, dy - \iint_{S^-} (y-x) \, dx \, dy$$

Posso calcolare entrambi in coord. polari, o calcolarne solo uno e poi moltiplicare per 2 (giustificare per bene la simmetria)

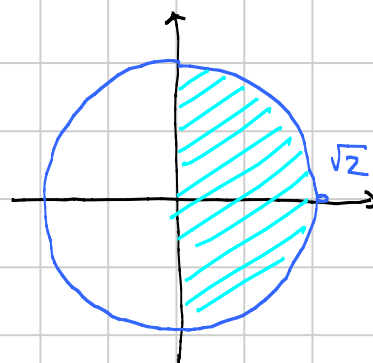
$$\iint_{S^+} (y-x) \, dx \, dy = \underbrace{\int_0^1 dp \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dy}_{S^+} \underbrace{p(\sin\theta - \cos\theta)}_{y-x} \cdot \underbrace{p}_{\text{pag.}}$$

$$= \underbrace{\int_0^1 p^2 dp}_{\frac{1}{3}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin\theta - \cos\theta) d\theta = \frac{1}{3} [-\cos\theta - \sin\theta]_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} = \text{si fa} \dots$$

Esempio 6 $A = \{(x,y): x^2 + y^2 \leq 2, x \geq 0\}$

$$\iint_A x^2 \, dx \, dy$$

$$\iint_A \sqrt{x^2 + y^2} \, dx \, dy$$



$$\iint_A x^2 \, dx \, dy = \int_0^{\sqrt{2}} dp \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, p^2 \cos^2\theta \, p$$

$$= 2 \int_0^{\sqrt{2}} dp \int_0^{\frac{\pi}{2}} d\theta \, p^3 \cos^2\theta = 2 \int_0^{\sqrt{2}} p^3 dp \underbrace{\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta}_{\frac{\pi}{4}} = \frac{\pi}{2}$$

$$= 2 \cdot \frac{1}{4} \cdot 4$$

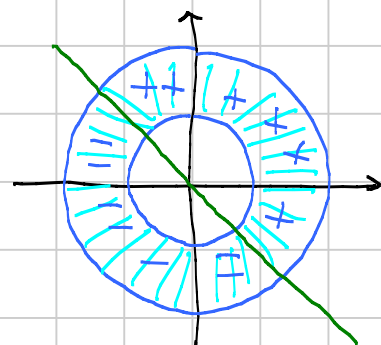
$$\iint_A \sqrt{x^2+y^2} dx dy = 2 \int_0^{\sqrt{2}} \rho d\rho \int_0^{\frac{\pi}{2}} d\theta \quad \rho \cdot \rho = 2 \cdot \frac{\pi}{2} \int_0^{\sqrt{2}} \rho^2 d\rho = \pi \frac{2\sqrt{2}}{3}$$

Esempio 7 $A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2+y^2 \leq 4\}$

$$\iint_A |x+y| dx dy = 2 \iint_{A^+} (x+y) dx dy$$

$$= 2 \int_1^2 \rho d\rho \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \rho (\cos\theta + \sin\theta) d\theta$$

$$= 2 \int_1^2 \rho^2 d\rho \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\theta + \sin\theta) d\theta$$



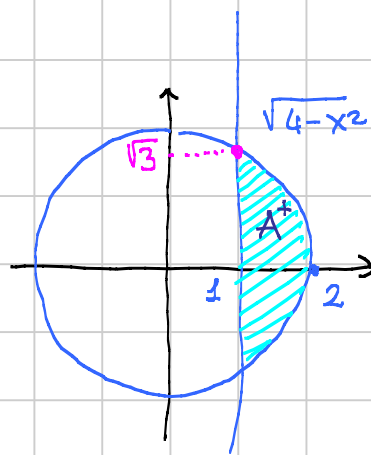
con qualche conto si fanno

Oss. Pensare a cosa verrebbe fatto spazzando in iussumi normali

Esempio 8 $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4, x \geq 1\}$

$$\iint_A x dx dy > 0 \quad \iint_A y dx dy = 0$$

↑
simmetria



1° modo $\iint_A x dx dy = 2 \iint_{A^+} x dx dy$

$$= 2 \int_1^2 dx \int_0^{\sqrt{4-x^2}} dy \quad x = 2 \int_1^2 x \sqrt{4-x^2} dx = \text{si fa come esempio prec.}$$

↑
normale
asse x

2° modo $2 \iint_{A^+} x dx dy = 2 \int_0^{\sqrt{3}} dy \int_1^{\sqrt{4-y^2}} dx \quad x = \int_0^{\sqrt{3}} dy [x^2]_{x=1}$

$$= \int_0^{\sqrt{3}} dy (4-y^2-1) = \text{si fa facile}$$

3° modo Per differenza

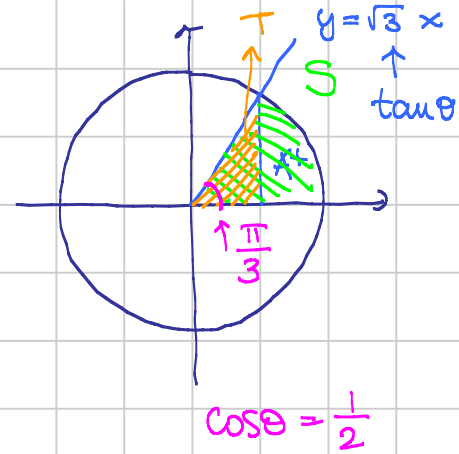
A^+ = tutto settore - triangolo

$$\iint_{A^+} x \, dx \, dy = \iint_S x \, dx \, dy - \iint_T x \, dx \, dy$$

polari cartesiane x

$$= \int_0^2 dp \int_0^{\frac{\pi}{3}} d\theta \, p \cos \theta \, p - \int_0^1 dx \int_0^{\sqrt{3}x} dy \, x$$

si fanno entrambi facilmente



4° modo Polari dirette !!

Il problema è descrivere A^+

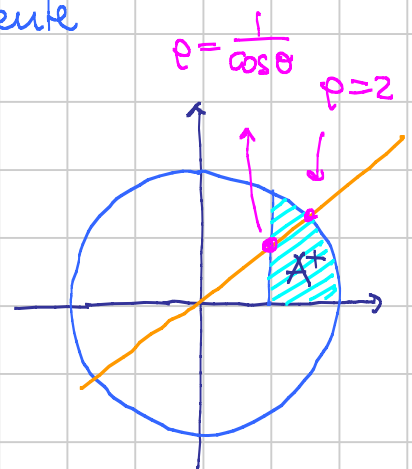
$$0 \leq \theta \leq \frac{\pi}{3}$$

$$x \geq 1$$

$$p \cos \theta \geq 1$$

$$p \geq \frac{1}{\cos \theta}$$

$$\frac{1}{\cos \theta} \leq p \leq 2$$



$$\iint_{A^+} x \, dx \, dy = \int_0^{\frac{\pi}{3}} d\theta \int_{\frac{1}{\cos \theta}}^2 dp \, p \cos \theta \cdot p$$

$$= \int_0^{\frac{\pi}{3}} \cos \theta \, d\theta \int_{\frac{1}{\cos \theta}}^2 p^2 \, dp = \int_0^{\frac{\pi}{3}} \cos \theta \, d\theta \left[\frac{1}{3} p^3 \right]_{p=\frac{1}{\cos \theta}}^{p=2}$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{3}} \cos \theta \left(8 - \frac{1}{\cos^3 \theta} \right) d\theta = \frac{1}{3} \int_0^{\frac{\pi}{3}} \left(8 \cos \theta - \frac{1}{\cos^2 \theta} \right) d\theta$$

= si fa facile (la primitiva di $\frac{1}{\cos^2 \theta}$ è $\tan \theta$)

[Controllare che nei 4 modi venga uguale e imparare per bene i 4 modi !!!]

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