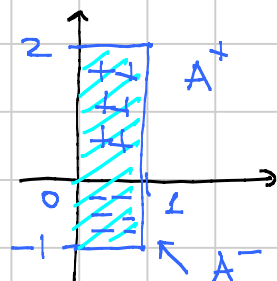


Integrali doppi con valori assoluti

Esempio 1 $\iint_A |y| dx dy$

$A = [0, 1] \times [-1, 2]$

Domanda: qual è il segno di y ?

$$\iint_A |y| dx dy = \iint_{A^+} y dx dy + \iint_{A^-} (-y) dx dy$$

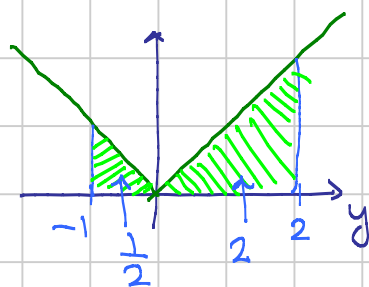
$$= \iint_{A^+} y dx dy - \iint_{A^-} y dx dy$$

$$= \int_0^1 dx \int_0^2 dy y - \int_0^1 dx \int_{-1}^0 dy y = \int_0^1 dx \left[\frac{1}{2} y^2 \right]_{y=0}^{y=2} - \int_0^1 dx \left[\frac{1}{2} y^2 \right]_{y=-1}^{y=0}$$

$$= \frac{1}{2} \int_0^1 (4 - 0 - 0 + 1) dx = \frac{5}{2}$$

Oss. In questo caso semplice si poteva fare direttamente

$$\iint_A |y| dx dy = \int_0^1 dx \int_{-1}^2 dy |y| = \frac{5}{2} \int_0^1 dx = \frac{5}{2}$$

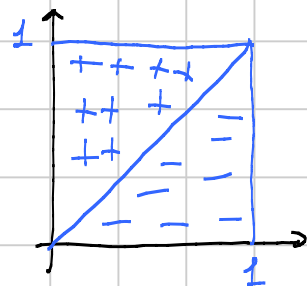


Esempio 2 $A = [0, 1] \times [0, 1]$

$\iint_A |y-x| dx dy$

$$\begin{aligned} \iint_{A^+} (y-x) dx dy &= \int_0^1 dx \int_x^1 dy (y-x) \\ &= \int_0^1 dy \int_0^y dx (y-x) \end{aligned}$$

asse x
↓
asse y



$$\iint_{A^-} (x-y) dx dy = \int_0^1 dx \int_0^x dy (x-y) \quad \text{Così si fa.}$$

Alla fine i 2 risultati vanno sommati.

Oss. In questo caso dico che $\iint_{A^+} |y-x| = \iint_{A^-} |y-x|$, quindi basta calcolare uno dei 2 e moltiplicare per 2 !!

Formalmente: Se scambio x e y (simmetria risp. bisettrice)
 $\rightarrow A^+$ diventa A^- e viceversa
 $\rightarrow$ L'integrandola resta la stessa

$$\iint_A |y-x| dx dy = 2 \int_0^1 dx \int_0^x dy (x-y)$$

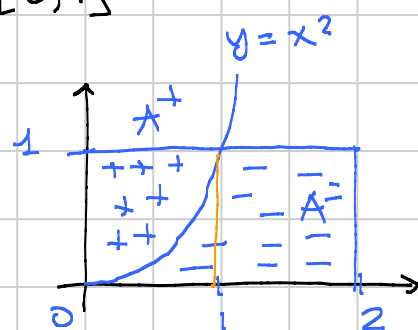
Esempio 3 $\iint_A |y-x^2| \quad A = [0, 2] \times [0, 1]$

1° modo Solito spezzamento in A^+ e A^-

$$\iint_A |y-x^2| dx dy = \iint_{A^+} (y-x^2) dx dy$$

$$- \iint_{A^-} (y-x^2) dx dy$$

$$= \int_0^1 dx \int_{x^2}^1 dy (y-x^2) - \int_0^1 dx \int_0^{x^2} dy (y-x^2) - \int_1^2 dx \int_0^1 dy (y-x^2)$$



Questi 3 integrali si fanno facilmente.

2° modo $\iint_A (y-x^2) dx dy = \iint_{A^+} (y-x^2) dx dy + \iint_{A^-} (y-x^2) dx dy$

da cui $\iint_{A^-} (y-x^2) = \iint_A (y-x^2) - \iint_{A^+} (y-x^2)$ e quindi

$$\iint_A |y-x^2| dx dy = 2 \iint_{A^+} (y-x^2) dx dy - \iint_A (y-x^2) dx dy$$

e in conclusione

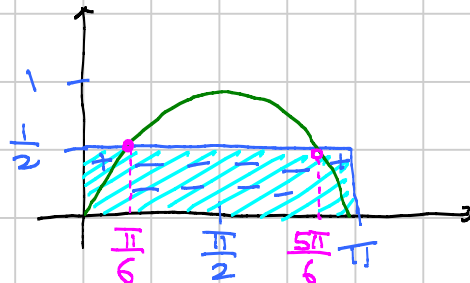
$$\iint_A |y-x^2| dx dy = 2 \int_0^1 dx \int_{x^2}^1 dy (y-x^2) - \int_0^1 dx \int_0^{x^2} dy (y-x^2)$$

Esempio 4 $\iint_A |y-\sin x| dx dy$

$$A = [0, \pi] \times [0, \frac{1}{2}]$$

A^+ è fatto dai 2 pezzi "Laterali"

A^- è fatto da 3 pezzi "sotto"



Quindi dovrai spezzare in 5 parti

$$\iint_A |y-\sin x| = \iint_{A^+} (y-\sin x) - \iint_{A^-} (y-\sin x)$$

$$= \int_0^{\frac{\pi}{6}} dx \int_{\sin x}^{\frac{1}{2}} dy (y-\sin x) + \int_{\frac{5\pi}{6}}^{\pi} dx \int_{\sin x}^{\frac{1}{2}} dy (y-\sin x) \quad A^+$$

$$- \int_0^{\frac{\pi}{6}} dx \int_0^{\sin x} dy (y-\sin x) - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} dx \int_0^{\frac{1}{2}} dy (...) - \int_{\frac{5\pi}{6}}^{\pi} dx \int_0^{\sin x} dy (y-\sin x)$$

così sono 5 integrali da calcolare.

Con ragionamenti di simmetria si vede che $1^o = 2^o$ e $3^o = 5^o$, ma restano comunque 3.

C'è però un modo di farne solo due.

$$\begin{aligned} \iint_A |f| &= \iint_{A^+} f - \iint_{A^-} f \quad ; \quad \iint_A f = \iint_{A^+} f + \underbrace{\iint_{A^-} f}_{\text{ricavo e sostituisco nella } 1^a} \\ &= 2 \iint_{A^+} f - \iint_A f \end{aligned}$$

e quindi nel nostro caso

$$\iint_A |y-\sin x| dx dy = 2 \iint_{A^+} (y-\sin x) dx dy - \iint_A (y-\sin x) dx dy$$

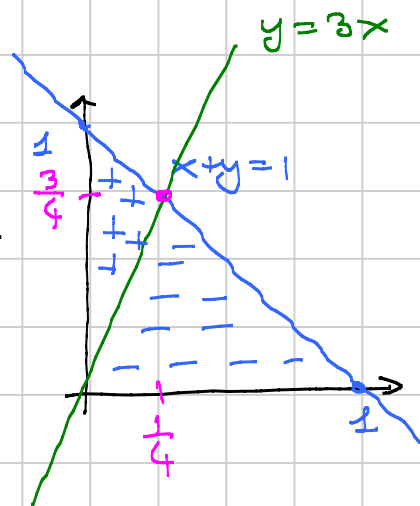
$$= 4 \int_0^{\frac{\pi}{6}} dx \int_{\sin x}^{\frac{1}{2}} dy (y - \sin x) - \int_0^{\pi} dx \int_0^{\frac{1}{2}} dy (y - \sin x)$$

Questi si fanno e sono solo due.

Fatto generale: lavorare per differenza può essere utile !!

Esempio 5 $\iint_A |y - 3x| dx dy$

$$A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$$



$$\iint_{A^+} (y - 3x) dx dy - \iint_{A^-} (y - 3x) dx dy$$

$$\iint_{A^+} (y - 3x) dx dy = \int_0^{\frac{1}{4}} dx \int_{3x}^{1-x} dy (y - 3x)$$

$$\iint_{A^-} (y - 3x) dx dy = \int_0^{\frac{1}{4}} dx \int_0^{3x} dy (y - 3x) + \int_{\frac{1}{4}}^1 dx \int_0^{1-x} dy (y - 3x)$$

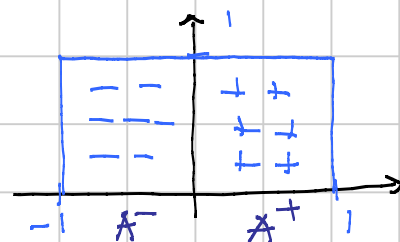
$$= \int_0^{\frac{3}{4}} dy \int_{\frac{1}{3}y}^{1-y} dx (y - 3x) \quad \text{così viene un integrale solo!}$$

asse x
asse y

Esempio 6 $\iint_A (y - 3|x|) dx dy \quad [-1, 1] \times [0, 1]$

$$\iint_A y dx dy - 3 \iint_A |x| dx dy$$

$$= 2 \iint_{A^+} y dx dy - 6 \iint_{A^+} x dx dy$$



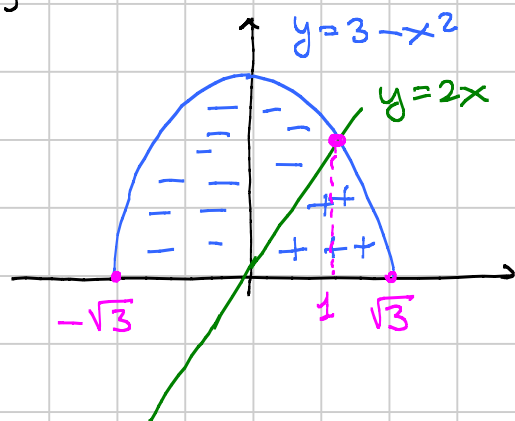
Ora osservo per simmetria che $\iint_{A^+} y dx dy = \iint_{A^+} x dx dy$
↑
 scambio x con y

quindi

$$\begin{aligned}\iint_A (y-3|x|) dx dy &= -4 \iint_{A^+} x dx dy = -4 \int_0^1 x dx \int_0^1 dy \\ &= -4 \int_0^1 x dx = -2.\end{aligned}$$

Esempio 7 $A = \{(x,y) \in \mathbb{R}^2: 0 \leq y \leq 3-x^2\}$

$$\iint_A |2x-y| dx dy$$



$$= \iint_{A^+} (2x-y) dx dy - \iint_{A^-} (2x-y) dx dy$$

oppure $= 2 \iint_{A^+} (2x-y) dx dy - \iint_A (2x-y) dx dy$

Ora

$$\iint_A (2x-y) dx dy = \int_{-\sqrt{3}}^{\sqrt{3}} dx \int_0^{3-x^2} dy (2x-y)$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} dx \int_0^{3-x^2} (-y) dy$$

l'integrale di x
viene = 0 per
simmetria

$$\iint_{A^+} (2x-y) dx dy = \int_0^1 dx \int_0^{2x} dy (2x-y) + \int_1^{\sqrt{3}} dx \int_0^{3-x^2} dy (2x-y)$$

e questi conti ora vanno fatti.

— o — o —