

COORDINATE CILINDRICHE

Coordinate polari nel piano e z
inalterata : (ρ, θ, z)

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta$$

$$z = z$$

Il J della trasformazione è ρ

$$\iiint_A f(x, y, z) dx dy dz = \iiint_B f(\rho \cos \theta, \rho \sin \theta, z) \rho d\rho d\theta dz$$

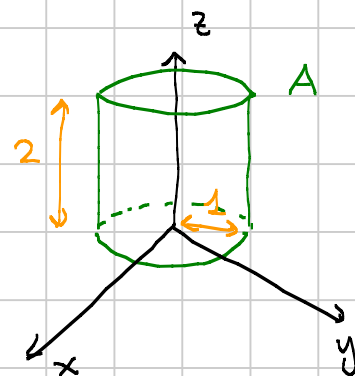
$\uparrow$
 B
 $\uparrow$
 A in coord. cilindriche

Esempio 1 $\iiint_A x^2 z dx dy dz$

$$B : \rho \in [0, 1], \theta \in [0, 2\pi], z \in [0, 2]$$

$$= \int_0^1 \rho d\rho \int_0^{2\pi} d\theta \int_0^2 dz \rho^2 \cos^2 \theta z \cdot \rho$$

$$= \underbrace{\int_0^1 \rho^3 d\rho}_{\frac{1}{4}} \underbrace{\int_0^{2\pi} \cos^2 \theta d\theta}_{\frac{\pi}{1}} \underbrace{\int_0^2 z dz}_{\frac{1}{2}} = \frac{\pi}{2}$$

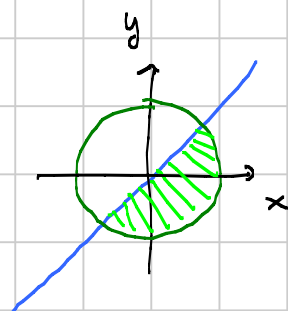


Esempio 2 $C = \{(x, y, z) \in A : y \leq x\}$

In coord. cilindriche diventa

$$\rho \in [0, 1], z \in [0, 2], \theta \in \left[-\frac{3}{4}\pi, \frac{\pi}{4}\right]$$

opp. $\theta \in \left[\frac{5\pi}{4}, \frac{9\pi}{4}\right]$



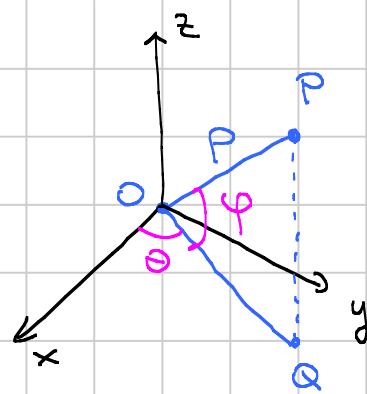
COORD. SFERICHE (Versione dei geografi)

$$\rho = \text{distanza } OP = \sqrt{x^2 + y^2 + z^2}$$

Q = proiezione di P su piano xy

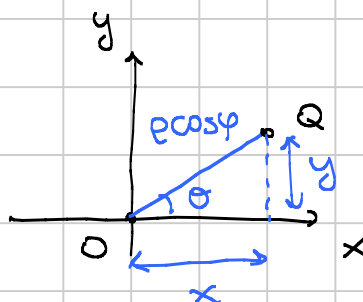
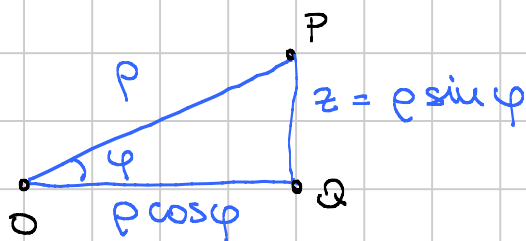
φ = latitudine = angolo tra OP e OQ
= angolo tra OP e piano xy

θ = "vecchio θ delle polari di Q " = angolo tra semiasse pos. x e OQ
= longitudine



Formule di passaggio:

$$x = \rho \cos \varphi \cos \theta \quad y = \rho \cos \varphi \sin \theta \quad z = \rho \sin \varphi$$



Come calcolo J ? Devo fare il val. assoluto del Det della matrice

$$\begin{pmatrix} x_\varphi & x_\theta & x_\rho \\ y_\varphi & y_\theta & y_\rho \\ z_\varphi & z_\theta & z_\rho \end{pmatrix}$$

esercizio
↓
~

$$J = \rho^2 \cos \varphi$$

Dove variano i parametri?

$$\rho \geq 0, \quad \theta \in [0, 2\pi], \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Achtung! Per i fisici il φ non è la latitudine, ma l'angolo tra OP e semiasse positivo delle z . Conseguenze: $\varphi \in [0, \pi]$ e $\sin$ e $\cos$ di φ si scambiano ovunque, J compreso

Achtung! In coord. sferiche

- il θ ed il φ dell'origine non sono ben definiti
- il θ dei p.ti dell'asse z non è ben definito.

Esempio 3 $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 3, \underbrace{z \leq 0}_{\substack{\uparrow \\ \text{parte} \\ \text{bassa}}}, \underbrace{x \geq 0}_{\substack{\uparrow \\ \text{parte dx} \\ \text{vista} \\ \text{dall'alto}}}\}$

$\uparrow$ $\frac{1}{4}$ di sfera

In coord. sferiche S diventa

$$\rho \in [0, \sqrt{3}]$$

$$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\varphi \in [-\frac{\pi}{2}, 0]$$

$\iiint_S xz \, dx \, dy \, dz$ ce lo aspettiamo negativo

$$\begin{aligned} &= \int_0^{\sqrt{3}} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{-\frac{\pi}{2}}^0 d\varphi \underbrace{\rho \cos\varphi \cos\theta}_x \underbrace{\rho \sin\varphi}_z \underbrace{\rho^2 \cos\varphi}_J \\ &= \underbrace{\int_0^{\sqrt{3}} \rho^4 d\rho}_{\frac{1}{5} 9\sqrt{3}} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta}_2 \underbrace{\int_{-\frac{\pi}{2}}^0 \cos^2\varphi \sin\varphi d\varphi}_{\substack{= [-\frac{1}{3} \cos^3\varphi]_{-\frac{\pi}{2}}^0 = -\frac{1}{3}}} = -\frac{6\sqrt{3}}{5} \end{aligned}$$

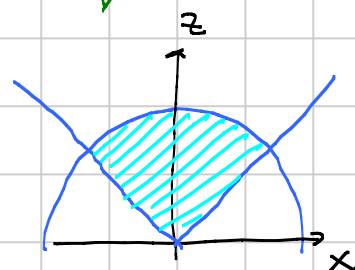
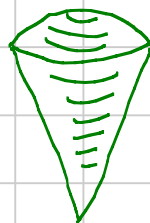
Esempio 4 $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, z \geq \sqrt{x^2 + y^2}\}$

$$\text{Vol}(A) = \iiint_A 1 \, dx \, dy \, dz$$

$$A: \rho \in [0, 2], \theta \in [0, 2\pi], \varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$$

$$= \int_0^2 d\rho \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \, \rho^2 \cos\varphi$$

= si fa...



Cambi di variabile in generale : funzionano come in 2 variabili

$$\iiint_A f(x,y,z) dx dy dz = \iiint_B g(u,v,w) J(u,v,w) du dv dw$$

$\swarrow$ A nelle nuove coord. $\nwarrow$ g vecchia f nelle nuove coord.

J si calcola con la solita procedura :

1 - ricavo x, y, z in funz. di u, v, w

2 - calcolo la matrice
$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

3 - $J = |\text{Det}|$.

Esempio 5 A = sfera con centro in $(1, 3, -2)$ e raggio 1

$$\iiint_A x^2 dx dy dz$$

Equazione sfera: $(\underbrace{x-1}_u)^2 + (\underbrace{y-3}_v)^2 + (\underbrace{z+2}_w)^2 \leq 1$

$$x = u+1, \quad y = v+3, \quad z = w-2 \quad \leadsto \quad J = 1$$

$$= \iiint_{u^2+v^2+w^2 \leq 1} (u^2 + 2u + 1) du dv dw$$

$\underbrace{\quad}_{0}$ (simmetria) $\leadsto$ Volume = $\frac{4}{3}\pi$

Resta da integrare u^2

$$\iiint_{u^2+v^2+w^2 \leq 1} u^2 = \int_0^1 \rho d\rho \int_0^{2\pi} d\sigma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \quad \rho^2 \sin^2 \varphi \cdot \rho^2 \cos \varphi$$

$$= \int_0^1 \rho^4 d\rho \int_0^{2\pi} d\sigma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi \cos \varphi d\varphi = \frac{1}{5} \cdot 2\pi \cdot \frac{2}{3} = \frac{4}{15} \pi$$

Oss. Data una sfera S , vale che

$$\iiint_S x^2 dx dy dz = \iiint_S y^2 \dots = \iiint_S z^2 \dots$$

↑ centro in (0,0,0)

per simmetria.

Ottima alternativa Li sommo e viene $\iiint_S \rho^2$

$$= \underbrace{\int_0^1 \rho^2 \rho^2 d\rho}_{\frac{1}{5}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \cos\phi}_2 = \frac{4}{5} \pi \quad \text{poi divido per 3.}$$

Esempio 6 $A = \{(x,y,z) \in \mathbb{R}^3 : 2x^2 + 4y^2 + z^2 \leq 4, z \geq 1\}$

$$\iiint_A x^2 dx dy dz \quad (\underbrace{\sqrt{2}x}_u)^2 + (\underbrace{2y}_v)^2 + (\underbrace{z}_w)^2 \leq 4$$

Nelle nuove coord. A insieme diventa

$$u^2 + v^2 + w^2 \leq 4 \rightsquigarrow \text{sfera}$$

$$w \geq 1$$

Calcolo J : $x = \frac{1}{\sqrt{2}}u \quad y = \frac{1}{2}v \quad z = w \quad J = \frac{1}{2\sqrt{2}}$

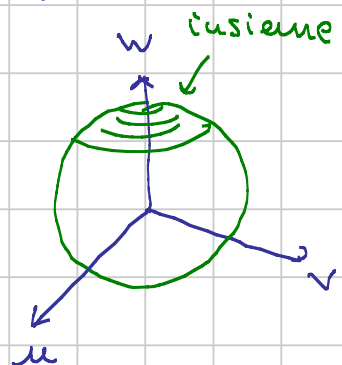
$$\iiint_A x^2 dx dy dz = \iiint_{\substack{u^2+v^2+w^2 \leq 4 \\ w \geq 1}} \frac{1}{2} u^2 \cdot \frac{1}{2\sqrt{2}} du dv dw$$

↑ corretto dopo video

$$= \frac{1}{4\sqrt{2}} \iiint u^2 du dv dw$$

1° modo: per sezioni

$$\begin{aligned} \iiint u^2 du dv dw &= \int_1^2 dw \iint_{S_z} u^2 du dv = (\text{polari su } S_z) \\ &= \int_1^2 dw \int_0^{\sqrt{4-w^2}} \underbrace{\rho}_{\text{raggio di } S_z} d\rho \int_0^{2\pi} d\theta \rho^2 \cos^2\theta \cdot \rho \end{aligned}$$



In realtà ho usato le coord. cilindriche

2° modo: per colonne

$$\iiint_{\Omega} u^2 du dv dw = \iint_{\Omega} du dv \int_1^{\sqrt{4-u^2-v^2}} u^2 dw$$

La base Ω è $\{(x,y) \in \mathbb{R}^2: x^2+y^2 \leq 3\}$

Esercizio: controllare che venga lo stesso.

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