

INTEGRALI CON VALORI ASSOLUTI

Strategie : $\rightarrow$ cavarsela con le simmetrie
 $\rightarrow$ spezzare la zona di integrazione

Esempio 1 $\iint_A |y| dx dy = \iint_A y dx dy$



$$= 2 \iint_B y dx dy = 2 \int_0^1 dp \int_0^{\pi/2} p \sin \theta \cdot p d\theta$$

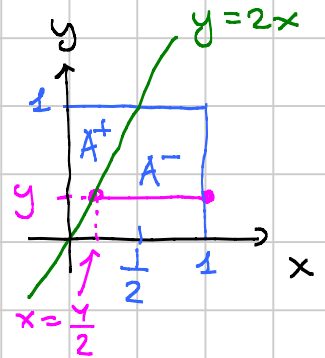
$$= 2 \underbrace{\int_0^1 p^2 dp}_{\frac{1}{3}} \underbrace{\int_0^{\pi/2} \sin \theta d\theta}_1 = \frac{2}{3}$$

Esempio 2 $\iint_A |x| dx dy$ $A = \text{semicerchio di prima}$

$$= 2 \iint_B |x| dx dy = 2 \iint_B x dx dy = \frac{2}{3} \text{ come prima}$$

Esempio 3 $\iint_A |y-2x| dx dy$ $A = [0,1] \times [0,1]$

$$|y-2x| = \begin{cases} y-2x & \text{in } A^+ \\ -y+2x & \text{in } A^- \end{cases}$$



$$\iint_A |y-2x| dx dy = \iint_{A^+} (y-2x) dx dy + \iint_{A^-} (2x-y) dx dy$$

$$= \int_0^{\frac{1}{2}} dx \int_{2x}^1 dy (y-2x) + \int_0^{\frac{1}{2}} dx \int_0^{2x} dy (2x-y) + \int_{\frac{1}{2}}^1 dx \int_0^1 dy (2x-y)$$

$$= \text{conti}$$

$$\int_0^1 dy \int_{\frac{y}{2}}^1 dx (2x-y)$$

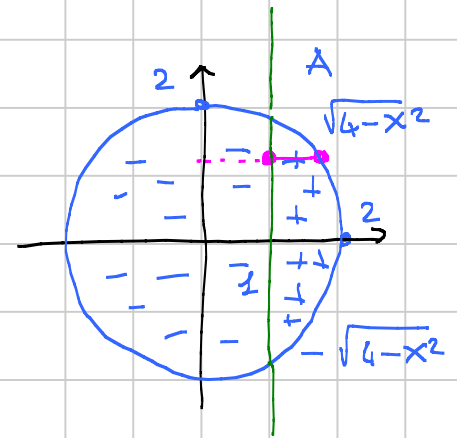
Oss. Si poteva fare

$$\iint_A |y-2x| dx dy = \underbrace{\iint_A (2x-y) dx dy}_{\text{quadrato}} + 2 \underbrace{\iint_{A^+} (y-2x) dx dy}_{\text{triangolo}}$$

$$\iint_{A^+} (2x-y) + \iint_{A^-} (2x-y)$$

Esempio 4 $\iint_A |x-1| dx dy$

$$= \iint_{A^+} (x-1) dx dy - \iint_{A^-} (x-1) dx dy$$



oppure come nell'osservazione precedente

$$\underbrace{\iint_A (1-x) dx dy}_{\text{Banale: Area}} + 2 \iint_{A^+} (x-1) dx dy$$

= 4π

Resta da calcolare $\iint_{A^+} (x-1) dx dy$. Tre strategie possibili

1° modo Normale asse x :

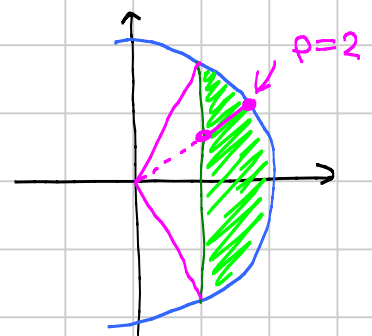
$$\int_1^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x-1) dy = \int_1^2 (x-1) \cdot 2\sqrt{4-x^2} dx = \text{analisi a seccante}$$

2° modo Normale asse y : $\int_{-\sqrt{3}}^{\sqrt{3}} dy \int_1^{\sqrt{4-y^2}} dx (x-1) = \text{si fa}$

3° modo Polari dirette $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$

$$x=1 \quad \rho \cos \theta = 1 \quad \rho = \frac{1}{\cos \theta}$$

$$\frac{1}{\cos \theta} \leq \rho \leq 2$$



L'integrale diventa

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\cos\theta}}^1 dp [p \cos\theta - 1] p$$

$\times -1$ $\uparrow$

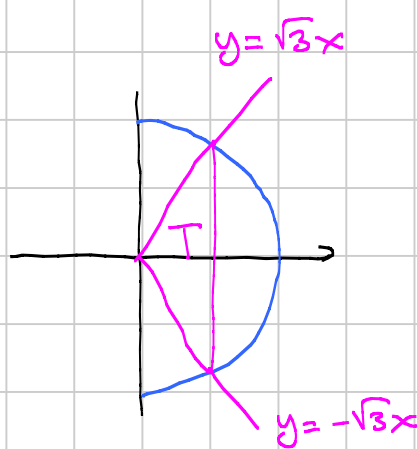
$$= 2 \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\cos\theta}}^1 dp [p^2 \cos\theta - p] = 2 \int_0^{\frac{\pi}{2}} d\theta \left[\frac{1}{3} p^3 \cos\theta - \frac{1}{2} p^2 \right]_{\frac{1}{\cos\theta}}^1$$

$$= 2 \int_0^{\frac{\pi}{2}} d\theta \left(\frac{1}{3} \cos\theta - \frac{1}{2} - \frac{1}{3} \frac{1}{\cos^2\theta} + \frac{1}{2} \frac{1}{\cos^2\theta} \right) = \text{analisi 1 facile}$$

$$\int \frac{1}{\cos^2\theta} = \tan\theta$$

4° modo Settore - triangolo

sette



$$\iint_{A^+} = \int_0^2 dp \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta (p \cos\theta - 1) p = \int_0^1 dx \int_{-\sqrt{3}x}^{\sqrt{3}x} dy (x-1)$$

$\underbrace{\hspace{10em}}_{\text{triangolo}} \quad \triangle \text{ in cartesiane}$

Esempio 5 $\iiint_S |x-1| dx dy dz$

$S =$ sfera con centro nell'origine e raggio 2

Osservo che è uguale a $\iiint_S |z-1| dx dy dz$ e questo lo faccio per sezioni lungo asse z .

$$\iiint_S |z-1| dx dy dz = \int_{-2}^2 dz |z-1| \cdot \text{Area}(S_z)$$

$$= \int_{-2}^2 dz |z-1| \cdot \pi R_z^2 = \pi \int_{-2}^2 |z-1| (4-z^2) dz$$

R_z^2
 $\downarrow$

= analisi 1 con valore assoluto.

Esempio 6 $\iint_Q \sqrt{x^2+y^2} dx dy$ $Q = [0,1] \times [0,1]$



$$= \int_0^1 dx \int_0^1 dy \sqrt{x^2+y^2}$$

si fa con la comoda sostituzione

$$y = x \sinh t$$

Provo in polari. Per simmetria

$$\iint_Q \sqrt{x^2+y^2} dx dy = 2 \iint_T \sqrt{x^2+y^2} dx dy$$

$$\rho \cos \theta = 1 \quad \rho = \frac{1}{\cos \theta}$$

$$= 2 \int_0^{\pi/4} d\theta \int_0^{\frac{1}{\cos \theta}} \underbrace{\rho \cdot \rho}_{\sqrt{x^2+y^2}} d\rho$$

$$= \frac{2}{3} \int_0^{\pi/4} d\theta \left[\rho^3 \right]_0^{\frac{1}{\cos \theta}} = \frac{2}{3} \int_0^{\pi/4} \frac{1}{\cos^3 \theta} d\theta$$

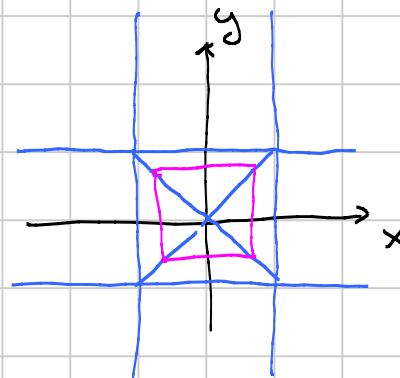
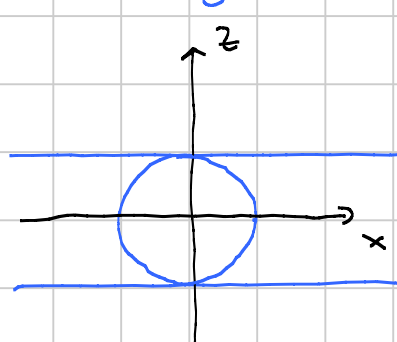
$$\int \frac{1}{\cos^3 \theta} d\theta = \int \frac{\cos \theta}{\cos^4 \theta} d\theta = \int \frac{\cos \theta}{(1-\sin^2 \theta)^2} d\theta = \int \frac{dt}{(1-t^2)^2} = \text{si fa.}$$

Esempio 7 $S = \{ (x,y,z) \in \mathbb{R}^3 : \underbrace{x^2+z^2 \leq 1}_{\text{cilindro asse } y}, \underbrace{y^2+z^2 \leq 1}_{\text{cilindro asse } x} \}$

$$\text{Vol}(S) =$$

$$= \int_{-1}^1 dz \text{Area}(S_z)$$

Chi è S_z ?



Fisso z , e voglio capire dove variano x e y : dalle equazioni

$$x^2 \leq 1-z^2, \quad y^2 \leq 1-z^2, \quad \text{cioè} \quad |x| \leq \sqrt{1-z^2} \quad |y| \leq \sqrt{1-z^2}$$

La sezione è un quadrato di lato $2\sqrt{1-z^2}$

In conclusione

$$\int_{-1}^1 \text{Area}(S_z) dz = 4 \int_{-1}^1 (1-z^2) dz = \text{si fa banale}$$

Oss. L'area $4(1-z^2)$ è, a meno di fattori π , la stessa che si ottiene sezionando una sfera.

Quindi per il principio di Cavalieri

$$\text{Vol}(S) = \frac{4}{\pi} \text{Vol}(\text{sfera di } R=1) = \frac{4}{\pi} \frac{4}{3} \pi = \frac{16}{3}$$

Esercizio vero : aggiungere il terzo cilindro lungo asse z e trovare

→ volume

→ numero di "facce"

→ superficie (in futuro)

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