

Calcolo di integrali doppi

SLOGAN: un integrale doppio = 2 integrali singoli (in una variabile)

FORMULE DI RIDUZIONE

1. Caso dei rettangoli
2. Caso degli insiemi NORMALI
3. Caso generale

Formula di riduzione su un rettangolo

Sia $A = [a, b] \times [c, d]$

Allora

$$\iint_A f(x, y) dx dy = \int_a^b dx \int_c^d f(x, y) dy = \int_c^d dy \int_a^b f(x, y) dx$$

Esempio 1 $A = [0, 1] \times [2, 4]$

$f(x, y) = x + x^2 y$

$$\iint_A (x + x^2 y) dx dy = \int_0^1 dx \int_2^4 (x + x^2 y) dy$$

Devo calcolare questo: calcolo la primitiva rispetto alla variabile y e poi sostituire gli estremi di integrazione $y=2$ e $y=4$

$$= \int_0^1 dx \left[xy + \frac{1}{2} x^2 y^2 \right]_{y=2}^{y=4}$$

Derivato rispetto ad y
da $x + x^2 y$

$$= \int_0^1 dx (4x + 8x^2 - 2x - 2x^2)$$

$$= \int_0^1 (2x + 6x^2) dx = \left[x^2 + 2x^3 \right]_0^1 = 1 + 2 = 3$$

In alternativa:

$$\begin{aligned}\iint_A (x+x^2y) dx dy &= \int_2^4 dy \underbrace{\int_0^1 dx (x+x^2y)}_{\text{faccio la primitiva rispetto ad } x} \\&= \int_2^4 dy \left[\frac{1}{2}x^2 + \frac{1}{3}x^3y \right]_{x=0}^{x=1} \\&= \int_2^4 dy \left[\frac{1}{2} + \frac{1}{3}y - 0 - 0 \right] = \int_2^4 \left(\frac{1}{2} + \frac{1}{3}y \right) dy \\&= \left[\frac{y}{2} + \frac{1}{6}y^2 \right]_{y=2}^{y=4} \\&= \underbrace{\frac{4}{2}}_0 + \underbrace{\frac{16}{6}}_0 - \underbrace{\frac{2}{2}}_0 - \underbrace{\frac{4}{6}}_0 = 2 + 2 - 1 = 4 - 1 = 3.\end{aligned}$$

Esempio 2 $\iint_A x^2 \sin y \, dx dy$ $A = [0, 3] \times [0, \pi]$

$$\begin{aligned}&= \int_0^3 dx \int_0^\pi dy \, x^2 \sin y = \int_0^3 dx \, x^2 \int_0^\pi \sin y \, dy = \int_0^3 x^2 dx [-\cos y]_0^\pi \\&= 2 \int_0^3 x^2 dx = \frac{2}{3} [x^3]_0^3 = 18\end{aligned}$$

è una costante quando integro rispetto ad y, quindi può andare fuori

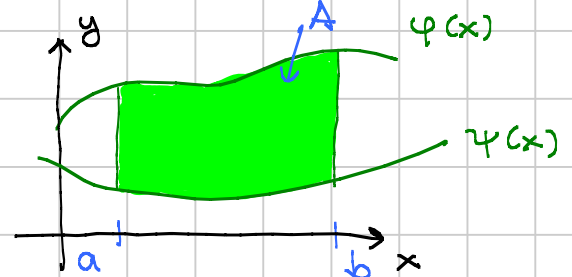
In alternativa

$$\begin{aligned}\iint_A x^2 \sin y \, dx dy &= \int_0^\pi dy \int_0^3 dx \, x^2 \sin y = \int_0^\pi \sin y \, dy \int_0^3 x^2 dx \\&= \int_0^\pi \sin y \, dy \left[\frac{1}{3}x^3 \right]_0^3 = 9 \int_0^\pi \sin y \, dy = \dots = 18.\end{aligned}$$

INSIEMI NORMALI

Def. Un insieme $A \subseteq \mathbb{R}^2$ si dice normale rispetto all'asse x se è del tipo

$$A = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], \varphi(x) \leq y \leq \psi(x)\}$$

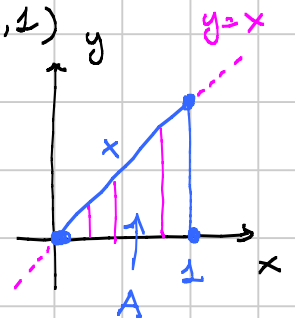


Formula di riduzione:

$$\iint_A f(x, y) dx dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x, y) dy$$

Quando calcolo questo faccio la primitiva risp. a y e come estremi sostituisco $y = \varphi(x)$ e $y = \psi(x)$

Esempio 3 $A =$ triangolo con vertici in $(0,0)$, $(1,0)$, $(1,1)$
 $f(x, y) = x + x^2 y$



Scrivo A come insieme normale risp. asse x

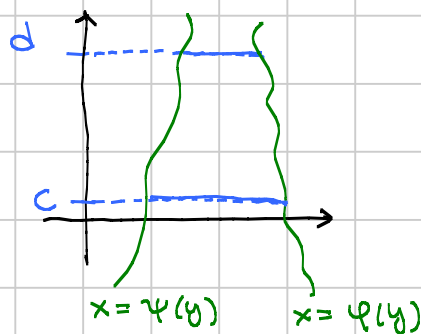
$$A = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], \underline{0 \leq y \leq x}\}$$

↑
fissato x, y varia tra 0 e x

$$\begin{aligned} \iint_A (x + x^2 y) dx dy &= \int_0^1 dx \int_0^x dy (x + x^2 y) \\ &= \int_0^1 dx \left[xy + \frac{1}{2} x^2 y^2 \right]_{y=0}^{y=x} \quad (\text{sostituisco } y=x \text{ e } y=0) \\ &= \int_0^1 dx \left[x^2 + \frac{1}{2} x^4 \right] = \left[\frac{1}{3} x^3 + \frac{1}{10} x^5 \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{10} = \frac{13}{30} \end{aligned}$$

Def. Un insieme si dice normale rispetto all'asse y se è del tipo

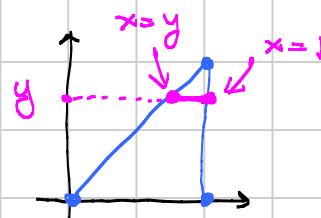
$$A = \{(x, y) \in \mathbb{R}^2 : y \in [c, d], \psi(y) \leq x \leq \varphi(y)\}$$



Formula di riduzione:

$$\iint_A f(x, y) dx dy = \int_c^d dy \int_{\psi(y)}^{\varphi(y)} dx f(x, y)$$

Esempio 3 (revisited)



Il triangolo, visto come normale rispetto all'asse y diventa

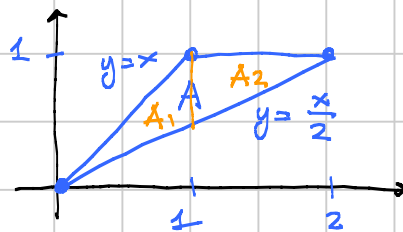
$$A = \{(x, y) \in \mathbb{R}^2 : y \in [0, 1], y \leq x \leq 1\}$$

$$\begin{aligned} \iint_A (x + x^2 y) dx dy &= \int_0^1 dy \int_y^1 dx (x + x^2 y) \\ &= \int_0^1 dy \left[\frac{1}{2} x^2 + \frac{1}{3} x^3 y \right]_{x=y}^{x=1} \\ &= \int_0^1 dy \left(\underbrace{\frac{1}{2} + \frac{1}{3} y}_{x=1} - \underbrace{\frac{1}{2} y^2 - \frac{1}{3} y^3}_{x=y} \right) \\ &= \int_0^1 \left(\frac{1}{2} + \frac{1}{3} y - \frac{1}{2} y^2 - \frac{1}{3} y^3 \right) dy \\ &= \left[\frac{1}{2} y + \frac{1}{6} y^2 - \frac{1}{6} y^3 - \frac{1}{15} y^4 \right]_{y=0}^{y=1} \\ &= \frac{1}{2} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{6}} - \frac{1}{15} = \frac{15-2}{30} = \frac{13}{30} \end{aligned}$$

Esempio 4

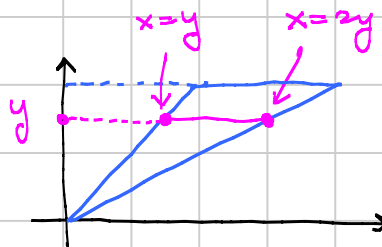
Rispetto asse x :

$$\begin{aligned}\iint_A f &= \iint_{A_1} f + \iint_{A_2} f \\ &= \int_0^1 dx \int_{\frac{x}{2}}^x dy f(x,y) + \int_1^2 dx \int_{\frac{x}{2}}^1 dy f(x,y)\end{aligned}$$



Rispetto asse y :

$$\iint_A f = \int_0^1 dy \int_y^{2y} dx f(x,y)$$



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