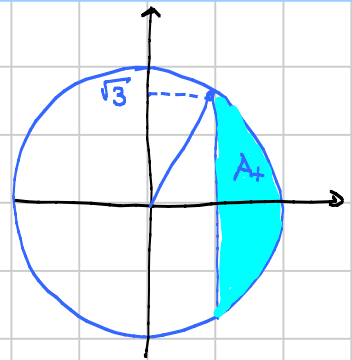


Esempio 1 $A = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 4, x \geq 1\}$

$$\iint_A |y| dx dy = 2 \iint_{A_+} y dx dy$$



1° modo: normale risp. asse x

$$\begin{aligned} \iint_{A_+} y dx dy &= \int_1^2 dx \int_0^{\sqrt{4-x^2}} dy y = \int_1^2 dx \left[\frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{4-x^2}} \\ &= \int_1^2 dx \frac{1}{2} (4-x^2) = \text{banale} \end{aligned}$$

2° modo: normale risp. asse y

$$\begin{aligned} \iint_{A_+} y dx dy &= \int_0^{\sqrt{3}} dy \int_1^{\sqrt{4-y^2}} dx y = \int_0^{\sqrt{3}} y dy \int_1^{\sqrt{4-y^2}} dx \\ &= \int_0^{\sqrt{3}} dy y (\sqrt{4-y^2} - 1) = \int_0^{\sqrt{3}} (y \sqrt{4-y^2} - y) dy \\ &= \left[\frac{2}{3} (4-y^2)^{\frac{3}{2}} \left(-\frac{1}{2}\right) - \frac{1}{2} y^2 \right]_0^{\sqrt{3}} = \text{controllare che venga uguale} \end{aligned}$$

↑
Derivare per controllo

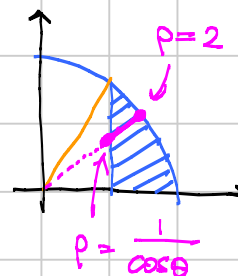
3° modo: settore - triangolo

$$\begin{aligned} \iint_{\text{Settore}} y dx dy &= \int_0^2 dp \int_0^{\frac{\pi}{3}} d\theta p \sin \theta \cdot p = \int_0^2 p^2 dp \int_0^{\frac{\pi}{3}} \sin \theta d\theta = \text{banale} \\ \iint_{\text{triangolo}} y dx dy &= \int_0^1 dx \int_0^{\sqrt{3}x} dy y = \int_0^1 dx \left[\frac{1}{2} y^2 \right]_0^{\sqrt{3}x} \\ &= \frac{3}{2} \int_0^1 x^2 dx \end{aligned}$$

Controllare che venga lo stesso

4° modo: coordinate polari secche

$$\begin{aligned} x^2 + y^2 &\leq 4 & y &\geq 0 & x &\geq 1 \\ \rho &\leq 2 & 0 &\leq \theta < \pi & \rho \cos \theta &\geq 1 & \rho &\geq \frac{1}{\cos \theta} \end{aligned}$$



Allora la descrizione di A_+ in coordinate polari è la seguente

$$0 \leq \theta \leq \frac{\pi}{3} \quad \text{Fissato } \theta, \text{ dove varia } \rho? \quad \frac{1}{\cos \theta} \leq \rho \leq 2$$

$$\text{Quindi } \iint_{A_+} y \, dx \, dy = \underbrace{\int_0^{\frac{\pi}{3}} d\theta}_{A_+ \text{ in coord. polari}} \underbrace{\int_{\frac{1}{\cos \theta}}^2}_{\rho} \underbrace{\rho \sin \theta}_{y} \underbrace{\rho}_{\rho} \, d\rho$$

$$= \int_0^{\frac{\pi}{3}} \sin \theta \, d\theta \int_{\frac{1}{\cos \theta}}^2 \rho^2 \, d\rho = \int_0^{\frac{\pi}{3}} \sin \theta \, d\theta \left[\frac{1}{3} \rho^3 \right]_{\rho=\frac{1}{\cos \theta}}^{\rho=2}$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{3}} \sin \theta \left(8 - \frac{1}{\cos^3 \theta} \right) d\theta = \frac{1}{3} \int_0^{\frac{\pi}{3}} \left(8 \sin \theta - \frac{\sin \theta}{\cos^3 \theta} \right) d\theta$$

$$= \frac{1}{3} \left[-8 \cos \theta - \frac{1}{2} \frac{1}{\cos^2 \theta} \right]_{\theta=0}^{\theta=\frac{\pi}{3}}$$

Derivare per credere (e per essere sicuri di non aver sbagliato segni o costanti)

Controllare che risulti lo stesso.

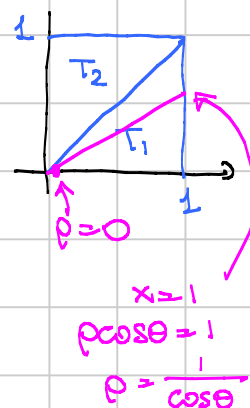
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Esempio 2 $A = [0, 1] \times [0, 1]$

$$\iint_A \sqrt{x^2 + y^2} \, dx \, dy$$

1° modo: in cartesiane diventa

$$\int_0^1 dx \int_0^1 dy \sqrt{x^2 + y^2} \quad \text{e questo (volendo) si fa con la sostituzione } y = x \sin t$$



2° modo: in polari $\iint_A \sqrt{x^2+y^2} dx dy = 2 \iint_{T_1} \sqrt{x^2+y^2} dx dy$

Devo descrivere T_1 in polari $0 \leq \theta \leq \frac{\pi}{4}$ (questo si vede)

Fissato θ , dove varia ρ ? $0 \leq \rho \leq \frac{1}{\cos \theta}$ Quindi

$$\iint_{T_1} \sqrt{x^2+y^2} dx dy = \underbrace{\int_0^{\frac{\pi}{4}} d\theta}_{T_1 \text{ in coord. polari}} \underbrace{\int_0^{\frac{1}{\cos \theta}} \rho}_{\sqrt{x^2+y^2}} d\rho = \int_0^{\frac{\pi}{4}} d\theta \left[\frac{1}{3} \rho^3 \right]_{\rho=0}^{\rho=\frac{1}{\cos \theta}}$$

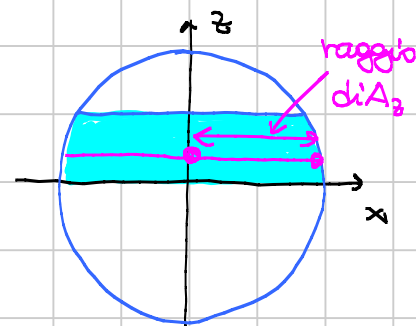
$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 \theta} d\theta \quad \text{Come fare la primitiva di } \frac{1}{\cos^3 \theta} ?$$

$$\int \frac{1}{\cos^3 \theta} d\theta = \int \frac{\cos \theta}{\cos^4 \theta} d\theta = \int \frac{\cos \theta}{(1-\sin^2 \theta)^2} d\theta$$

Ponendo $t = \sin \theta$ diventa $\int \frac{dt}{(1-t^2)^2}$ e questo si sa fare.

Esempio 3 $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, 0 \leq z \leq 1\}$

Volume di $A = \iiint_A 1 dx dy dz =$



Lo rispondo per sezioni

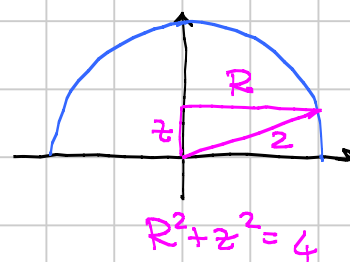
$$= \int_0^1 dz \iint_{A_z} 1 dx dy = \int_0^1 \text{Area}(A_z) dz$$

A_z è una sezione di A ad altezza z

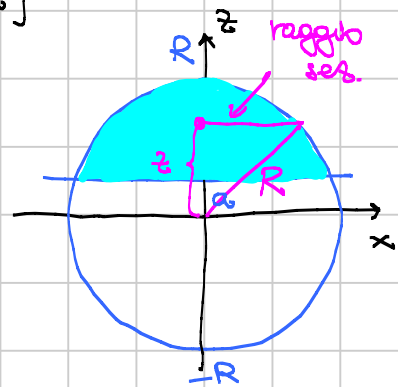
A_z è un cerchio di raggio $\sqrt{4-z^2}$

$$= \int_0^1 \pi (4-z^2) dz = \pi \left[4z - \frac{1}{3} z^3 \right]_0^1$$

$$= \pi \left(4 - \frac{1}{3} \right) = \frac{11}{3} \pi$$



Esempio 4 $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, z \geq a\}$
 con $a \in [-R, R]$



Volume di $A = \iiint_A 1 \, dx \, dy \, dz =$

$\stackrel{\substack{\uparrow \\ \text{sezioni}}}{=} \int_a^R dz \iint_{A_z} 1 \, dx \, dy = \int_a^R \text{Area}(A_z) \, dz$

Ora come prima A_z è un cerchio di raggio $\sqrt{R^2 - z^2}$ (Pitagora)

$$= \int_a^R \pi (R^2 - z^2) \, dz = \pi \left[R^2 z - \frac{1}{3} z^3 \right]_a^R = \pi \left[R^3 - \frac{R^3}{3} - R^2 a + \frac{1}{3} a^3 \right]$$

Controllare che per $a = -R$ venga $\frac{4}{3} \pi R^3$ (tutta la sfera)
 per $a = 0$ venga $\frac{2}{3} \pi R^3$ (semisfera)

Osservazione Non cambia nulla se avremo

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x \geq a\}$$

oppure

$$y \geq a$$

Volendo, più di cambiare le variabili, si può sempre fare sezioni rispetto all'asse z .

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