

INTEGRALI IMPROPRI IN PIÙ VARIABILI

Un integrale in 2 o + variabili può essere improprio

- ↗ perchè la zona di integrazione non è limitata
- perchè la funzione non è limitata
- ↘ per tutti e 2 i motivi:
— 0 — 0 —

$$\iint_{\mathbb{R}^2} \frac{1}{x^2+y^2+1} dx dy = \lim_{R \rightarrow +\infty} \iint_{B_R} \frac{1}{x^2+y^2+1} dx dy$$

↑
insieme non limitato

↑
funzione senza problemi

↑
cerchio con centro in (0,0) e raggio R

In generale se $\iint_{\mathbb{R}^2} |f(x,y)| dx dy$ esiste fatto con i cerchi, allora

$\iint_{\mathbb{R}^2} f(x,y) dx dy$ viene lo stesso numero con qualunque tipo di figura io vado a fare il limite

$$\begin{aligned} \iint_{B_R} \frac{1}{x^2+y^2+1} dx dy &= \int_0^R dp \int_0^{2\pi} d\theta \frac{1}{p^2+1} \quad \begin{array}{c} \uparrow \\ \text{coord.} \\ \text{polari} \end{array} \quad \begin{array}{c} p \\ \uparrow \\ J \end{array} \\ &= 2\pi \int_0^R \frac{p}{p^2+1} dp \end{aligned}$$

Quindi

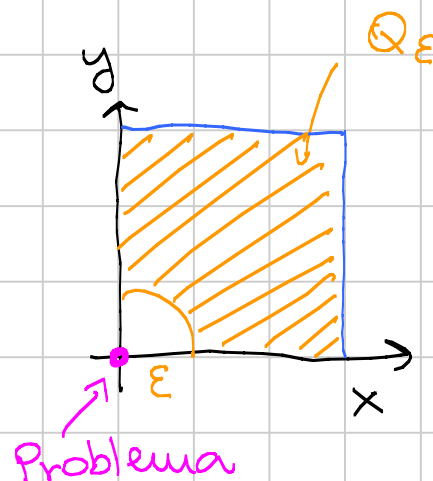
$$\iint_{\mathbb{R}^2} \frac{dx dy}{x^2+y^2+1} = 2\pi \underbrace{\int_0^{+\infty} \frac{p}{p^2+1} dp}$$

Diverge perché si comporta come $\frac{1}{p}$

$$Q = [0, 1] \times [0, 1]$$

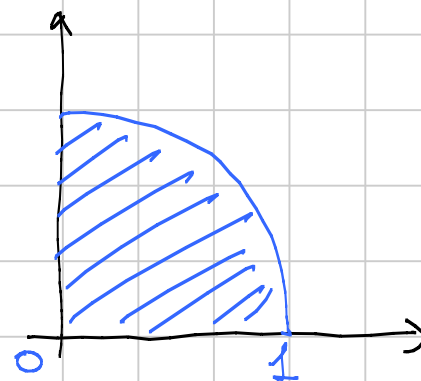
$$\iint_Q \frac{dx dy}{(x^2 + y^2)^\alpha} =$$

↑
Funzione non
limitata vicino
a (0,0)



$$= \lim_{\epsilon \rightarrow 0^+} \iint_{Q_\epsilon} \frac{dx dy}{(x^2 + y^2)^\alpha}$$

DEFINIZIONE



Oss. L'integrale dato converge $\Leftrightarrow$ converge l'integrale sul
quarto di cerchio

$$\iint_D \frac{dx dy}{(x^2 + y^2)^\alpha} \stackrel{\text{POLARI}}{=} \int_0^1 \rho d\rho \int_0^{\pi/2} d\theta \frac{1}{\rho^{2\alpha}} \cdot \rho = \frac{\pi}{2} \int_0^1 \frac{d\rho}{\rho^{2\alpha-1}}$$

converge $\Leftrightarrow 2\alpha - 1 < 1$

$\Leftrightarrow 2\alpha < 2 \Leftrightarrow \alpha < 1$

$$\iiint_A \frac{x \, dx \, dy \, dz}{(x^2 + y^2 + z^2)^\alpha} =$$

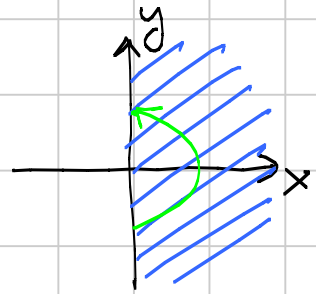
$$A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 1\}$$

esterno della
sfera di raggio 1

prendo
solo $x \geq 0$

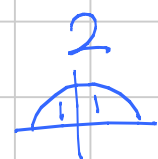
In coord. sferiche l'insieme
A si descrive come:

$$\rho \geq 1, \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \underbrace{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{x \geq 0}$$



$$= \int_{-1}^{+\infty} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \frac{\rho \cos \varphi \cos \theta}{\rho^{2\alpha}} \underbrace{\rho^2 \cos \varphi}_J =$$

$$= \int_{-1}^{+\infty} \frac{1}{\rho^{2\alpha-3}} d\rho \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \, d\theta}_2 \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \varphi \, d\varphi}_{\frac{\pi}{2} \text{ (poi vediamo perché)}}$$

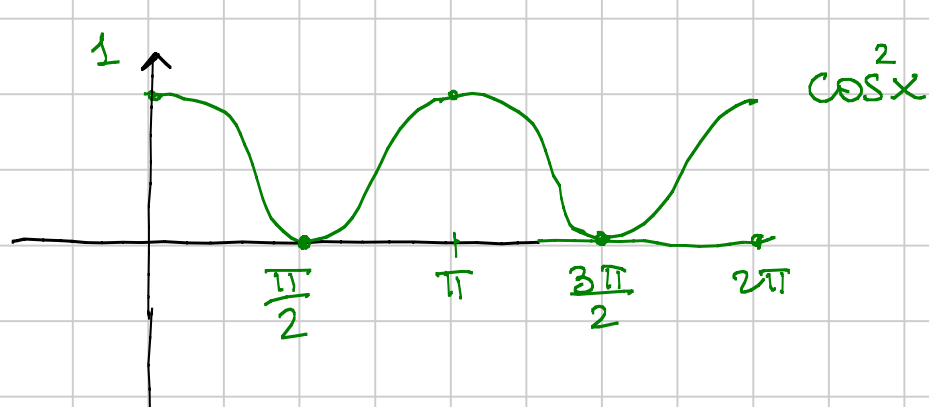


$$= \pi \int_1^{+\infty} \frac{dp}{p^{2\alpha-3}}$$

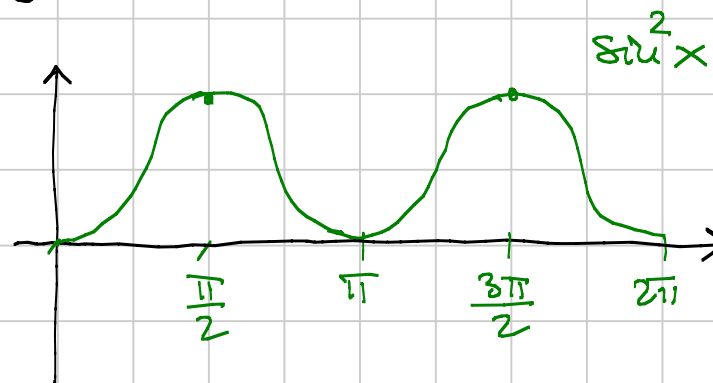
$$\text{converge} \Leftrightarrow 2\alpha - 3 > 1$$

$$\Leftrightarrow 2\alpha > 4 \Leftrightarrow \alpha > 2$$

N.B. Per gli $\alpha > 2$ si può calcolare



$$I_1 = \int_0^{\pi/2} \cos^2 x \, dx$$



$$I_2 = \int_0^{\pi/2} \sin^2 x \, dx$$

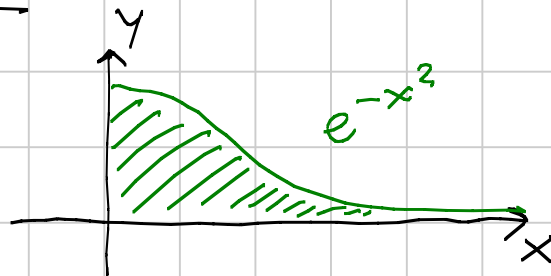
$I_1 = I_2$ per simmetria

$$I_1 + I_2 = \int_0^{\pi/2} (\cos^2 x + \sin^2 x) \, dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$$

$$\Rightarrow I_1 = I_2 = \frac{\pi}{4}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \, dx = 2 \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$$

$$\int_0^{+\infty} e^{-x^2} \, dx = I$$



Converge per mille motivi...

Quanto è il suo valore? N.B. Non si sa fare la primitiva di e^{-x^2} !!!

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2} \, dx \, dy = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, e^{-x^2-y^2}$$

$\swarrow$ e^{-x^2} $\cdot e^{-y^2}$

$$= \int_{-\infty}^{+\infty} e^{-x^2} \, dx \underbrace{\int_{-\infty}^{+\infty} e^{-y^2} \, dy}_{2I} = 2I \int_{-\infty}^{+\infty} e^{-x^2} \, dx = 4I^2$$

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy \stackrel{\text{POLAR}}{=} \int_0^{+\infty} dp \int_0^{2\pi} d\theta e^{-p^2} \cdot p$$

$$= \int_0^{+\infty} p e^{-p^2} \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= \pi \int_0^{+\infty} 2p e^{-p^2} dp$$

$$= \pi \lim_{A \rightarrow +\infty} \left[-e^{-p^2} \right]_{p=0}^{p=A}$$

$$= \pi \lim_{A \rightarrow +\infty} \left\{ \underbrace{-e^{-A^2}}_0 + 1 \right\} = \pi$$

$$\pi = 4 I^2 \Rightarrow I^2 = \frac{\pi}{4} \Rightarrow \boxed{I = \frac{\sqrt{\pi}}{2}}$$