

Esempi di cambi di variabile

Solidi di rotazione e teorema di GULDINO

Cambi di variabile in generale

A = sfera con centro in $(1, 2, 0)$ e raggio 1

x_0, y_0, z_0

R

$$\iiint_A z \, dx \, dy \, dz = 0 \quad \text{per simmetria}$$

$$\iiint_A y \, dx \, dy \, dz$$

y varia tra 1 e 3 nella sfera

$\Rightarrow$ l'integrale di sicuro è > 0

Equazione sfera

$$\overset{u}{(x-1)}^2 + \overset{v}{(y-2)}^2 + \overset{w}{z}^2 \leq 1$$

$x-x_0 \quad y-y_0 \quad z-z_0 \quad R^2$

Pongo $u = x - 1$ $v = y - 2$ $w = z$

$$\iiint_A y \, dx \, dy \, dz = \iiint_{\substack{u^2 + v^2 + w^2 \leq 1 \\ \text{INSIEME} \\ \text{NELLE VARIABILI} \\ u, v, w}} (v+2) \cdot 1 \cdot du \, dv \, dw$$

y $J(u, v, w)$

Come si calcola $J(u, v, w)$?

- ① Ricavare x, y, z in funzione di u, v, w
- ② Costruire la matrice JACOBIANA $\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$
- ③ $J(u, v, w) = |\det(\text{Matrice})|$

Nel nostro caso abbiamo una traslazione:

$$x = u + 1 \qquad y = v + 2 \qquad z = w$$

$$\Rightarrow \text{Matrice} = \text{identità} \Rightarrow J = 1$$

$$\iiint_{---} (v+2) du dv dw = \iiint_{\substack{0 \\ \text{(simmetria)}}} v du dv dw + 2 \iiint_{\substack{1 \\ 2 \text{ Vol(sfera)}}} du dv dw$$

$$= 2 \cdot \frac{4}{3} \pi 1^3 = \frac{8}{3} \pi$$

$$\iiint_A y^2 dx dy dz = \iiint_{u^2+v^2+w^2 \leq 1} (v+2)^2 du dv dw$$

$\xrightarrow{\substack{\uparrow \\ \text{SOLTO} \\ \text{CAMBIO}}}$

$$= \iiint v^2 du dv dw + 4 \iiint_{\substack{0 \\ \text{}}} v du dv dw + 4 \iiint_{\substack{1 \\ 4 \text{ Vol(sfera)}}} du dv dw$$

Osservazione preliminare: sulla sfera $u^2 + v^2 + w^2 \leq R^2$

$$\iiint u^2 du dv dw = \iiint v^2 \text{ --- } = \iiint w^2 \text{ ---}$$

Coord. sferiche ($R=1$) $\rho \in [0, 1]$, $\theta \in [0, 2\pi]$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$u = \rho \cos \varphi \cos \theta, \quad v = \rho \cos \varphi \sin \theta, \quad w = \rho \sin \varphi$$

$$\iiint v^2 du dv dw = \iiint w^2 du dv dw$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \underbrace{\rho^2 \sin^2 \varphi}_{w^2} \underbrace{\rho^2 \cos \varphi}_J$$

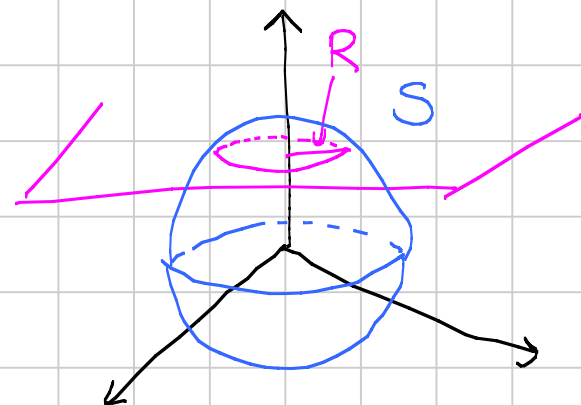
$$= \int_0^1 \rho^4 d\rho \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi \cos \varphi d\varphi$$

$$= 2\pi \int_0^1 \rho^4 d\rho \left[\frac{\sin^3 \varphi}{3} \right]_{\varphi = -\frac{\pi}{2}}^{\varphi = \frac{\pi}{2}}$$

$$= 2\pi \cdot \frac{2}{3} \left[\frac{\rho^5}{5} \right]_{\rho=0}^{\rho=1} = \frac{4}{15} \pi$$

Esempio 2 Volume sfera di raggio 1.

$$\text{vol}(S) = \iiint_S 1 \, dx \, dy \, dz = (\star)$$



Presentazione di S per sezioni

$$S = \{ (x, y, z) \in \mathbb{R}^3 : z \in [-1, 1], (x, y) \in S_z \}$$

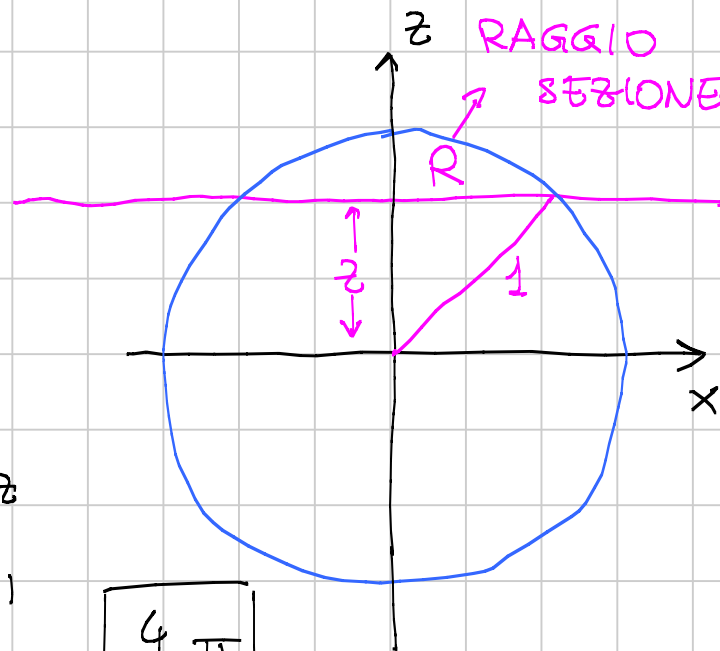
S_z = cerchio con centro in $(0,0)$ e raggio ...

$$z^2 + R^2 = 1 \Rightarrow R = \sqrt{1 - z^2}$$

SEZIONI

$$(\star) \stackrel{\text{SEZIONI}}{=} \int_{-1}^1 dz \iint_{S_z} 1 \, dx \, dy = \int_{-1}^1 \text{Area}(S_z) \, dz$$

$$= \pi \int_{-1}^1 (1 - z^2) \, dz = \pi \left[z - \frac{z^3}{3} \right]_{z=-1}^{z=1} = \boxed{\frac{4}{3} \pi}$$



Esempio 3 Volume elissoide

$$a^2 x^2 + b^2 y^2 + c^2 z^2 \leq R^2 \quad a, b, c \text{ reali } > 0$$

$$\underbrace{(ax)^2}_{u^2} + \underbrace{(by)^2}_{v^2} + \underbrace{(cz)^2}_{w^2} \leq R^2$$

$$\text{Vol (Elissoide)} = \iiint_{\text{Eliss.}} 1 \, dx \, dy \, dz = \iiint_{u^2+v^2+w^2 \leq R^2} \underbrace{1 \frac{1}{abc}}_J \, du \, dv \, dw$$

$$x = \frac{u}{a} \quad y = \frac{v}{b} \quad z = \frac{w}{c}$$

$$J(u, v, w) = \left| \det \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix} \right| = \frac{1}{abc}$$

$$\text{Vol} = \frac{1}{abc} \text{ Vol (Sfera)} = \boxed{\frac{1}{abc} \frac{4}{3} \pi R^3}$$

SOLIDI DI ROTAZIONE

Volume solido di rotazione

$$\text{Vol}(S) = \iiint_S 1 \, dx \, dy \, dz$$

SEZIONI

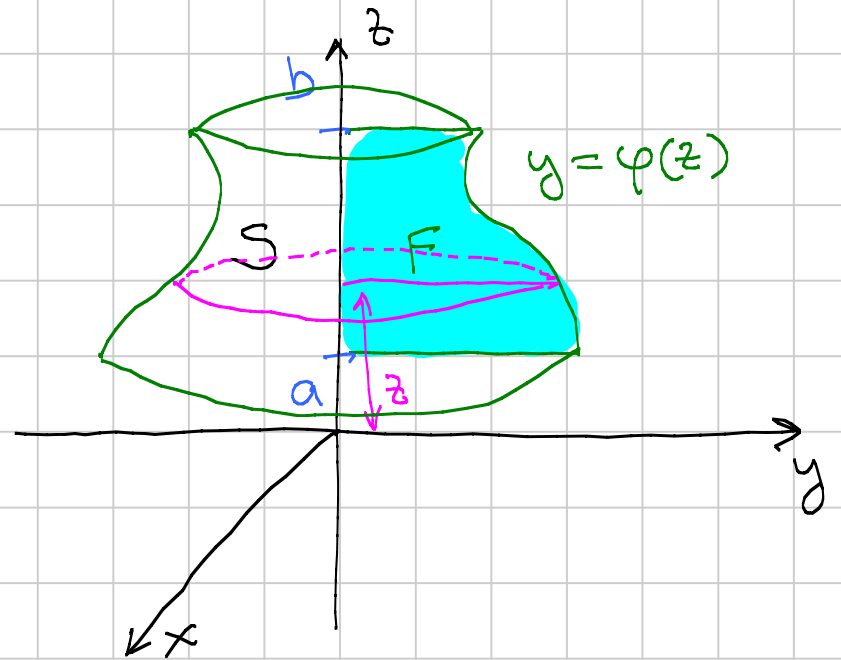
$$= \int_a^b dz \iint_{S_z} dx \, dy$$

$$= \int_a^b \text{Area}(S_z) \, dz$$

$$= \pi \int_a^b [\varphi(z)]^2 \, dz$$

Esempio Cilindro

$$\text{Vol} = \pi \int_0^R r^2 \, dz = \boxed{\pi r^2 R}$$



S_z = cerchio con centro $(0,0)$
e raggio $\varphi(z)$

