

ANALISI II CIN (NUCL)

ORA 30

Esempio 1 Usiamo la formula generale per $J(u, v)$ nel caso delle coordinate polari

① Ricavare x e y in funzione di ρ e θ

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta$$

② Matrice
$$\begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{pmatrix}$$

③ Determinante
$$J = |\text{Det}| = |\rho \cos^2 \theta + \rho \sin^2 \theta| = \rho$$

Casi facili di cambio di variabili

* Traslazioni $u = x + a$ $v = y + b$ a, b numeri

Ricavare : $x = u - a$, $y = v - b$

Matrice : $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{identità}$

$$J(u, v) = 1$$

* Dilatazioni sugli assi $u = ax$ $v = by$ a, b numeri > 0

Ricavare $x = \frac{u}{a}$ $y = \frac{v}{b}$

Matrice : $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} = \text{matrice diagonale}$

$$J(u,v) = |\text{Det}| = \frac{1}{ab}$$

* Trasformazioni affini

$$u = \alpha x + \beta y, \quad v = \delta x + \epsilon y$$

Ricavare: $x = au + bv$ $y = cu + dv$

(a, b, c, d si ricavano in funzione di $\alpha, \beta, \delta, \epsilon$)

Matrice: $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Matrice indep.
da u e v

$$J(u,v) = |ad - bc|$$

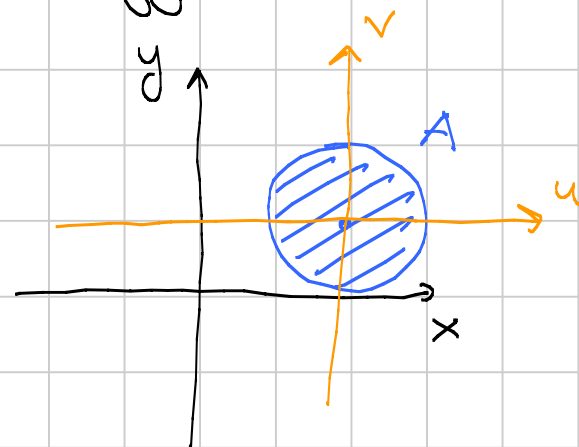
Esempio 1

$$\iint_A xy \, dx \, dy$$

A = cerchio con centro in $(2,1)$ e raggio 1

L'equazione di A è

$$\underbrace{(x-2)}_u^2 + \underbrace{(y-1)}_v^2 \leq 1$$



Nelle coord. (u,v) l'insieme A diventa $u^2 + v^2 \leq 1$

$$\iint_A xy \, dx \, dy = \iint_{\substack{u^2+v^2 \leq 1 \\ \uparrow \\ \text{insieme A} \\ \text{nelle coord. } u,v}} \underbrace{(u+2)(v+1)}_{x \cdot y} \underbrace{1}_{J(u,v)} \, du \, dv$$

$$= \iint_{u^2+v^2 \leq 1} (u+2)(v+1) du dv \quad \odot \quad \text{uso coord. polari}$$

$$u = \rho \cos \theta$$

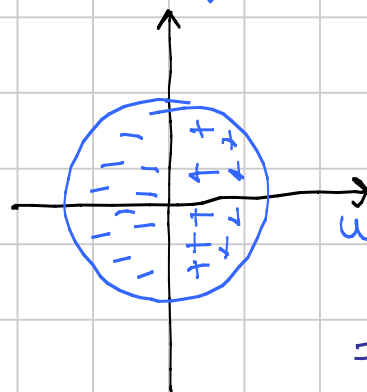
$$v = \rho \sin \theta$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\theta \underbrace{(\rho \cos \theta + 2)}_{u+2} \underbrace{(\rho \sin \theta + 1)}_{v+1} \underbrace{\rho}_{\text{pagamento}}$$

$$\odot \quad \iint (u+2)(v+1) = \iint uv + \iint u + 2 \iint v + 2 \iint 1$$

↑
svolgo
il
prodotto

↑
l'integrale
della somma
è la somma
degli integrali



segno di u
nel cerchio

$$= 2 \text{ Area (cerchio)}$$

$$= 2\pi$$

$$\iint_A 1 \, dx \, dy = \text{Area}(A)$$

In 1 variabile

$$\int_a^b 1 \, dx = b - a$$

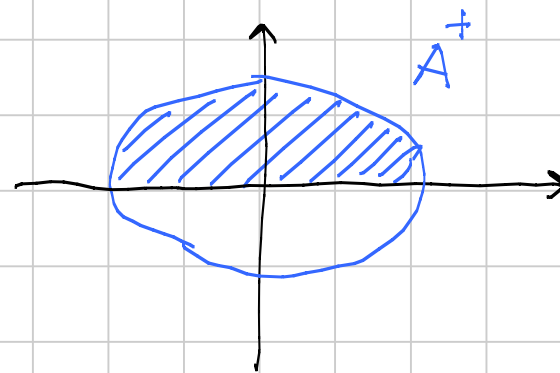
= lungh. del segmento

Esempio 2 $A = \{ (x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 \leq 5 \}$

$$\iint_A x \, dx \, dy = 0$$

$$\iint_{A^+} x \, dx \, dy = 0 \quad ; \quad \iint_{A^+} y \, dx \, dy$$

di sicuro positivo



$$\iint_{A^+} y \, dx \, dy =$$

$$2x^2 + 3y^2 \leq 5$$

$$\underbrace{(\sqrt{2}x)^2}_u + \underbrace{(\sqrt{3}y)^2}_v \leq 5$$

$$= \iint_{\substack{u^2 + v^2 \leq 5, \\ \text{insieme in} \\ u \text{ e } v}}, v \geq 0$$

↑
perché
siamo in A^+

$$\frac{v}{\sqrt{3}} \quad \frac{1}{\sqrt{6}} \, du \, dv =$$

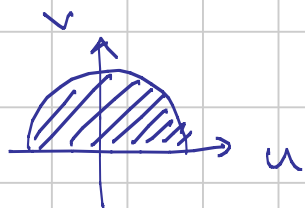
↑ ↑
 y $J(u, v)$

Ricavare $u = \sqrt{2}x$ $v = \sqrt{3}y$ $x = \frac{1}{\sqrt{2}}u$ $y = \frac{1}{\sqrt{3}}v$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$J(u, v) = \frac{1}{\sqrt{6}}$$

$$= \frac{1}{3\sqrt{2}} \iint_{\substack{u^2+v^2 \leq 5 \\ v \geq 0}} v \, du \, dv$$



$$v \, du \, dv$$

$$= \frac{1}{3\sqrt{2}} \int_0^{\sqrt{5}} dp \int_0^{\pi} d\theta \, p \sin \theta \, p$$

↑
p dθ

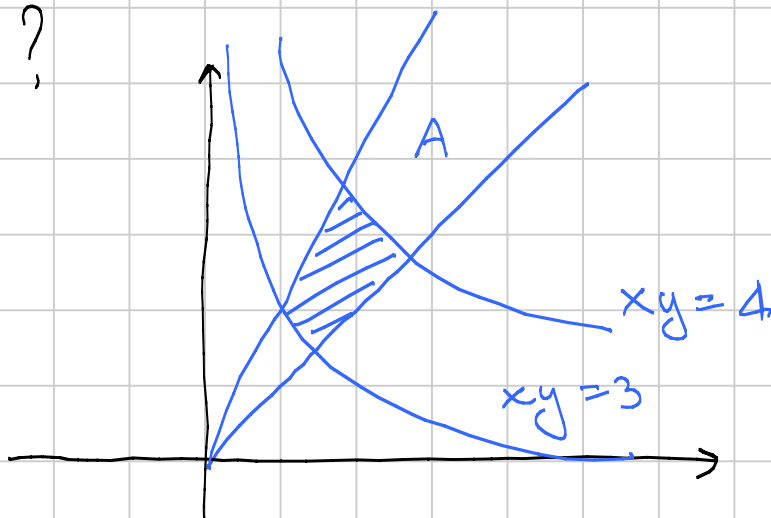
$$= \dots$$

— 0 — 0 —

Esempio 3

$$A = \left\{ (x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x \leq y \leq 2x, 3 \leq xy \leq 4 \right\}$$

Area (A) = ?



$$x \leq y \leq 2x$$

$$1 \leq \frac{y}{x} \leq 2$$

$\downarrow$
 u

$$3 \leq xy \leq 4$$

$$\downarrow$$

 v

$$\text{Area (A)} = \iint_A 1 \, dx \, dy = \underbrace{\int_1^2 du \int_3^4 dv}_{\substack{\text{insieme in} \\ u \text{ e } v}} \underbrace{1}_{\text{}} \cdot \underbrace{\frac{1}{2u}}_{J(u,v)} = \dots$$

① Ricavare

$$u = \frac{y}{x}$$

$$v = xy$$

Moltiplico : $y^2 = u \cdot v$; $y = \sqrt{uv} = \sqrt{u} \sqrt{v}$

Divido

$$x^2 = \frac{v}{u}$$

$$x = \frac{\sqrt{v}}{\sqrt{u}}$$

② Matrice $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} =$

$$\begin{pmatrix} -\frac{\sqrt{v}}{2u\sqrt{u}} & \frac{1}{2\sqrt{u}\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{pmatrix}$$

③ $|\text{Det}| = \left| -\frac{1}{4u} - \frac{1}{4u} \right| = \frac{1}{2u}$

[Provare a calcolare $\text{Det} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = 2u]$