

ANALISI 2 CIV

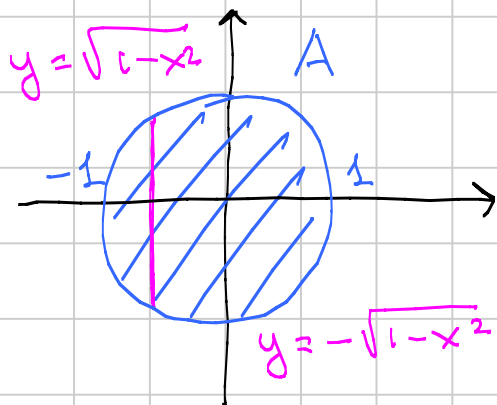
ORA 29

Titolo nota

15/11/2006

Utilizzo coordinate polari / cambi di variabile.

DESCRIZIONE DI INSIEMI IN COORD. POLARI



A è normale rispetto asse x

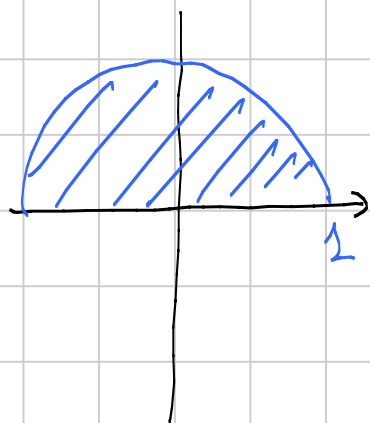
$$A = \{ (x, y) \in \mathbb{R}^2 : x \in [-1, 1], -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \}$$

$$\iint_A f(x, y) dx dy = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy$$

SCOMODO

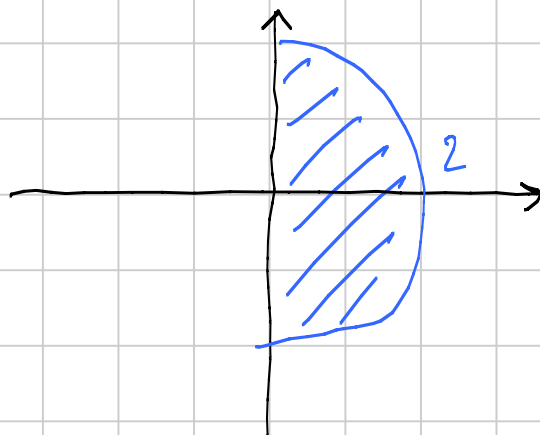
In coordinate polari $\mathbb{R}^2$ insieme A si descrive come

$$A = \{ (p \cos \theta, p \sin \theta) \in \mathbb{R}^2 : p \in [0, 1], \theta \in [0, 2\pi] \}$$



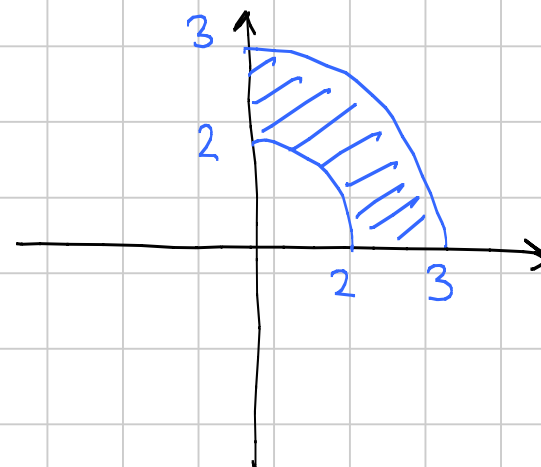
$$p \in [0, 1]$$

$$\theta \in [0, \pi]$$



$$p \in [0, 2]$$

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$p \in [2, 3]$$

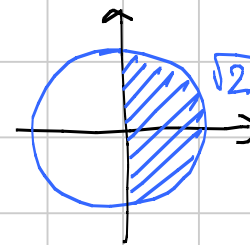
$$\theta \in \left[0, \frac{\pi}{2}\right]$$

SI $\theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$

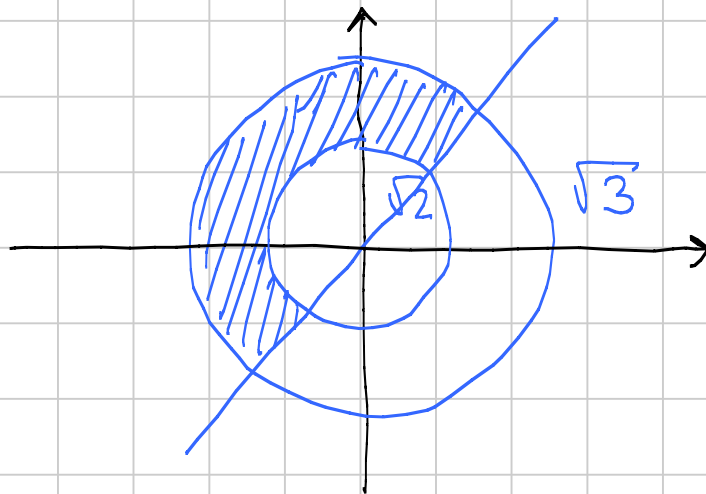
$\theta \in \left[\frac{3\pi}{2}, \frac{\pi}{2}\right]$ NO!

$$A = \{ (x,y) \in \mathbb{R}^2 : x \geq 0, x^2 + y^2 \leq 2 \}$$

$$\rho \in [0, \sqrt{2}], \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$B = \{ (x,y) \in \mathbb{R}^2 : 2 \leq x^2 + y^2 \leq 3, y \geq x \}$$



$$\rho \in [\sqrt{2}, \sqrt{3}]$$

$$\theta \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Calcolo di integrali doppi mediante coord. polari

$$\iint_A f(x,y) dx dy = \iint_B g(\rho, \theta) \cdot \boxed{\rho} d\rho d\theta$$

$B = \mathbb{R}^2$ insieme A descritto mediante coord. polari

$g(\rho, \theta) =$ funzione $f(x,y)$ descritta in coord. polari

$$g(\rho, \theta) = f(\rho \cos \theta, \rho \sin \theta)$$

ρ è il "raggio"

Esempio 1

$$\iint_A x \, dx \, dy =$$

$$= \int_0^2 dp \int_0^{\pi/2} d\theta$$

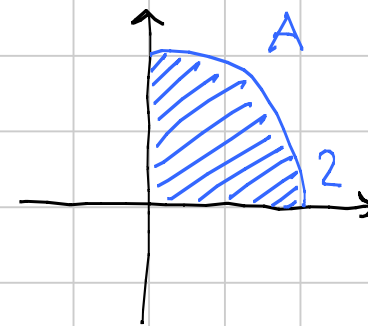
Descrizione
di A in
coord. polari

$$\rho \cos \theta$$

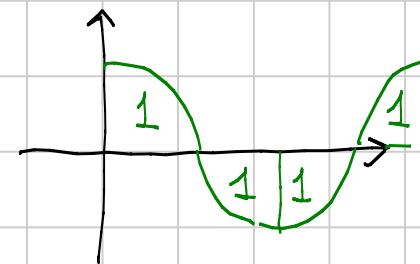
x in
coord.
polari

$$\rho =$$

↑
raggiamento



$$A : \rho \in [0, 2], \theta \in [0, \frac{\pi}{2}]$$



$$= \int_0^2 dp \int_0^{\pi/2} d\theta \, \rho^2 \cos \theta = \int_0^2 p^2 dp \int_0^{\pi/2} \underbrace{\cos \theta}_{=1} d\theta = \int_0^2 p^2 dp = \left[\frac{p^3}{3} \right]_0^2 = \boxed{\frac{8}{3}}$$

Esempio 2

A = cerchio con centro in $(0,0)$ e raggio 3

$$\iint_A (x^2 + y^2) dx dy =$$

In coord. polari $A = \begin{matrix} \rho \in [0, 3] \\ \theta \in [0, 2\pi] \end{matrix}$

$$= \int_0^3 d\rho \int_0^{2\pi} d\theta$$

insieme in
coord. polari

ρ^2
 $x^2 + y^2$
in coord.
polari

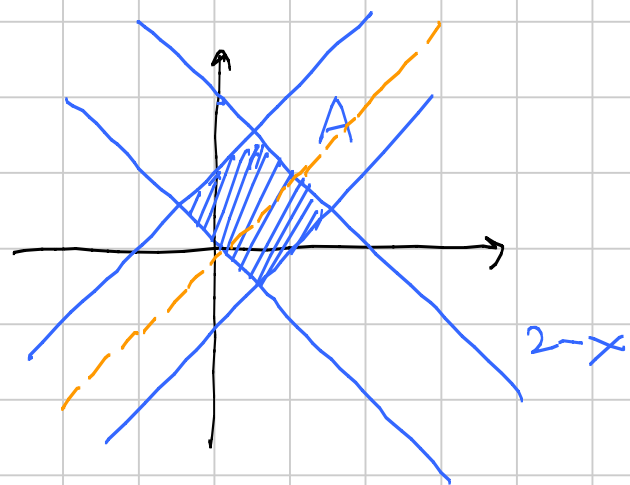
ρ
 $\uparrow$
rag.

$$= \int_0^3 d\rho \int_0^{2\pi} d\theta \cdot \rho^3$$
$$= \int_0^3 \rho^3 d\rho \int_0^{2\pi} d\theta$$

$$= 2\pi \int_0^3 \rho^3 d\rho = 2\pi \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\rho=3} = \boxed{\pi \frac{81}{2}}$$

CAMBIO DI VARIABILI IN GENERALE

$$A = \{ (x, y) \in \mathbb{R}^2 : 0 \leq x+y \leq 2, -1 \leq x-y \leq 1 \}$$



$$\begin{aligned} y &\leq 2-x \\ y &\geq -x \end{aligned}$$

$$\begin{aligned} x-y &\leq 1; & y &\geq x-1 \\ x-y &\geq -1 & y &\leq x+1 \end{aligned}$$

$$\begin{aligned} 0 \leq x+y \leq 2, & \quad -1 \leq x-y \leq 1 \\ \parallel & \quad \parallel \\ 0 \leq u \leq 2 & \quad -1 \leq v \leq 1 \end{aligned}$$

$$\iint_A (x^2 - y^2) dx dy = \underbrace{\int_0^2 du \int_{-1}^1 dv}_{\text{insieme } A \text{ descritto nelle variabili } u \text{ e } v} \underbrace{u v}_{x^2 - y^2 = (x+y)(x-y)} \underbrace{J(u, v)}_{\text{pagamento}}$$

$J(u, v)$ è una funzione di u, v detta DETERMINANTE JACOBIANO e si calcola in 3 tappe

① RICAVARE

② MATRICE

③ DETERMINANTE

① RICAVARE

$$\begin{aligned}x + y &= u \\ x - y &= v\end{aligned}$$

Ricavo x e y in funzione di u e v

Somma: $2x = u + v \Rightarrow x = \frac{u + v}{2}$

Sottrazione: $2y = u - v \Rightarrow y = \frac{u - v}{2}$

② **MATRICE** Alla fine dalla ① mi ritrovo

$$x = \varphi(u, v)$$

$$y = \psi(u, v)$$

MATRICE JACOBIANA :

$$\begin{pmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

Nell' esempio

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

③ **DETERMINANTE**

$$J(u, v) = | \det(\text{matrice jacobiana}) |$$

Nell' esempio

$$J(u,v) = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

$$\iint_A (x^2 - y^2) dx dy = \underbrace{\int_0^2 du \int_{-1}^1 dv}_{\text{insieme A descritto nelle variabili } u \text{ e } v} \underbrace{uv}_{\substack{x^2 - y^2 = \\ (x+y)(x-y)}} \underbrace{\frac{1}{2}}_{\substack{J(u,v) \\ \text{pagamento}}}$$

$$\begin{aligned} &= \frac{1}{2} \int_0^2 du \int_{-1}^1 dv uv \\ &= \frac{1}{2} \int_0^2 u du \underbrace{\int_{-1}^1 v dv}_0 = 0 \end{aligned}$$