

Cambi di Variabile negli integrali tripli + coordinate cilindriche
+ coord. sferiche

COORD. CILINDRICHE

Cartesiane (x, y, z)

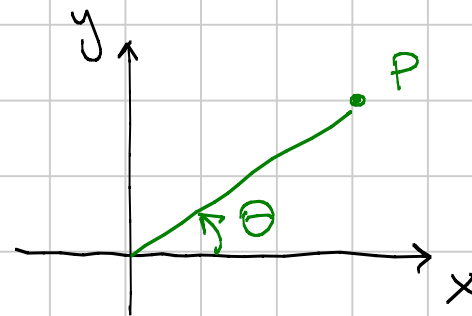
Cilindriche (ρ, θ, z)

z è la stessa, (ρ, θ) sono le polari corrispondenti a (x, y)

z è la quota di P ,

ρ = distanza della proiezione
di P dalla torre di controllo

θ = angolo solito



Quando si integra in coord. cilindriche $J = \rho$

Esempio A = solito cilindro con raggio base = 1 e alt. = 2

In coord. cilindriche A diventa : $\rho \in [0, 1]$, $\theta \in [0, 2\pi]$, $z \in [0, 2]$

$$\iiint_A z \, dx \, dy \, dz = \underbrace{\int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^2 dz}_{\text{Descriz. di } A \text{ in COORD. CILINDR.}} \underbrace{z}_{\text{Sempre } z \text{ è}} \underbrace{\rho}_{\text{pag.}}$$

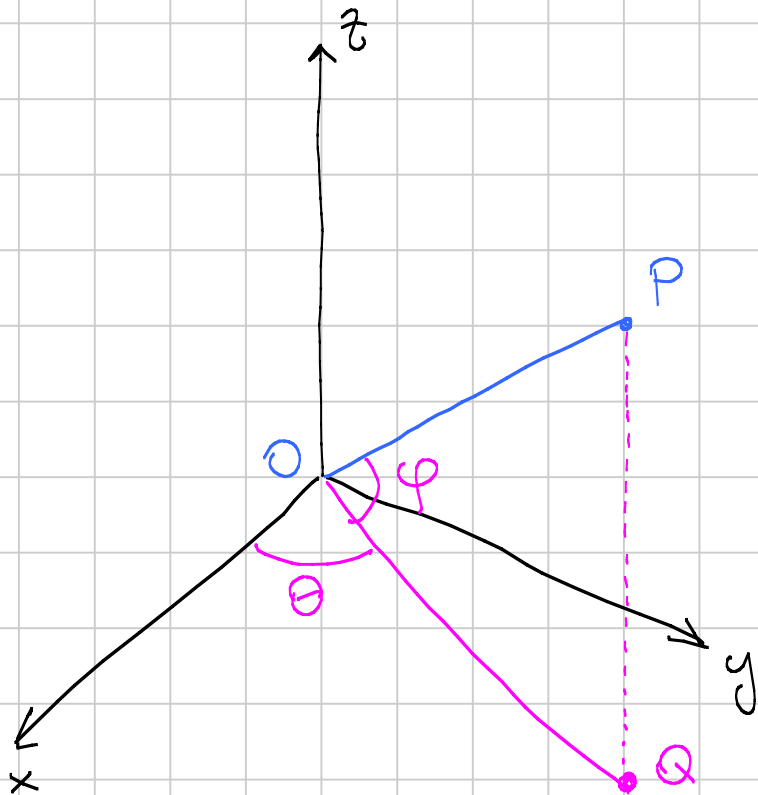
$$= \int_0^1 \rho \, d\rho \int_0^{2\pi} d\theta \int_0^2 z \, dz =$$

$$= \int_0^1 \rho \, d\rho \int_0^{2\pi} d\theta \left[\frac{z^2}{2} \right]_{z=0}^{z=2} = 2 \int_0^1 \rho \, d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= 4\pi \int_0^1 \rho \, d\rho = 2\pi$$

COORDINATE SFERICHE

(ρ, θ, φ)



ρ = lunghezza di OP =
= distanza di P dall'origine

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Sia Q la proiezione di P sul
piano base xy

[OQ sarebbe il ρ delle coord.
cilindriche]

φ = angolo tra OQ e OP = angolo tra OP e piano base xy

θ = angolo tra OQ e semiasse positivo
della x (solito)

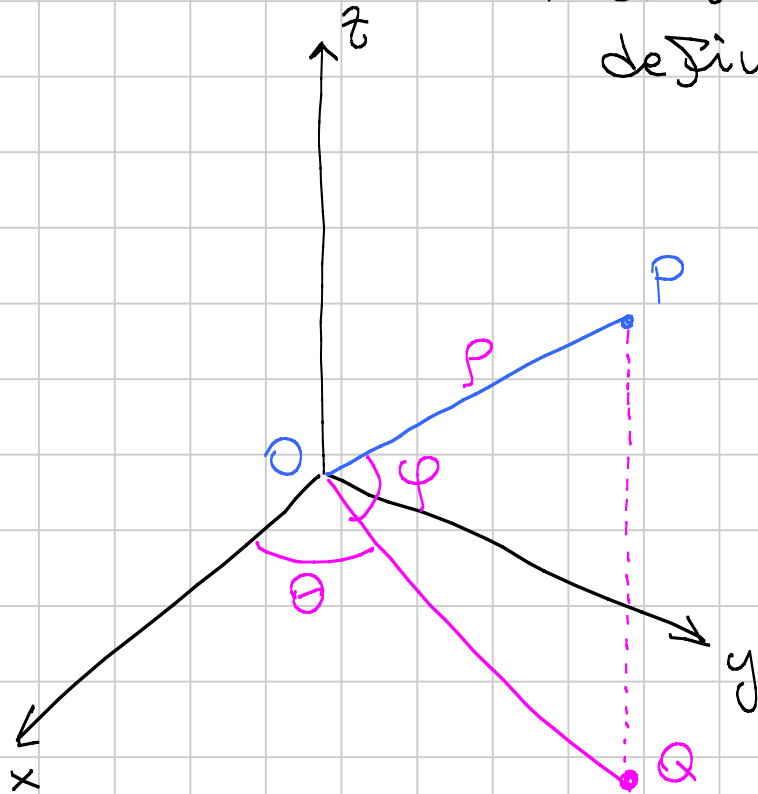
Dove variano? $\rho \geq 0$, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\theta \in [0, 2\pi]$

LATITUDINE

LONGITUDINE

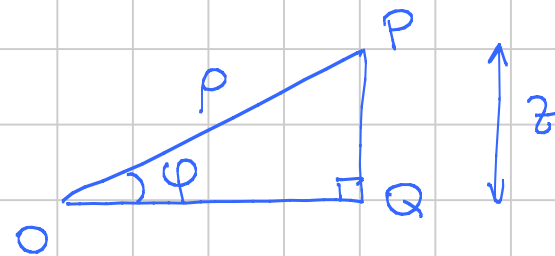
Casi particolari: * nell'origine si ha $\rho = 0$, mentre θ e φ non sono definite

* su tutti i punti dell'asse z non è definita θ



FORMULE DI PASSAGGIO

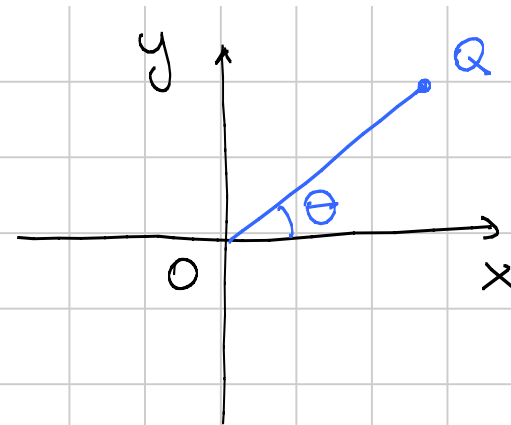
$$z = \text{lungh. con segno di } PQ \\ = \rho \sin \varphi$$



Analogamente $OQ = \rho \cos \varphi$

$$x = OQ \cdot \cos \theta = \rho \cos \varphi \cos \theta$$

$$y = OQ \cdot \sin \theta = \rho \cos \varphi \sin \theta$$



$$x = \rho \cos \varphi \cos \theta, \quad y = \rho \cos \varphi \sin \theta, \quad z = \rho \sin \varphi$$

FORMULE DI PASSAGGIO

Come si calcola J ? 1. Ricavare x, y, z in funzione di ρ, θ, φ
(appena fatto)

2. Matrice jacobiana

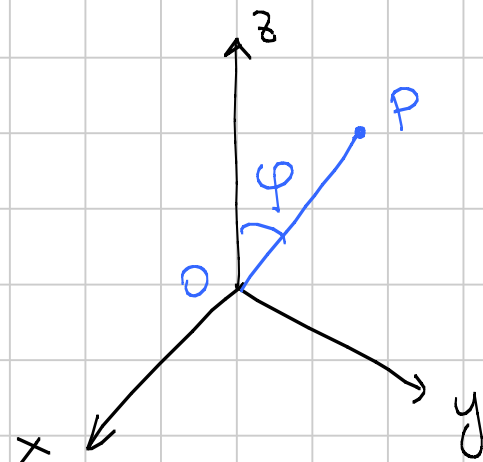
$$\begin{pmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{pmatrix}$$

$$3. J(\rho, \theta, \varphi) = |\text{Det}(\text{matrice})| = \boxed{\rho^2 \cos \varphi}$$

ACHTUNG!!! I fisici talvolta definiscono diversamente le coord. sferiche.

Definita come in figura
 φ varia da 0 a π .

In tutte le formule che
 contengono φ si scambiano
 $\sin \varphi$ e $\cos \varphi$ (compreso
 nel J .)



Esempio 1 $\overset{\circ}{\text{---}} \overset{\circ}{\text{---}} \overset{\circ}{\text{---}}$ $A = \text{"semisfera alta"} \quad x^2 + y^2 + z^2 \leq 1, z \geq 0.$

In coord. sferiche si descrive come

$$\rho \in [0, 1], \quad \theta \in [0, 2\pi], \quad \varphi \in [0, \frac{\pi}{2}]$$

$$\iiint_A z \, dx \, dy \, dz = \int_0^1 \rho \, d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \quad \underbrace{\rho \sin \varphi}_{z} \quad \underbrace{\rho^2 \cos \varphi}_{\rho AG}$$

Descrizione di A in
coord. sferiche

$$= \int_0^1 \rho^3 \, d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \underbrace{\frac{2 \cos \varphi \cdot \sin(2\varphi)}{2}}_{\sin(2\varphi)} \, d\varphi$$

2π

$$= 2\pi \int_0^1 \rho^3 \, d\rho \left[-\frac{\cos(2\varphi)}{4} \right]_{\varphi=0}^{\varphi=\frac{\pi}{2}}$$

$$= 2\pi \int_0^1 \rho^2 \, d\rho \left\{ \frac{-(-1) + 1}{4} \right\} = \pi \int_0^1 \rho^3 \, d\rho =$$

$$= \pi \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\rho=1} = \frac{\pi}{4}$$

Esempio 2 $A =$ sfera con centro in $(0,0,0)$ e raggio 1

$$\iiint_A x^2 dx dy dz = \int_0^1 \rho d\rho \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \quad \underbrace{\rho^2 \cos^2 \theta \cos^2 \varphi}_{x^2} \underbrace{\rho^2 \cos \varphi}_J$$

TUTTA LA SFERA

$$= \int_0^1 \rho^4 d\rho \int_0^{2\pi} \cos^2 \theta d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 \varphi d\varphi = \text{provare e viene}$$

$$\iiint_A x^2 dx dy dz = \iiint_A y^2 dx dy dz = \iiint_A z^2 dx dy dz$$

$$\iiint_A z^2 dx dy dz = \int_0^1 \rho d\rho \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \quad \underbrace{\rho^2 \sin^2 \varphi}_{z^2} \underbrace{\rho^2 \cos \varphi}_J$$

$$= \int_0^1 \rho^4 d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_{-\frac{H}{2}}^{\frac{H}{2}} \sin^2 \varphi \cdot \cos \varphi d\varphi}_{\left[\frac{\sin^3 \varphi}{3} \right]_{\varphi = -\frac{H}{2}}^{\varphi = \frac{H}{2}} = \frac{2}{3}}$$

$$= 2\pi \cdot \frac{2}{3} \cdot \int_0^1 \rho^4 d\rho = 2\pi \cdot \frac{2}{3} \cdot \frac{1}{5} = \frac{4}{15} \pi$$

In alternativa bastava calcolare

$$\frac{1}{3} \iiint_A \underbrace{(x^2 + y^2 + z^2)}_{\rho^2} dx dy dz$$