

Formula di integrazione per SEZIONI

$$A = \{ (x, y, z) \in \mathbb{R}^3 : z \in [a, b], (x, y) \in S_z \}$$

minima
quota

max.
quota

Sezione di A
ad altezza z

$$\iiint_A f(x, y, z) dx dy dz = \int_a^b dz \iint_{S_z} dx dy f(x, y, z)$$

FORMULA DI INT. PER SEZIONI

Esempio 1

$$\iiint_A z dx dy dz =$$

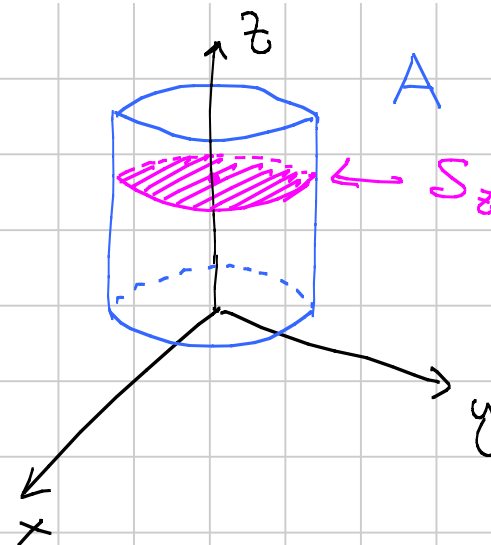
A = cilindro prec,

sezioni



$$= \int_0^2 dz \iint_{S_z} dx dy \cdot z =$$

S_z = sezione cilindro ad altezza z
 = cerchio con centro in $(0,0)$
 e raggio 1



$$= \int_0^2 z dz \iint_{S_z} dx dy = \int_0^2 z \cdot \text{Area}(S_z) dz = \pi \int_0^2 z dz = 2\pi$$

— o — o —

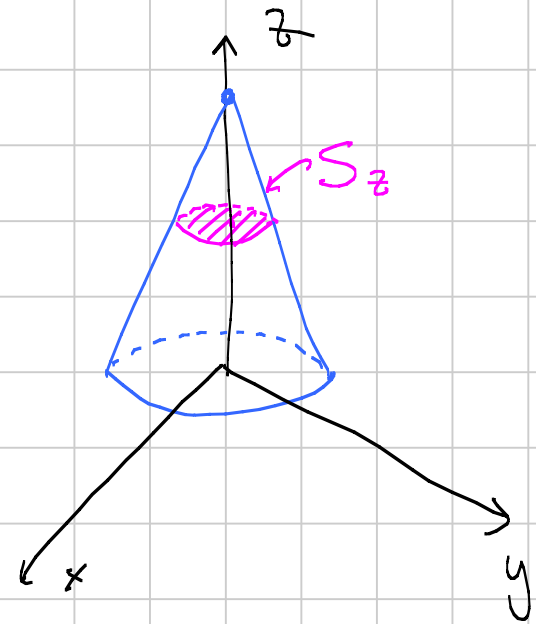
Esempio 2 A = cono con base cerchio con centro in $(0,0)$
 e raggio 2, altezza = 3

$$f(x,y,z) = z^2$$

$$\iiint_A z^2 dx dy dz \stackrel{\text{set.}}{=} \int_0^3 dz \iint_{S_z} dx dy \cdot z^2$$

$$= \int_0^3 z^2 dz \iint_{S_z} dx dy =$$

$$= \int_0^3 z^2 dz \cdot \text{Area}(S_z) =$$



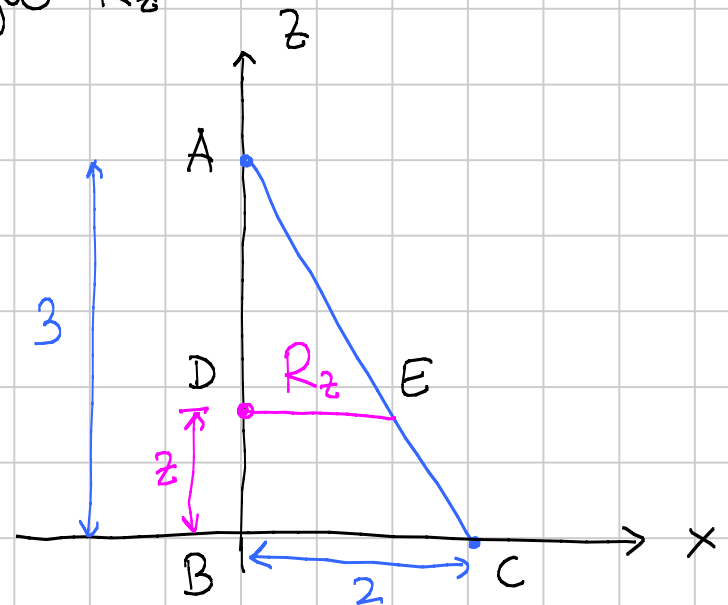
S_z = cerchio con centro in $(0,0)$ e raggio R_z

Dalla similitudine $ADE \sim ABC$
ho che

$$AD : DE = AB : BC$$

$$(3-z) : R_z = 3 : 2$$

$$R_z = \frac{2(3-z)}{3} = \boxed{2 - \frac{2}{3}z}$$



$$R_z = 2 - \frac{2}{3}z \quad \text{controllo} \quad \begin{array}{l} \nearrow z=0 \rightarrow R_z = 2 \\ \searrow z=3 \rightarrow R_z = 0 \end{array}$$

$$= \int_0^3 z^2 \underbrace{\pi \left(2 - \frac{2}{3}z\right)^2}_{\text{Area}(S_z)} dz = \text{si fa facilmente.}$$

$$\text{Area}(S_z) = \pi (R_z)^2$$

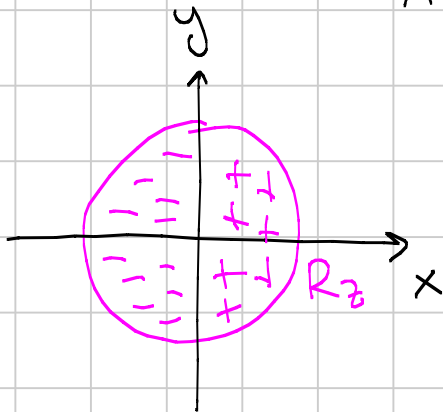
— 0 — 0 —

Esempio 2.5 $A =$ corpo dell'esempio 2

$$\iiint_A x z^2 dx dy dz = \int_0^3 dz \iint_{S_z} dx dy x \underbrace{z^2}_{\text{costante}}$$

$\uparrow$
sez.

$$= \int_0^3 z^2 dz \underbrace{\iint_{S_z} x dx dy}_0 = 0$$



Esempio 3 $A =$ semisfera di raggio 1

$$\iiint_A z \, dx \, dy \, dz$$

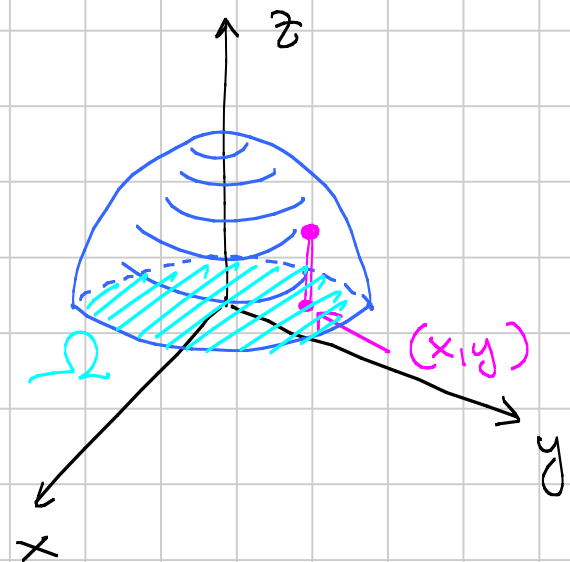
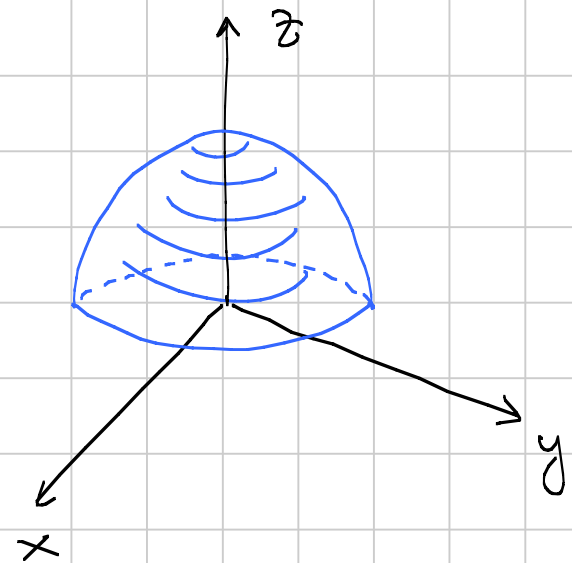
$$A = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0 \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 : (x, y) \in \Omega, \\ 0 \leq z \leq \sqrt{1 - x^2 - y^2} \}$$

↑
Descrizione come
insieme normale
risp. piano base xy

$$= \{ (x, y, z) \in \mathbb{R}^3 : z \in [0, 1], (x, y) \in S_z \}$$

↑
Descrizione per
sezioni



$$\iiint_A z \, dx \, dy \, dz = \iint_{\Omega} dx \, dy \int_0^{\sqrt{1-x^2-y^2}} dz = z$$

↑
colonne

$$= \iint_{\Omega} dx \, dy \left[\frac{z^2}{2} \right]_{z=0}^{z=\sqrt{1-x^2-y^2}}$$

$$= \frac{1}{2} \iint_{\Omega} (1-x^2-y^2) \, dx \, dy$$

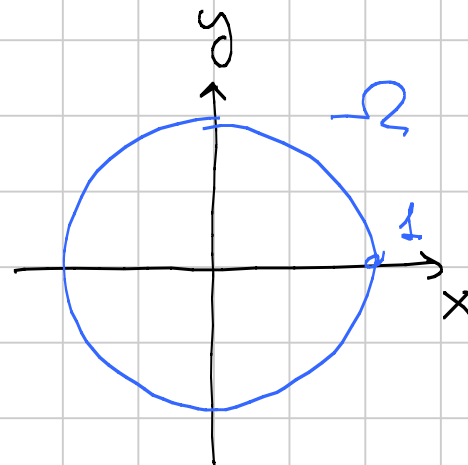
COORDINATE
POLARI

$$\rightarrow = \frac{1}{2} \int_0^1 dp \int_0^{2\pi} d\theta \underbrace{(1-p^2)}_{1-x^2-y^2} \underbrace{p}_{\text{PAG}}$$

Ω in coord.
polari

$$= \frac{1}{2} \int_0^1 (p-p^3) \, dp \int_0^{2\pi} d\theta = \pi \int_0^1 (p-p^3) \, dp =$$

2π



$$= \pi \left[\frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_{\rho=0}^{\rho=1} = \pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{4}$$

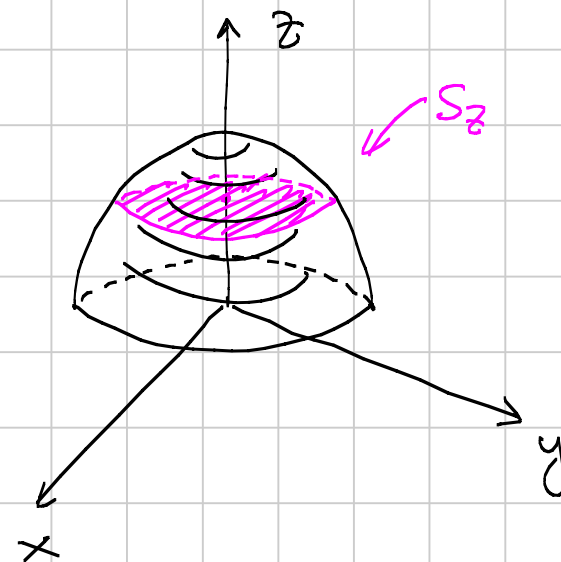
$$\iiint_A z \, dx \, dy \, dz \stackrel{\substack{\text{sezioni} \\ \downarrow}}{=} \int_0^1 dz \iint_{S_z} dx \, dy \cdot \textcircled{z}$$

$$= \int_0^1 z \, dz \iint_{S_z} dx \, dy =$$

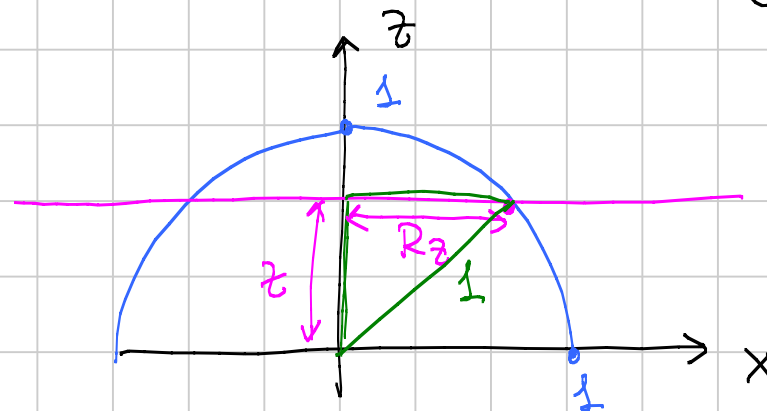
$$= \int_0^1 z \cdot \text{Area}(S_z) \, dz =$$

$$\text{Pitagora} \Rightarrow z^2 + R_z^2 = 1$$

$$\Rightarrow R_z = \sqrt{1 - z^2}$$



S_z è un cerchio di raggio R_z



S_z è un cerchio di raggio $\sqrt{1-z^2}$ $\rightarrow z=0 \rightsquigarrow R=1$
 $\rightarrow z=1 \rightsquigarrow R=0$

$$= \int_0^1 z \cdot \underbrace{\pi R_z^2}_{\text{Area}(S_z)} dz = \pi \int_0^1 z(1-z^2) dz = \pi \int_0^1 (z-z^3) dz$$
$$= \frac{\pi}{4}.$$