

CAMBI DI VARIABILE PER INT. DOPPI

Esempio $A = \{ (x, y) \in \mathbb{R}^2 : 0 \leq x+y \leq 1, 2 \leq x-y \leq 3 \}$

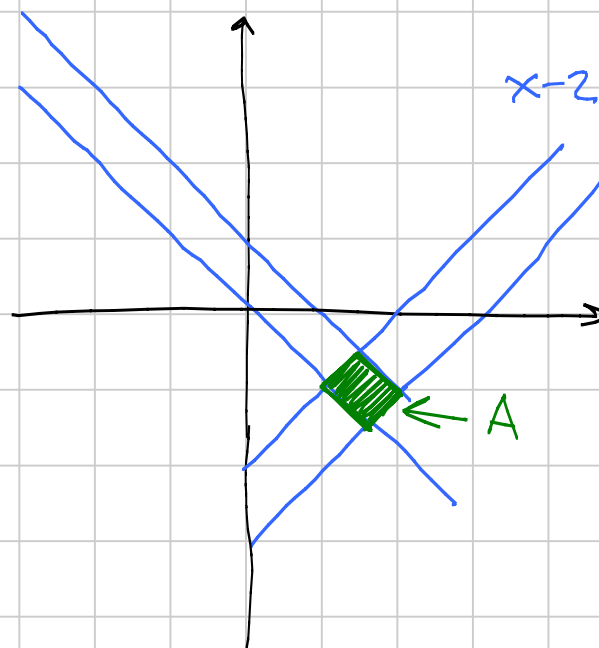
$$\iint_A (x^2 - y^2) dx dy$$

$$x+y \geq 0 \rightarrow y \geq -x$$

$$x+y \leq 1 \rightarrow y \leq 1-x$$

$$x-y \leq 3 \rightarrow y \geq x-3$$

$$x-y \geq 2 \rightarrow y \leq x-2$$



$$0 \leq x+y \leq 1$$

u

$$2 \leq x-y \leq 3$$

v

$$(x^2 - y^2) = (x+y)(x-y) = uv$$

$$\iint_A (x^2 - y^2) dx dy = \int_0^1 du \int_2^3 dv$$

$\uparrow$
 descrizione insieme
 A usando le
 coord. u e v

uv
 Funzione
 $x^2 - y^2$ nelle
 variabili u e v

$\frac{1}{2} =$
 $J(u, v)$
 pagamento

Come calcolare il pagamento $J(u, v)$

① Ricavare x e y (in funzione di u e v)

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

Somma: $2x = u + v \Rightarrow$

Sottr.: $2y = u - v \Rightarrow$

$$\begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \end{aligned}$$

② Costruzione matrice JACOBIANA

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

③ Determinante JACOBIANO

$$J(u, v) = | \text{Det (Matrice)} |$$

$$\text{Nell'esempio: } J(u, v) = \left| -\frac{1}{4} - \frac{1}{4} \right| = \left| -\frac{1}{2} \right| = \frac{1}{2}.$$

Nell'esempio l'integrale è diventato

$$\frac{1}{2} \int_0^1 du \int_2^3 dv (uv)$$

— o — o —

La procedura indicata nel caso delle coord. polari diventa:

① Ricavare x e y in funzione di ρ e θ :

$$x = \rho \cos \theta \quad y = \rho \sin \theta$$

② Matrice jacobiana:

$$\begin{pmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{pmatrix}$$

③ Det. jacobiano

$$J(\rho, \theta) = \rho \cos^2 \theta + \rho \sin^2 \theta = \rho (\cos^2 \theta + \sin^2 \theta) = \rho$$

Esempi importanti di cambio di variabile

1. Traslazioni $u = x + a$ $v = y + b$ (a, b numeri)

① $x = u - a, y = v - b$

② $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Identità}$

③ $J = |\text{Det}| = 1$

2. Dilatazioni degli assi $u = ax$ $v = by$ (a, b numeri $\neq 0$)

① $x = \frac{u}{a}$ $y = \frac{v}{b}$

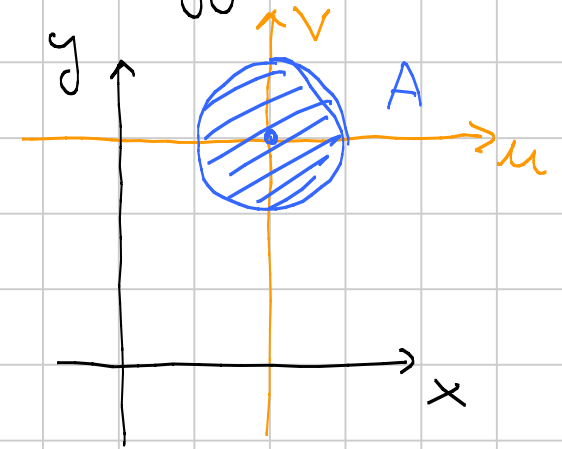
② $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$ Matrice DIAGONALE

③ $J = |\text{Det}| = \frac{1}{|ab|}$

Esempio 1

A = cerchio con centro in $(2,3)$ e raggio 1

$$\iint_A x \, dx \, dy \quad (\text{di sicuro sarà } > 0)$$



$$A = \{ (x, y) \in \mathbb{R}^2 : \underbrace{(x-2)}_u^2 + \underbrace{(y-3)}_v^2 \leq 1^2 \}$$

Nelle coordinate (u, v) l'equazione del cerchio è $u^2 + v^2 \leq 1$

$$\iint_A x \, dx \, dy = \iint_{u^2 + v^2 \leq 1} (u+2) \quad 1 \quad du \, dv =$$

↑
Insieme A nelle
coord. u e v

↑
 x ricavato
risp. a u e v

↑
1

$$= \iint_{u^2 + v^2 \leq 1} (u+2) \, du \, dv$$

Per questo posso usare coord. polari

Oppure, più brevemente,

$$= \iint_{u^2+v^2 \leq 1} u \, du \, dv + \iint_{u^2+v^2 \leq 1} 2 \, du \, dv = 2\pi$$

||
0
per simmetria

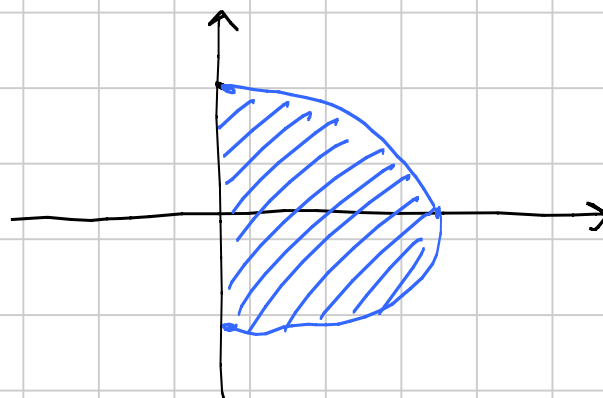
||
2 Area cerchio
||
2π

— 0 — 0 —

Esempio 2 $A = \{ (x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 \leq 1, x \geq 0 \}$

$$\iint_A y^2 \, dx \, dy \quad (\text{di sicuro } > 0)$$

$$2x^2 + 3y^2 \leq 1$$
$$(\underbrace{\sqrt{2}}_u x)^2 + (\underbrace{\sqrt{3}}_v y)^2 \leq 1$$



Posto $u = \sqrt{2}x$, $v = \sqrt{3}y$ l'insieme diventa

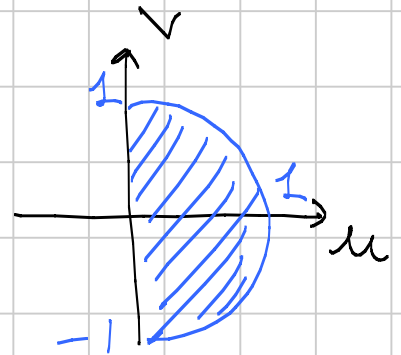
$$u^2 + v^2 \leq 1, \quad u \geq 0$$

$$y = \frac{v}{\sqrt{3}}$$

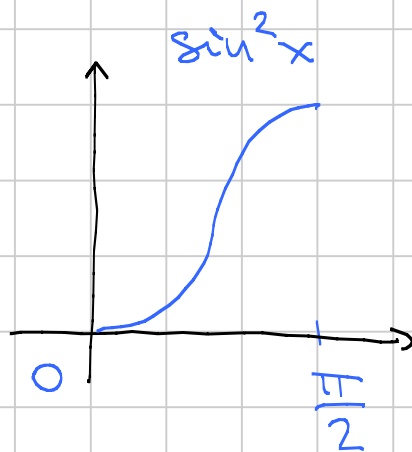
$$\iint_A y^2 dx dy = \iint_{\substack{u^2+v^2 \leq 1 \\ u \geq 0}} \underbrace{\frac{v^2}{3}}_{y^2} \underbrace{\frac{1}{\sqrt{6}}}_{J \text{ calcolato prima}} du dv$$

$$= \frac{1}{3\sqrt{6}} \iint_{\substack{u^2+v^2 \leq 1 \\ u \geq 0}} v^2 du dv = (\text{POLARI})$$

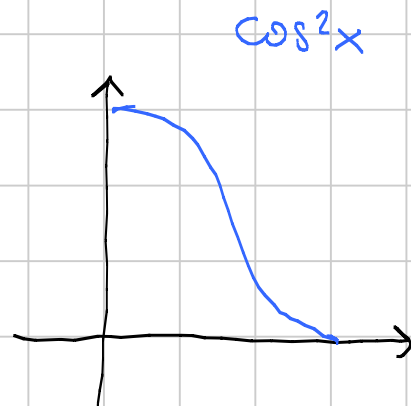
$$= \frac{1}{3\sqrt{6}} \int_0^1 \rho d\rho \int_{-\pi/2}^{\pi/2} d\theta \underbrace{\rho^2 \sin^2 \theta}_{v^2} \quad \begin{matrix} \rho \\ \uparrow \\ J \end{matrix}$$



$$= \frac{1}{3\sqrt{6}} \int_0^1 \rho^3 d\rho \underbrace{\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta d\theta}_{\frac{\pi}{2}} = \text{si fa}$$



$$I_1 = \int_0^{\pi/2} \cos^2 x dx$$



$$I_2 = \int_0^{\pi/2} \sin^2 x dx \Rightarrow I_1 = I_2$$

$$I_1 + I_2 = \int_0^{\pi/2} (\cos^2 x + \sin^2 x) dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I_1 = I_2 = \frac{\pi}{4}$$