

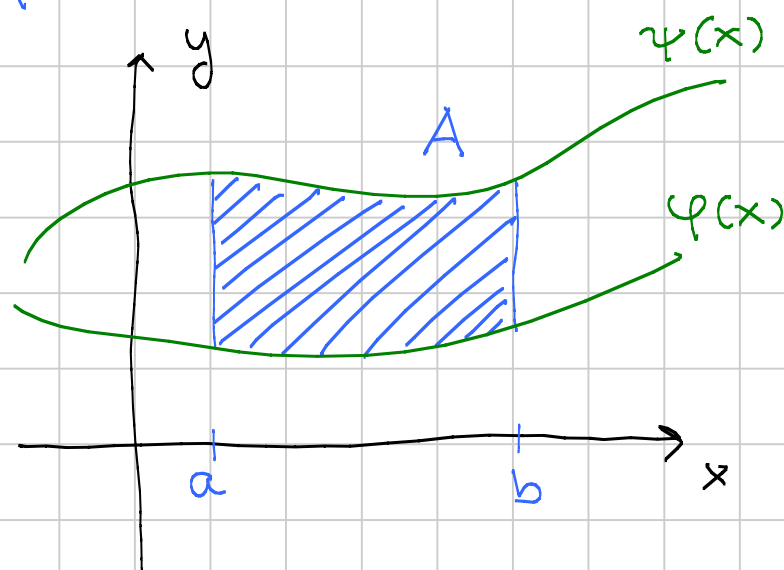
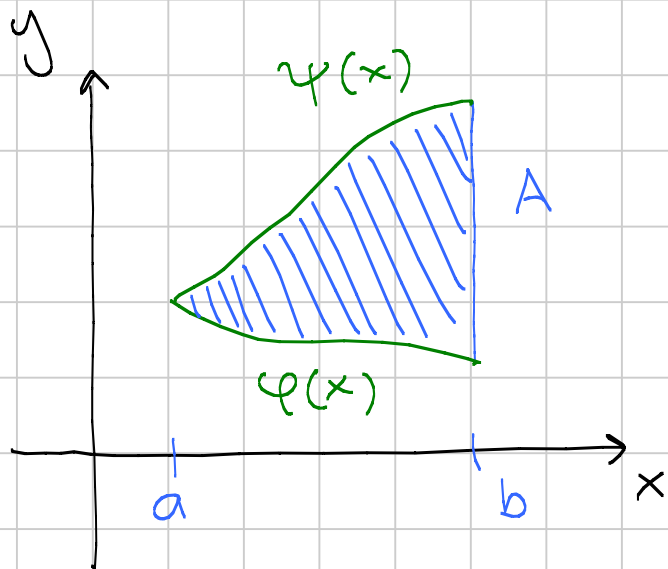
INSIEMI NORMALI RISPETTO ASSE x

Un insieme $A \subseteq \mathbb{R}^2$ si dice norm. risp. asse x se è della forma

$$A = \{ (x, y) \in \mathbb{R}^2 : x \in [a, b], \varphi(x) \leq y \leq \psi(x) \}$$

↑
intervallo dato

↑
funzioni date



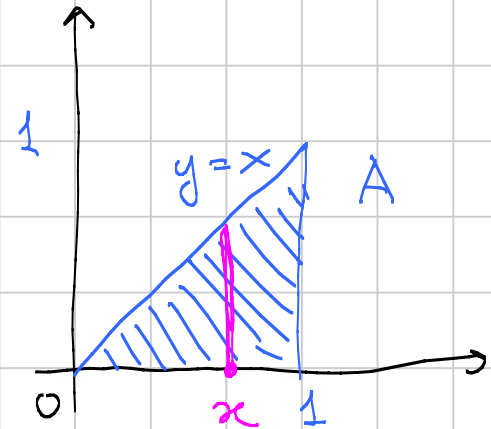
FORMULA DI RIDUZIONE su insiemi normali rispetto asse x

A = ins. norm. asse x $f(x,y)$ = funzione limitata qualunque

$$\iint_A f(x,y) dx dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y) dy$$

Esempio 1 $\iint_A xy dx dy$

Scriviamo A come insieme norm. risp.
asse x



$$A = \{ (x,y) \in \mathbb{R}^2 : x \in [0,1], 0 \leq y \leq x \}$$

$$\iint_A xy dx dy = \int_0^1 dx \int_0^x dy \quad xy = \int_0^1 x dx \int_0^x y dy =$$

$$= \int_0^1 x dx \left[\frac{y^2}{2} \right]_{y=0}^{y=x}$$

↑
primitiva di
y risp. a y

$$= \int_0^1 x dx \left\{ \frac{x^2}{2} - 0 \right\} =$$

↑
y=x ↑
y=0

$$= \frac{1}{2} \int_0^1 x^3 dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_{x=0}^{x=1} = \frac{1}{8}$$

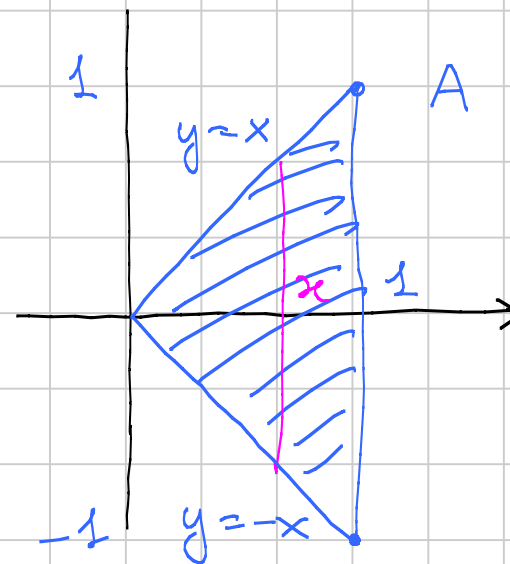
— 0 — 0 —

Esempio 2

$$\iint_A x^2 dx dy =$$

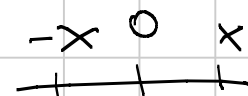
$$A = \{ (x, y) \in \mathbb{R}^2 : x \in [0, 1], -x \leq y \leq x \}$$

$$= \int_0^1 dx \int_{-x}^x dy \quad x^2 = \int_0^1 x^2 dx \int_{-x}^x dy$$



$$= \int_0^1 x^2 dx \left[y \right]_{y=-x}^{y=x} = \int_0^1 x^2 \left\{ \underset{\substack{\uparrow \\ y=x}}{x} - \underset{\substack{\uparrow \\ y=-x}}{(-x)} \right\} dx =$$

$$= 2 \int_0^1 x^3 dx = 2 \left[\frac{x^4}{4} \right]_{x=0}^{x=1} = \frac{1}{2}.$$

In generale $\int_a^b dx = b-a = \text{lunghezza intervallo}$ 

$\int_{-x}^x dy = \text{lungh. intervallo } [-x, x] = 2x$

Analogamente: $\iint_A dx dy = \iint_A 1 dx dy = \text{Area}(A)$

Si può usare per calcolare l'area di A

Esempio 3

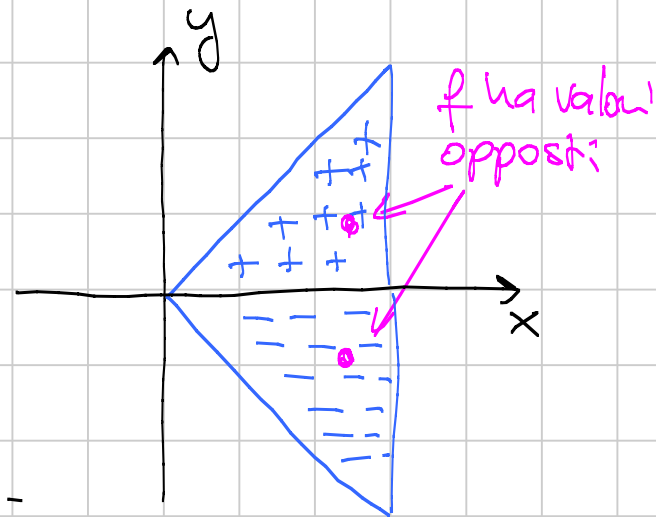
A = come esempio precedente

$$\begin{aligned}\iint_A y \, dx \, dy &= \int_0^1 dx \int_{-x}^x dy \, y = \int_0^1 dx \left[\frac{y^2}{2} \right]_{y=-x}^{y=x} = \\ &= \int_0^1 dx \left\{ \frac{x^2}{2} - \frac{(-x)^2}{2} \right\} \left[= \int_0^1 0 \, dx \right] = 0\end{aligned}$$

$\uparrow$ $y=x$ $\uparrow$ $y=-x$

Potevamo scrivere 0 direttamente?

$\iint_A y \, dx \, dy$ = volume positivo nella
parte +++
sommato ad un volume
negativo uguale nella parte ---



= 0

Rigorosamente

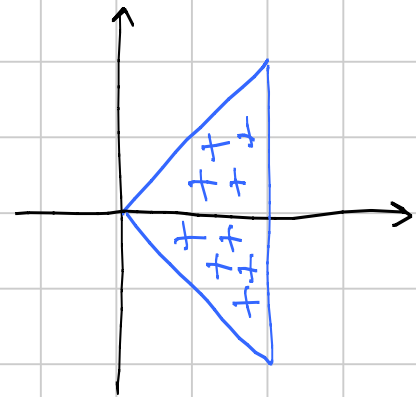
$\nearrow f(x, -y) = -f(x, y)$ $(x, y) \rightarrow (x, -y)$
 $\searrow$ insieme simmetrico rispetto alla trasf.

Esempio 4

$A = \text{sdito}$

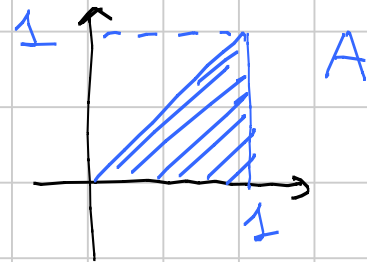
$$\iint_A x \, dx \, dy > 0$$

perché è to
integrando una
funzione > 0



$$\begin{aligned} \iint_A x \, dx \, dy &= \int_0^1 dx \int_{-x}^x dy \quad \text{(x is circled)} = \int_0^1 x \, dx \underbrace{\int_{-x}^x dy}_{2x = \text{lungh. interv.}} = \\ &= 2 \int_0^1 x^2 \, dx = \frac{2}{3}. \end{aligned}$$

Esempio 5 $\iint_A x \, dx \, dy = \frac{1}{3}$



$$\iint_{\triangle} x \, dx \, dy = \iint_{\triangle} x \, dx \, dy$$

Esempio 6

$$A = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1 - x^2\}$$

Non è ancora
una scrittura
come insieme
normale

$$\iint_A y \, dx \, dy = \text{(di sicuro sarà } > 0 \text{)}$$

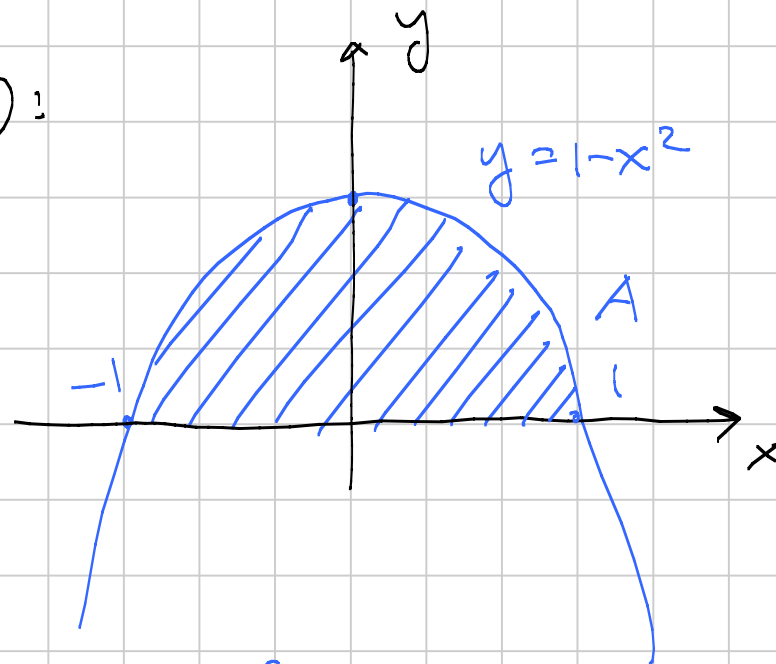
Scrivo A come insieme normale (asse x):

$$A = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 1], 0 \leq y \leq 1 - x^2\}$$

$$= \int_{-1}^1 dx \int_0^{1-x^2} dy \, y = \int_{-1}^1 dx \left[\frac{y^2}{2} \right]_{y=0}^{y=1-x^2}$$

$$= \frac{1}{2} \int_{-1}^1 (1-x^2)^2 dx = \frac{1}{2} \int_{-1}^1 (1 - 2x^2 + x^4) dx$$

= ...



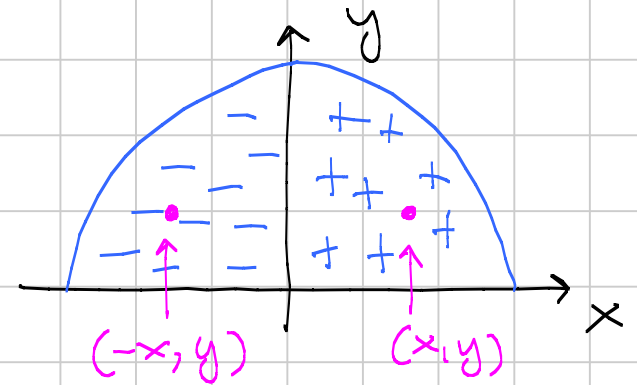
$$1 - x^2 = 0$$

$$x^2 = 1 \Rightarrow x = \pm 1$$

Esempio 7 $A =$ come esempio 6

$$\iint_A x \, dx \, dy = 0$$

dispari risp.
variabile x



Rigoroso: $\nearrow f(-x, y) = -f(x, y)$

$\searrow$ l'insieme A è simmetrico rispetto alla trasformazione $(x, y) \rightarrow (-x, y)$

Esempio 8 $A =$ come 6 e 7

$$\begin{aligned} \iint_A |x| \, dx \, dy &= 2 \iint_{AD} x \, dx \, dy = 2 \int_0^1 dx \int_0^{1-x^2} dy \quad \text{(per simmetria)} = \\ &= 2 \int_0^1 x \, dx \int_0^{1-x^2} dy \quad \text{(parte destra di A)} = \\ &= 2 \int_0^1 x (1-x^2) \, dx = 2 \int_0^1 (x - x^3) \, dx = \dots \end{aligned}$$

$1-x^2 = \text{lungh. interv.}$