Algebra lineare 6/7/2011 (1)

Traccia sol.

(1.2) $A_t = \begin{pmatrix} 1 & 1 & 2 & t \\ 0 & 0 & t & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

$L_{A_t} : \mathbb{R}^4 \to \mathbb{R}^3$

A_t matrice 3×4 I col = II col

$\Downarrow$

$rk(A_t) = rk\begin{pmatrix} 1 & 2 & t \\ 0 & t & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$rk\begin{pmatrix} 1 & 2 & t \\ 0 & t & 1 \\ 1 & 2 & 1 \end{pmatrix} = 3$ per $t \neq 0, 1$

$rk\begin{pmatrix} 1 & 2 & t \\ 0 & t & 1 \\ 1 & 2 & 1 \end{pmatrix} = 2$ per $t = 0, 1$

i) $\begin{cases} \dim \mathrm{Im} = 3 \\ \dim \mathrm{Ker} = 4-3 \end{cases}$ per $t \neq 0, 1$

$\begin{cases} \dim \mathrm{Im} = 2 \\ \dim \mathrm{Ker} = 2 \end{cases}$ per $t = 0, 1$

ii) Per $t \neq 0,1$ (2)

$$rk(A_t) = 3 = rk(A_t \vdots b)$$

$\Rightarrow \exists$ sol. & $\dim\{\text{soluz.}\} = 4-3$

• Per $t = 0$

$$A_0 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \qquad rk = 2$$

$$(A_0 \vdots b) = \left(\begin{array}{cccc:c} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right) \qquad b = \text{IV col.}$$

$$\Downarrow$$

$$rk(A_0) = rk(A_0 \vdots b) = 2$$

$\Rightarrow \exists$ sol. & $\dim\{\text{soluzioni}\} = 4-2$

• Per $t = 1$

$$A_1 = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} \qquad rk = 2$$

$$(A_1 \vdots b) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix} \qquad \det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = 1 \neq 0$$

$$\Downarrow$$

$$rk = 3$$

• Per $t = 1$ non $\exists$ sol.

iii) Per $t \neq 0,1$ $\dim\{sol.\} = 1$

Per $t = 0$ $\dim\{sol.\} = 2$

Per $t = 1$ non $\exists$ sol.

La sol. non è mai unica

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2 $V = \{X \in \mathbb{R}^4 : x_1 + x_2 + 2x_3 + 2x_4 = 0\}$

$\dim V = 4 - 1 = 3$

$V = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \oplus W \quad \Rightarrow \dim W = 2$

$W = \langle w_1, w_2 \rangle \quad w_1, w_2 \in V$

&

$\left\langle w_1, w_2, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$ base di U

Per es. poniamo $x_1 = 0$

$x_4 = 0$

$x_3 = t$

$x_2 = -2x_3 = -2t$

$x_1 = 0$

$x_3 = 0$

$x_4 = s$

$x_2 = -2s$

$w_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

3 cerco righe $(e_1\ e_2\ e_3)$ t.c.

$$(e_1\ e_2\ e_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix} = (0\ \ 0\ \ 0)$$

sol. $= (0\ \ 1\ \ -1)$

Posta $B = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

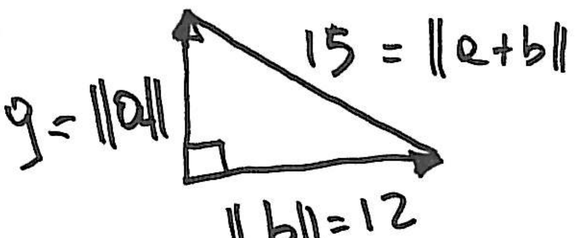
si ha

$$B \cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

cioè $\mathcal{L}_B \circ \mathcal{L}_A = 0$

(4) N.B. i vettori a, b devono $\in V$!

$\|a+b\| = 15 = 3 \cdot 5 \Rightarrow$ cerco a di norma $\|a\| = 3 \cdot 3 = 9$



b di norma $\|b\| = 3 \cdot 4 = 12$

$\tilde{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ soddisfa $2 \cdot 1 + 1 \cdot 2 + 2 \cdot (-2) = 0$

$a \in V$

$\|\tilde{a}\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$

$a = 3 \cdot \tilde{a} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$

b deve essere $\perp$ ad a :

$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\quad 3 \cdot b_1 + 6 b_2 - 6 b_3 = 0$

(5)

$b \in V$

$b \perp a$ $\iff$ $\begin{cases} 2b_1 + b_2 + 2b_3 = 0 \\ 3b_1 + 6b_2 - 6b_3 = 0 \\ b_1^2 + b_2^2 + b_3^2 = 12^2 \end{cases}$

$\|b\| = 12$

$\iff \begin{cases} 2b_1 + b_2 + 2b_3 = 0 \\ 3b_1 + 3b_2 = 0 \\ b_1^2 + b_2^2 + b_3^2 = 12^2 \end{cases}$

$b_1 = -b_2$

$b_2 = 2b_3$ $\rightarrow$ $b = \begin{pmatrix} -2t \\ 2t \\ t \end{pmatrix}$

$b_3 = t$

$\|b\| = 12^2 \iff (-2t)^2 + (2t)^2 + t^2 = 12^2$

$9t^2 = 12^2 = 3^2 \cdot 4^2$

$t^2 = 4^2$

Per $t = 4$

$b = \begin{pmatrix} -8 \\ 8 \\ 4 \end{pmatrix}$

5.2 $A = \begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

i) $P_{car} = \lambda^2 \cdot (\lambda^2 - 4\lambda + 4)$

autovalori: 0 m.a. $= 2$

2 m.a. $= 1$

$m.g.(0) = 4 - rk(A) = 4 - 3 = 1$

$m.g.(2) = 4 - rk(A - 2\,Id) = 4 - 2 = 2$

ii) Autovettori relativi a $\lambda_0 = 0$ $= \left\{ t \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$

Autovettori relativi a $\lambda_1 = 2$: $\left\{ t \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} : t, s \in \mathbb{R} \right\}$

iii) A è triang.

A non è diag.

⑥ $\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$X, Y \mapsto Y^T \cdot A \cdot X$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$\det(A) = 0 \Rightarrow$ A è degenere

$a_{11} = 1$

$\det \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} > 0$ $\Rightarrow \exists$ sottospazio di dim = 2 su cui $\langle , \rangle$ è def. positivo

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OPPURE:

autovalori di A = 0, 1, 3

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conclusione: $\langle , \rangle$ è semidefinito positivo