Corso di laurea in Ingegneria Gestionale/ Chimica
Esame di ALGEBRA LINEARE - anno accademico 2011/2012

Prova scritta del 21/02/2012
TEMPO A DISPOSIZIONE: 120 minuti

	MARCO	
(Cognome)	(Nome)	(Numero di matricola)

PRIMA PARTE

PUNTEGGIO : risposta mancante = 0 ; risposta esatta = +1 risposta sbagliata = -1
calcoli e spiegazioni non sono richiesti

- Sia $z = 1 - \sqrt{3}i$. Scrivere z nella rappresentazione trigonometrica $z = \varrho \cdot e^{i\vartheta}$: $z =$ $\boxed{2 \cdot e^{i \frac{5}{3}\pi}}$

- Dati W e Z i seguenti sottospazi di $\mathbb{R}^3$:

$$W = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \ : \ x_1 + x_2 + x_3 = 0\} \ , \ Z = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \ : \ x_2 - x_3 = 0\} \ .$$

$\dim(W + Z) = \boxed{3}$

Determinare una base di $W \cap Z$:

$$\boxed{\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}}$$

n.b.
$\dim(W) = 2$
$\dim(Z) = 2$
$\dim(W \cap Z) = 1$

- $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 6 & 3 & 0 \end{pmatrix} \implies \dim(Ker(l_A)) = \boxed{3}$ $\quad \mathrm{rg}(A) = \boxed{3}$

$6 - 3$

- $\det \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 \end{pmatrix} = \boxed{1}$

- $A = \begin{pmatrix} 5 & 0 \\ 1 & -2 \end{pmatrix} \implies A$ è diagonalizzabile | ~~vero~~ | falso |

- Il vettore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ è autovettore dell'applicazione lineare associata alla matrice (barrare la matrice giusta)

$$A_1 = \cancel{\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}} \qquad A_2 = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

- $A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \implies A \cdot B =$ $\begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$

21-2-2012 Treccie sol.

(1)
$$\begin{cases} z^3 = \pi^2 \cdot \bar{z} \\ e^z = e^{\bar{z}} \end{cases}$$

(1') $z = \rho \cdot e^{i\vartheta} \Rightarrow$

$$z^3 = \rho^3 \cdot e^{i3\vartheta} = \pi^2 \cdot \rho \cdot e^{-i\vartheta} = \pi^2 \cdot \bar{z}$$

$$\Leftrightarrow \begin{cases} \rho^3 = \pi^2 \cdot \rho \\ 3\vartheta = -\vartheta + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

SOL. distinte:

$\rho = 0 \ (\Leftrightarrow z = 0)$

$$\begin{cases} \rho = \pi \\ \vartheta = \dfrac{2k\pi}{4} \end{cases} \quad k = 0, 1, 2, 3$$

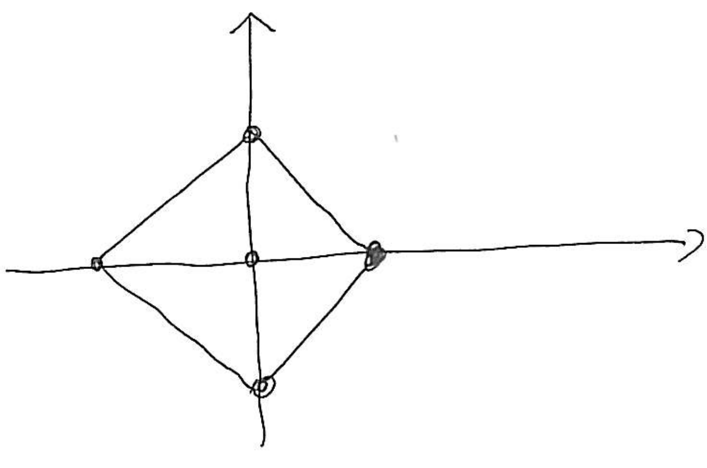
$z_0 = \pi$

$z_1 = i\pi$

$z_2 = -\pi$

$z_3 = -i\pi$

$z_4 = 0$



(20) $e^{z} = e^{\bar{z}} \iff z = \bar{z} + 2k\pi i \qquad k \in \mathbb{Z}$

Posto $z = x + iy$

$\bar{z} = x - iy$

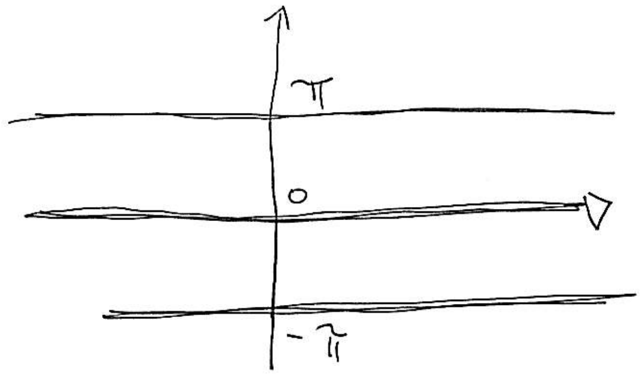
$$(x + iy) = x + i(-y + 2k\pi)$$

Re = Re

Im = Im

x qualsiasi:

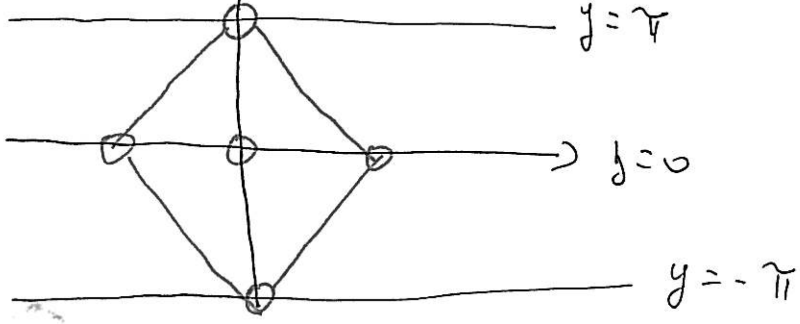
$y = k\pi, \ k \in \mathbb{Z}$



CONCLUSIONE:

SOLUZIONE SISTEMA $\quad \pi, \ i\pi, \ -\pi, \ -i\pi, \ 0$

(Tutte le sol. di 1°)



2

$$A_t = \begin{pmatrix} 1 & t & -1 \\ t & t & 1 \\ 1 & 0 & t \end{pmatrix}$$

$$\det(A_t) = -t \cdot (t^2 - t - 2)$$

$$\det(A_t) = 0 \iff t = 0, -1, 2$$

$t \neq 0, -1, 2$ $\quad rk(A) = 3 \quad \dim \ker = 0$

$t = 0$ $\quad A_0 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

I, III colonne lin ind.

$\Downarrow$

$rk(A_0) = 2$

$\dim(\ker) = 1$

$t = -1$ $\quad A_{-1} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ $\quad M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ ha $\det \neq 0$

$\Rightarrow rk(A_{-1}) = 2$

$\dim(\ker) = 1$

$t = 2$ $\quad A_2 = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ $\quad M = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$ ha $\det \neq 0$

$\Rightarrow rk(A_2) = 2$

$\dim(\ker) = 1$

ii) $$A_t \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

A_t matrice 3×3

$$(A_t \,\vdots\, b) = \left(\begin{array}{ccc|c} 1 & t & -1 & 1 \\ t & t & 1 & 0 \\ 1 & 0 & t & 1 \end{array}\right)$$ matrice 3×4

Per $t \neq 0, -1, 2$

$$rk(A_t) = 3 \leq rk(A_t \,\vdots\, b) \leq 3$$

$$\Rightarrow rk(A_t) = rk(A_t \,\vdots\, b)$$

$\Rightarrow \exists$ (unica) sol.

$t = 0$: $$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

I colonna = b

$\Downarrow$

$rk(A_0) = rk(A_0 \,\vdots\, b) = 2$

$\Rightarrow \exists$ soluzione

dim{soluzioni} $= 3 - 2 = 1$

$t = -1$

$$rk(A_{-1}) = 2 < 3 = rk(A_{-1} \vdots b)$$

$\Rightarrow$ non $\exists$ soluzione

$t = 2$ $\quad rk(A_2) = 2 < 3 = rk(A_2 \vdots b)$

$\Rightarrow$ non $\exists$ soluzione

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iii) $W = \left\langle \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} \right\rangle \qquad \dim(W) = 1$

Quindi $\quad \mathbb{R}^3 = \mathrm{Im}(f_t) \oplus W$

$$\Rightarrow \dim(\mathrm{Im}\, f_t) = rk(A_t) = 2$$

$\Rightarrow$ uniche possibilità sono $\quad t = 0, -1, 2$

$t = 0$: $\quad$ Base Im $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$\det \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 9 \\ 1 & 0 & 3 \end{pmatrix} \neq 0 \quad \Rightarrow \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} \right\}$$

è una base di $\mathbb{R}^3$

$\Rightarrow \mathbb{R}^3 = \mathrm{Im}(f_t) \oplus W$ $\boxed{\text{VERA}}$

$t = -1$:

$$\text{Base Im} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$\det \begin{pmatrix} 1 & -1 & 3 \\ -1 & -1 & 9 \\ 1 & 0 & 3 \end{pmatrix} \neq 0 \quad \Rightarrow \quad \mathbb{R}^3 = \operatorname{Im}(f_t) \oplus W \quad \boxed{\text{VERA}}$$

(come per $t=0$)

$t = 2$

$$\text{Base Im} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right\}$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 9 \\ 1 & 0 & 3 \end{pmatrix} \neq 0 \quad \Rightarrow \quad \mathbb{R}^3 = \operatorname{Im}(f_t) \oplus W \quad \boxed{\text{VERA}}$$

3 $f: \mathbb{R}^3 \to \mathbb{R}^3$ t.c.

$$Im(f) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \qquad Ker(f) \subset Im(f)$$

In particolare si ha $rk(f) = 2$

$$\dim(Ker(f)) = 1$$

Quindi dobbiamo cercare un vettore $v \in \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ tale che $f(v) = 0$.

2 modi

1° modo. PONIAMO: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = v$

Imponiamo:

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

CIOE'

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2° modo

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \text{I colonne di } A$$

$$f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{II colonne di } A$$

Poniamo $v = \begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathrm{Ker}(f)$

$$\underbrace{f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}}_{\text{III colonne di } A} \quad \text{con } \begin{pmatrix}a\\b\\c\end{pmatrix} \text{ da determinare}$$

Condizione $f\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff \begin{pmatrix}1 & 1 & a\\1 & 1 & b\\0 & 1 & c\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\iff \begin{cases} 1+1+a = 0 \\ 1+1+b = 0 \\ 1+c = 0 \end{cases} \qquad \begin{matrix} a = -2 \\ b = -2 \\ c = -1 \end{matrix}$$

$$A = \begin{pmatrix}1 & 1 & -2\\1 & 1 & -2\\0 & 1 & -1\end{pmatrix}$$

(4) $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 \end{pmatrix}$

$P_A(\lambda) = \det(A - \lambda I_d) = \lambda^4 - \lambda^2 = \lambda^2 \cdot (\lambda^2 - 1)$

$= \lambda^2 (\lambda - 1)(\lambda + 1)$

autovalori $\quad 0 \quad m.a. = 2$

$1 \quad m.a. = 1$

$-1 \quad m.a. = 1$

$1 \leq m.g. \leq m.a.$ $\forall$ autovalore $\Rightarrow$ $m.g.(1) = 1$
$m.g.(-1) = 1$

$m.g.(0) = 4 - rk(A)$ $\quad$ (n.b. $A - 0 I_d = A$!)

$rk(A) = 3$ poiché $M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ ha $\det \neq 0$ $\quad$ eliminiamo colonna 2 riga 3

CONCLUSIONE:

autovalori	0	$m.a. = 2$	$m.g. = 1$
	1	$m.a. = 1$	$m.g. = 1$
	-1	$m.a. = 1$	$m.g. = 1$

AUTOVETTORI

relativi a $\lambda_0 = 0$: $\left\{ t \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} : t \neq 0 \right\}$

AUTOVETTORI

relativi a $\lambda_1 = 1$: $\left\{ t \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix} : t \neq 0 \right\}$

AUTOVETTORI

relativi a $\lambda_2 = -1$: $\left\{ t \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \\ 3 \end{pmatrix} : t \neq 0 \right\}$

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iii) Poiché $m.a.(0) = 2 > 1 = m.g.(0)$

A non è diagonalizzabile