Corso di laurea in Ingegneria Gestionale
Esame di ALGEBRA LINEARE - anno accademico 2024/2025

Prova scritta del 8/1/2025
TEMPO A DISPOSIZIONE: 120 minuti

	MARCO	
(Cognome)	(Nome)	(Numero di matricola)

PRIMA PARTE

PUNTEGGIO : risposta mancante = 0 ; risposta esatta = +1 risposta sbagliata = -1
calcoli e spiegazioni non sono richiesti

- Sia $z = 4 \cdot e^{i\frac{2\pi}{3}}$. Scrivere z nella rappresentazione cartesiana $z = x + iy$: $z = \boxed{-2+i\ 2\sqrt{3}}$

- $z = 2 + -i$, $w = 3 - 2i \Rightarrow Re(z \cdot w) = \boxed{4}$

Dati W e Z i seguenti sottospazi di $\mathbb{R}^3$:

$$W = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \ : \ x_1 - 2x_2 + x_3 = 0\} \ , \ Z = \langle \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \rangle .$$

- Determinare una base di $W \cap Z$

$$\boxed{\left\{ \begin{pmatrix} 5 \\ 9 \\ 13 \end{pmatrix} \right\}}$$

- $A = \begin{pmatrix} 4 & 5 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 4 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \dim(Ker(\mathcal{L}_A)) = \boxed{2} \qquad rg(A) = \boxed{2}$

- $\det \begin{pmatrix} 0 & 2 & 0 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \end{pmatrix} = \boxed{-18}$

- $A = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix} \Longrightarrow A$ è diagonalizzabile | vero | falso X |

- $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \Longrightarrow A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{pmatrix}$

- $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -1 \end{pmatrix} \Longrightarrow A \cdot B = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$

- Sia $f : \mathbb{R}^2 \to \mathbb{R}^2$ lineare. Sapendo che $f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $f \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$, allora $f \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

8-1-2025

Traccia soluzioni

1

$$\begin{cases} z^3 = -8\cdot|z|\cdot\bar{z} \\ |z-4i| \le |z| \end{cases}$$

1ª eq. $z=\rho\cdot e^{i\vartheta} \rightarrow z^3=\rho^3\cdot e^{i\vartheta}$

$|z|=\rho$

$\bar{z}=\rho\cdot e^{-i\vartheta}$

$-8 = 8\cdot e^{i\pi}$

$z^3=-8|z|\cdot\bar{z} \iff \rho^3\cdot e^{i3\vartheta} = 8\cdot\rho^2\cdot e^{i(\pi-\vartheta)}$

$$\iff \begin{cases} \rho^3 = 8\rho^2 \\ 3\vartheta = \pi-\vartheta+2k\pi, \quad k\in\mathbb{Z} \end{cases}$$

SOL. DISTINTE

$\rho=0$
$(z=0)$

$$\begin{cases} \rho = 8 \\ \vartheta = \dfrac{\pi+2k\pi}{4} \end{cases}, \quad k=0,1,2,3$$

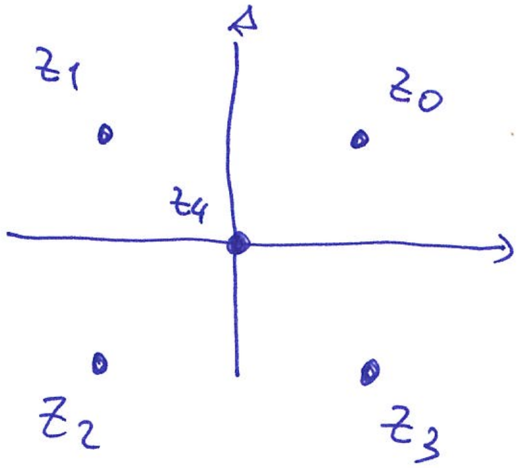
$z_0 = 4\sqrt{2} + i\,4\sqrt{2}$

$z_1 = -4\sqrt{2} + i\,4\sqrt{2}$

$z_2 = -4\sqrt{2} - i\,4\sqrt{2}$

$z_3 = 4\sqrt{2} - i\,4\sqrt{2}$

$z_4 = 0$



2ª eq: $|z - 4i| \leq |z|$

$z = x + iy$

$z - 4i = x + i(y-4)$

$$|z-4i| \leq |z| \iff \sqrt{x^2 + (y-4)^2} \leq \sqrt{x^2+y^2}$$

$$\iff 16 - 8y \leq 0$$

$$\iff \begin{cases} y \geq 2 \\ x \text{ qualsiasi} \end{cases}$$

SOLUZIONI SISTEMA: $z_0 = 4\sqrt{2} + i\,4\sqrt{2}$, $z_1 = -4\sqrt{2} + i\,4\sqrt{2}$

2 $\quad A_t = \begin{pmatrix} t & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 4 & -t \end{pmatrix}$

A_t matrice 3×3

i)

$\det(A_t) = t^2 + 2t$

QUINDI $\quad \det(A_t) \neq 0 \iff t \neq 0, -2$

$t \neq 0, 2 \quad \begin{cases} rg(A_t) = 3 \\ \dim(\ker) = 3-3 = 0 \end{cases}$

$t=0: \quad A_{t=0} : \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 4 & 0 \end{pmatrix}$

$M = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ ha $\det \neq 0 \Rightarrow \begin{cases} rg(A) = 2 \\ \dim(\ker) = 3-2 = 1 \end{cases}$

$t=-2: \quad A_{t=-2} : \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix}$

$M = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$ ha $\det \neq 0 \Rightarrow \begin{cases} rg(A) = 2 \\ \dim(\ker) = 3-2 = 1 \end{cases}$

ii) $$A_t \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

$t \neq 0, -2$ $\quad rg(A_t) = 3$

$rg(A_t) \leq rg(A_t \vdots b) \leq 3 \quad \Rightarrow rg(A_t) = rg(A_t \vdots b) = 3$

$\Rightarrow \exists!$ SOLUZIONE

casi particolari:

$t=0$: $\quad rg(A_t) = 2 \qquad rg(A_t \vdots b) = 3$

poiché $\det \begin{pmatrix} 2 & 0 & 6 \\ 1 & 1 & 2 \\ 4 & 0 & 4 \end{pmatrix} \neq 0$

$\Rightarrow$ non $\exists$ SOLUZIONE

$t=-2$: $\quad rg(A_t) = 2 \qquad rg(A_t \vdots b) = 3$

poiché $\det \begin{pmatrix} -2 & 2 & 6 \\ 1 & 1 & 2 \\ 0 & 4 & 4 \end{pmatrix} \neq 0$

$\Rightarrow$ non $\exists$ SOLUZIONE

iii) $t=0$ $\qquad A_{t=0} : \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 4 & 0 \end{pmatrix}$

$$\mathrm{Im}(\rho_{A_t}) = \mathrm{Span}(\text{colonne di } A)$$
$$= \mathrm{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}\right)$$

$$\mathrm{Ker}(\rho_{A_t}) \Leftrightarrow \quad A_t \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 2x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \\ 4x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = -x_3 \end{cases}$$

$$\Leftrightarrow \left\{ t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}}_{= \mathrm{Im}}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{= \mathrm{Ker}} \quad \text{sono lin. IND.}$$

$$\Downarrow$$

$$\mathbb{R}^3 = \mathrm{Im}(\rho_A) \oplus \mathrm{Ker}(\rho_A)$$

3 $\quad f: \mathbb{R}^3 \to \mathbb{R}^3$ t.c.

$$\mathrm{Im}(f) = \left\langle \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathrm{Ker}(f) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$\dim(\mathrm{Im}) = 2, \quad \dim(\mathrm{Ker}) = 1 \Rightarrow$ una tale appl. esiste

$\mathrm{Im}(f) = \mathrm{Span}$ (colonne di A)

PROVIAMO CON $A = \begin{pmatrix} 1 & 1 & a \\ 4 & -3 & b \\ 1 & 1 & c \end{pmatrix}$

condizione $\mathrm{Ker} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \iff A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\iff \begin{pmatrix} 1 & 1 & a \\ 4 & -3 & b \\ 1 & 1 & c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} 1 + 1 + a = 0 \\ 4 - 3 + b = 0 \\ 1 + 1 + c = 0 \end{cases} \iff \begin{cases} a = -2 \\ b = -1 \\ c = -2 \end{cases}$$

una possibile matrice A è

SOL. $\quad A = \begin{pmatrix} 1 & 1 & -2 \\ 4 & -3 & -1 \\ 1 & 1 & -2 \end{pmatrix}$

4 $A = \begin{pmatrix} 6 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix}$

i) $P_A(\lambda) = \det(A - \lambda Id) = -\lambda \cdot (4-\lambda) \cdot \det\begin{pmatrix} 6-\lambda & -4 \\ 1 & 2-\lambda \end{pmatrix}$

$= -\lambda(4-\lambda)(\lambda^2 - 8\lambda + 16) = -\lambda \cdot (4-\lambda) \cdot (4-\lambda)^2 = -\lambda \cdot (4-\lambda)^3$

$\Rightarrow$ radici: 0, 4

autovalori $\quad 0 \quad m.a.(0) = 1$

$\quad 4 \quad m.a.(4) = 3$

- $m.g.(0) = 1$ poiché $1 \leq m.g. \leq m.a.$

$m.g.(4) = \dim(\mathrm{Ker}(A - 4Id))$

$A - 4Id = \begin{pmatrix} 2 & 0 & -4 & 0 \\ 0 & -4 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$, $rK = 3 \Rightarrow m.g.(4) = 1$

ii) $V_0 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$; $V_4 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

iii) A è triang.^{le}

A non è diag.^{le} poiché $m.g.(4) = 1 \neq 3 = m.a.($