Corso di laurea in Ingegneria Gestionale
Esame di ALGEBRA LINEARE - anno accademico 2022/2023
Prova scritta del 21/01/2023
TEMPO A DISPOSIZIONE: 120 minuti

(Cognome) (Nome) (Numero di matricola)

PRIMA PARTE

PUNTEGGIO : risposta mancante = 0 ; risposta esatta = +1 risposta sbagliata = -1

calcoli e spiegazioni non sono richiesti

- $z = 2 + i3$, $w = 2 + i \Rightarrow Re(z \cdot w) =$ $\boxed{1}$

- Sia $z = -3\sqrt{3} + i3$. Scrivere z nella rappresentazione trigonometrica $z = \varrho \cdot e^{i\vartheta}$: $z =$ $\boxed{6 \cdot e^{i\frac{5}{6}\pi}}$

- Dati W e Z i seguenti sottospazi di $\mathbb{R}^3$:

$$W = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \ : \ x_1 + x_2 - 4x_3 = 0 \} , \ Z = \langle \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \rangle . \quad \text{Allora} \quad \mathbb{R}^3 = W + Z$$

~~vero~~	falso

- Determinare una base di $W \cap Z$

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}}$$

- $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 3 & 0 & 0 & 3 & 6 & 6 \\ 0 & 3 & 0 & 0 & 3 & 6 & 6 \end{pmatrix} \Longrightarrow \dim(Ker(\mathcal{L}_A)) =$ $\boxed{5}$ $\mathrm{rg}(A) =$ $\boxed{2}$

- $\det \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & -3 & 1 & 3 \end{pmatrix} =$ $\boxed{2}$

- $A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 4 \end{pmatrix} \Longrightarrow m.g.(4) =$ $\boxed{1}$

- $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \Longrightarrow A \cdot B =$ $\begin{pmatrix} 1 & 5 \\ -4 & 1 \end{pmatrix}$

- Sia $f : \mathbb{R}^2 \to \mathbb{R}^2$ lineare. Sapendo che $f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $f \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, allora $f \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

21/1/2023

(1) $$\begin{cases} z^2 = i\,|z|\cdot \bar{z} \\ |z+2i| = |z| \end{cases}$$

(i) $z = \rho \cdot e^{i\vartheta} \longrightarrow$ $z^2 = \rho^2 \cdot e^{i2\vartheta}$

$|z| = \rho$

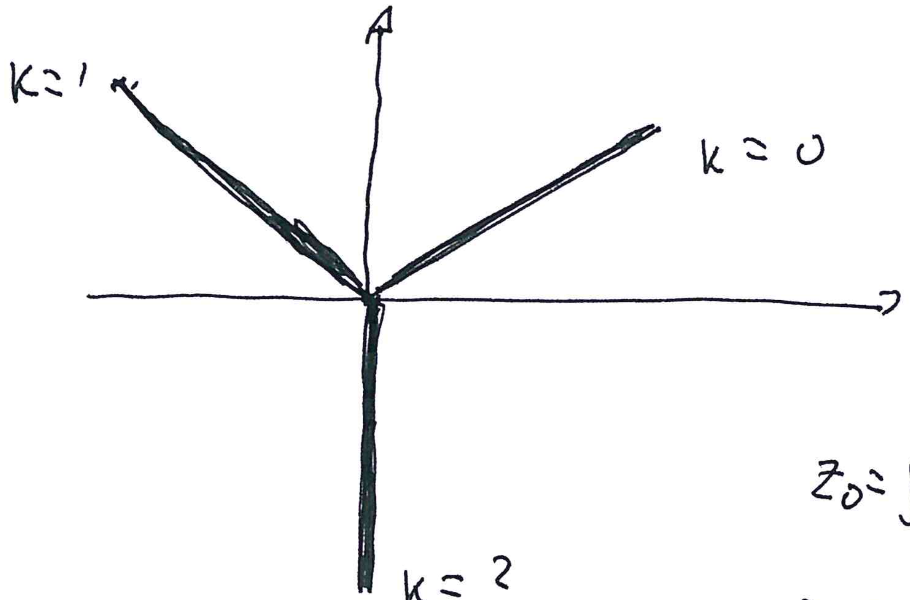
$\bar{z} = \rho \cdot e^{-i\vartheta}$

$i = 1 \cdot e^{i\frac{\pi}{2}}$

Quindi $z^2 = i\,|z|\,\bar{z} \iff \rho^2 \cdot e^{i2\vartheta} = e^{i\frac{\pi}{2}} \cdot \rho \cdot \rho \cdot e^{-i\vartheta}$

$$\iff \begin{cases} \rho^2 = \rho^2 \quad , \rho \in \mathbb{R}^+ \\ 2\vartheta = \frac{\pi}{2} - \vartheta + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

sol. distinte : $\begin{cases} \forall \rho \in \mathbb{R}^+ \\ \vartheta = \frac{\pi}{6} + \frac{2k\pi}{3} \qquad k = 0, 1, 2 \end{cases}$



$z_0 = \rho \cdot \left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \right)$

$z_1 = \rho \left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \right)$

$z_2 = \rho \cdot (-i)$

(ii) $z = x + iy$

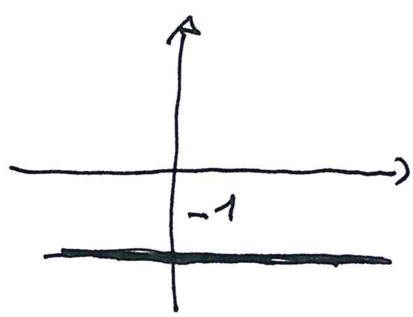
$z + 2i = x + i(y+2)$

$|z+2i| = |z| \iff \sqrt{x^2 + (y+2)^2} = \sqrt{x^2 + y^2}$

$\iff x^2 + (y+2)^2 = x^2 + y^2$

$\iff 4y + 4 = 0$

$\iff \begin{cases} x \text{ qualsiasi} \\ y = -1 \end{cases}$



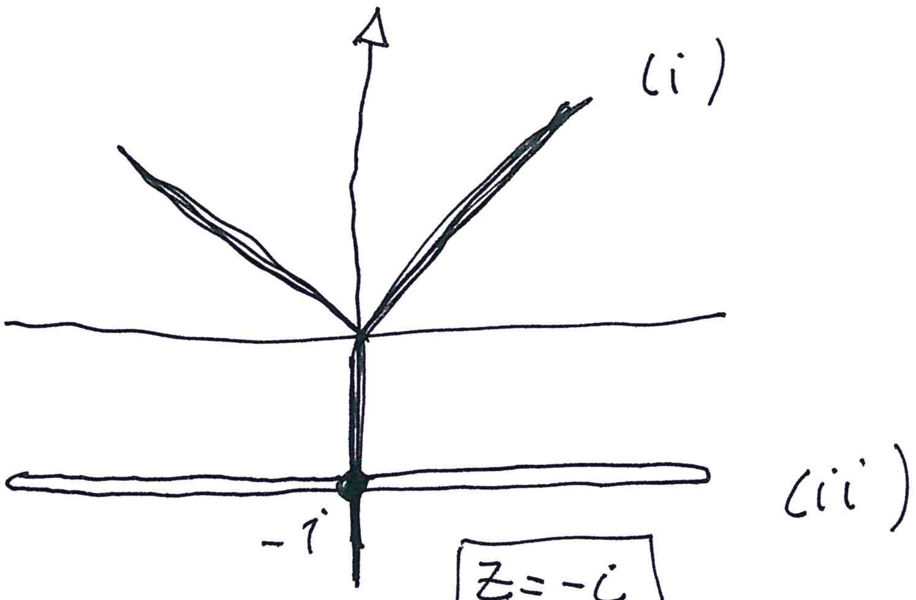
Sol. sisteme

Tra le sol. di (i) cerco

(ρ, ϑ) t.c. $\rho \cdot \sin\vartheta = -1$

$$\Rightarrow \quad \begin{cases} \rho = 1 \\ \vartheta = \frac{3}{2}\pi \end{cases} \quad \Leftrightarrow \quad z = -i$$

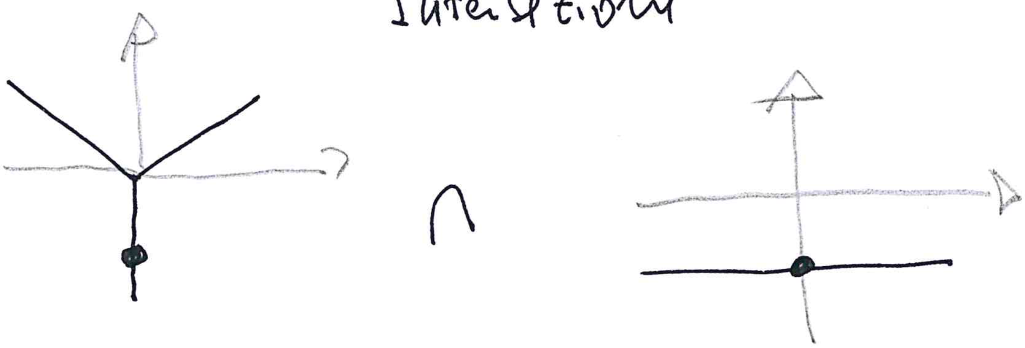
SOLUZIONE GRAFICA:



$\boxed{z = -i}$

$\Updownarrow$

Intersezione



(2) $A_t = \begin{pmatrix} t & 0 & -2 \\ 3 & t & 1 \\ 0 & 1 & 1 \end{pmatrix}$ (4)

i) $\det(A_t) = t^2 - t - 6$

$\det = 0 \iff t = -2, 3$

$t \neq -2, 3$ $\begin{cases} rk = 3 \\ \dim \ker = 3 - 3 = 0 \end{cases}$

$t = -2, 3$ $M = \begin{pmatrix} 3 & t \\ 0 & 1 \end{pmatrix}$ ha $\det \neq 0$

$\Rightarrow \begin{cases} rk = 2 \\ \dim \ker = 3 - 2 = 1 \end{cases}$

ii) $A_t \cdot X = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

$t \neq -2, 3$: $rk(A_t) = 3$

$(A_t \vdots b)$ matrice 3×4 $\Rightarrow rk \leq 3$

$rk(A) = 3 \leq rk(A_t \vdots b) \leq 3$

$\Rightarrow$ $rk(A) = rk(A_t \vdots b) = 3$

$\Leftrightarrow$ $\exists!$ SOL.

$t = -2$: $rk(A) = 2$

$$(A \vdots b) = \begin{pmatrix} -2 & 0 & -2 & 0 \\ 3 & -1 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

elimino colonna 1 : $M = \begin{pmatrix} 0 & -2 & 0 \\ -1 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

det $\neq 0$

$\Rightarrow rk(A \vdots b) = 3 > 2 = rk(A)$

non $\exists$ SOL.

$t=3$: $\quad rk(A) = 2$

$$(A \vdots b) = \begin{pmatrix} 3 & 0 & -2 & 0 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

elimino colonna 1: $\quad M = \begin{pmatrix} 0 & -2 & 0 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

$\det \neq 0$

$\Rightarrow rk(A \vdots b) = 3 > 2 = rk(A)$

non $\exists$ SOL.

(iii) $\quad t=3 \quad rk(A) = 2$

$\Leftrightarrow \dim \operatorname{Im}(\mathcal{L}_A) = 2$

colonna 1, colonna 2 sono lin. IND.

$\Rightarrow$ BASE $\operatorname{Im}(\mathcal{L}_A) = \left\{ \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

Eq. intrinseche di $\operatorname{Im}(\mathcal{L}_A)$ è data da $3-2=1$ equazioni

cioè: $\quad a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$

Imponiamo "passaggio" per

$$v_1 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{cases} 3a_1 + 3a_2 = 0 \\ 3a_2 + a_3 = 0 \end{cases}$$

$a_2 = t$

$a_3 = -3t$

$a_1 = -t$

$t = -1$: un'equazione è

$$x_1 - x_2 + 3x_3 = 0$$

3 $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$Im(f) = \left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle$$

$$Ker(f) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

A matrice 3×3

$rk(A) = 1$

tutte le colonne di A sono multiple di $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} \alpha & \beta & \gamma \\ -\alpha & -\beta & -\gamma \\ 2\alpha & 2\beta & 2\gamma \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \text{colonna 2 di } A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

CIOE' $A = \begin{pmatrix} \alpha & 0 & \gamma \\ -\alpha & 0 & -\gamma \\ 2\alpha & 0 & 2\gamma \end{pmatrix}$

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha + \gamma = 0 \\ -\alpha - \gamma = 0 \\ 2\alpha + 2\gamma = 0 \end{cases}$$

CIOE' $\alpha = -\gamma$

Posto $\alpha = 1$
una matrice è

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

4

$$A = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

$$P_A(\lambda) = \dots = \lambda^2 \cdot (\lambda - 2)^2$$

RADICI: 0 molt $= 2$
2 molt $= 2$

$rk(A) = 2 \Rightarrow m.g.(0) = 4 - 2 = 2$

$rk(A - 2\,Id) = 2 \Rightarrow m.g.(2) = 4 - 2 = 2$

AUTOVALORI: 0 $m.a. = 2$ $m.g. = 2$
2 $m.a. = 2$ $m.g. = 2$

iii) Tutte le radici di $P_A(\lambda) \in \mathbb{R}$

$\Rightarrow$ A è triang.le

$\forall \lambda_i$ m.a.$(\lambda_i) =$ m.g.(λ_i)
& A è triang.le

$\Rightarrow$ A è diag.le

ii) AUTOSPAZI

$$V_0 = \left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$V_2 = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$