Nonlinear Wave Equation

May 20, 2018

1 Introduction

In this Chapter we consider the nonlinear (so called cubic defocusing) wave equation

$$\begin{cases} \partial_t^2 u &= \Delta u - u^3 \\ u|_{t=0} &= u_0 \\ \partial_t u|_{t=0} &= v_0 \end{cases}$$

on the three-dimensional torus

$$x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3.$$

We may rewirte it as a first order system:

$$\begin{cases} \partial_t u &= v \\ \partial_t v &= \Delta u - u^3 \\ u|_{t=0} &= u_0 \\ v|_{t=0} &= v_0 \end{cases}$$

or in vector form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -u^3 \end{pmatrix}$$

with the initial condition $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

1.1 Conserved quantity

The next proposition describes a relevant conserved quantity. We have used the notation $-\langle \Delta u, u \rangle$ for the most common one $\left\| (-\Delta)^{1/2} u \right\|_{L^2}^2$ since we prefer to postpone the description of the operator $(-\Delta)^{1/2}$. Notice that $-\langle \Delta u, u \rangle$ is a non-negative quantity.

Proposition 1 *Assume* $\begin{pmatrix} u \\ v \end{pmatrix}$ *is a smooth solution. Then*

$$\frac{d}{dt}\left(-\frac{1}{2}\left\langle \Delta u, u\right\rangle + \frac{1}{2}\left\| v\right\|_{L^2}^2 + \int_{\mathbb{T}^3}\frac{u^4}{4}dx\right) = 0.$$

Proof. It is sufficient to notice that

$$-\frac{1}{2}\frac{d}{dt}\left\langle \Delta u, u\right\rangle = -\left\langle \Delta u, v\right\rangle$$

$$\frac{1}{2}\frac{d}{dt}\left\| v\right\|_{L^2}^2 = \left\langle v, \Delta u - u^3\right\rangle$$

$$\frac{d}{dt}\int_{\mathbb{T}^3}\frac{u^4}{4}dx = \int_{\mathbb{T}^3} u^3 v dx = \left\langle v, u^3\right\rangle$$

and then sum these identities. ■

1.2 The linear operator

Preliminarily, recall that the Laplacian operator on $\mathbb{T}^3$ may be described in Fourier variables as

$$\widehat{(-\Delta f)}(k) := |k|^{2\alpha}\,\widehat{f}(k), \qquad k \in \mathbb{Z}^3.$$

Based on this, one can define the fractional powers

$$(\widehat{(-\Delta)^{\alpha} f})(k) := |k|^{2\alpha}\,\widehat{f}(k)$$

where we may take $k \in \mathbb{Z}^3$ for $\alpha \geq 0$, but we have to exclude $k = 0$ for $\alpha < 0$, hence working on zero-mean functions. Moreover, as a particular case of the so called functional calculus for operators, we may define the operators

$$\sin\left((-\Delta)^{1/2} t\right), \qquad \cos\left((-\Delta)^{1/2} t\right)$$

for every real t as

$$\left(\sin\left(\widehat{(-\Delta)^{1/2}} t\right) f\right)(k) \quad : \quad = \sin(|k|\, t)\,\widehat{f}(k)$$
$$\left(\cos\left(\widehat{(-\Delta)^{1/2}} t\right) f\right)(k) \quad : \quad = \cos(|k|\, t)\,\widehat{f}(k).$$

For every $s \in \mathbb{R}$, consider the product space

$$\mathcal{H}^s = H^s \times H^{s-1}$$

the linear unbounded operator

$$\begin{aligned}\mathcal{D}(A_s) &= \mathcal{H}^{s+1} \\ A_s &: \mathcal{D}(A_s) \subset \mathcal{H}^s \to \mathcal{H}^s \\ A_s &= \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\end{aligned}$$

and the family of linear operators

$$e^{tA_s} : \mathcal{H}^s \to \mathcal{H}^s$$

given by

$$e^{tA_s}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \cos\left((-\Delta)^{1/2} t\right) u_0 + (-\Delta)^{-1/2} \sin\left((-\Delta)^{1/2} t\right) v_0 \\ -(-\Delta)^{1/2} \sin\left((-\Delta)^{1/2} t\right) u_0 + \cos\left((-\Delta)^{1/2} t\right) v_0 \end{pmatrix}.$$

Lemma 2 *The operators e^{tA_s} are bounded in $\mathcal{H}^s$, are a group ($e^{(t+\tau)A_s} = e^{tA_s}e^{\tau A_s}$ for every real t, τ, $e^{0A_s} = Id$) and are strongy continuous, with infinitesimal generator $A_s : \mathcal{D}(A_s) \subset \mathcal{H}^s \to \mathcal{H}^s$, hence*

$$\frac{d}{dt} e^{tA_s}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = A_s e^{tA_s}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

for every $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{H}^s$.

Proof. Full details are not difficult but we limit ourselves to check that e^{tA_s} are bounded in $\mathcal{H}^s$ and the differential equation holds. Boundedness holds because $\sin\left((-\Delta)^{1/2} t\right)$ and $\cos\left((-\Delta)^{1/2} t\right)$ are bounded operators in each H^s (easy to check), commute with $(-\Delta)^{\alpha}$, hence $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{H}^s$ implies $\begin{pmatrix} u_0 \\ (-\Delta)^{-1/2} v_0 \end{pmatrix} \in H^s \times H^s$, $\begin{pmatrix} (-\Delta)^{1/2} u_0 \\ v_0 \end{pmatrix} \in H^{s-1} \times H^{s-1}$. To show that the differential equation holds, notice that

$$\begin{aligned}\partial_t v &= \Delta \cos\left((-\Delta)^{1/2} t\right) u_0 - (-\Delta)^{1/2} \sin\left((-\Delta)^{1/2} t\right) v_0 \\ &= \Delta \cos\left((-\Delta)^{1/2} t\right) u_0 - (-\Delta)^{1/2} (-\Delta)^{1/2} (-\Delta)^{-1/2} \sin\left((-\Delta)^{1/2} t\right) v_0 \\ &= \Delta\left(\cos\left((-\Delta)^{1/2} t\right) u_0 + (-\Delta)^{-1/2} \sin\left((-\Delta)^{1/2} t\right) v_0\right) = \Delta u.\end{aligned}$$

We have used natural formulae for derivatives of sin and cos which easily follow from the Fourier representation. ■

1.3 Mild formula and local solution in $\mathcal{H}^1$

Having the semigroup associated to the linear homogeneous part, we may rewrite the nonlinear wave equation in the form (also called Duhamel formula)

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ -u^3(s) \end{pmatrix} ds.$$

For obvious reasons we do not write the subscript s anymore; the space $\mathcal{H}^s$ will be clear each time. In this section we work in $\mathcal{H}^1$.

Proposition 3 *Given* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{H}^1$*, there exists* $T > 0$ *and a unique solution* $\begin{pmatrix} u \\ v \end{pmatrix}$ *in* $C([0,T];\mathcal{H}^1)$*.*

Proof. For every $T > 0$, onsider the map $\Gamma_T : C([0,T];\mathcal{H}^1) \to C([0,T];\mathcal{H}^1)$ defined as

$$\left(\Gamma_T \begin{pmatrix} u \\ v \end{pmatrix}\right)(t) = e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ -u^3(s) \end{pmatrix} ds.$$

Let us check that this map is well defined between these spaces. The function $e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ belongs to $C([0,T];\mathcal{H}^1)$. Since $u \in C([0,T];H^1)$ and Sobolev embedding theorem states that

$$H^1 \subset L^6$$

(recall that $W^{s,p} \subset L^q$ for $\frac{1}{q} = \frac{1}{p} - \frac{s}{3}$), we have $u \in C([0,T];L^6)$, hence $u^3 \in C([0,T];L^2)$. Therefore

$$\begin{pmatrix} 0 \\ -u^3(\cdot) \end{pmatrix} \in C([0,T];\mathcal{H}^1)$$

and thus the integral in the definition of Γ_T is also in $C([0,T];\mathcal{H}^1)$. Precisely, we have

$$\begin{aligned} \left\| \begin{pmatrix} 0 \\ u^3 \end{pmatrix} \right\|_{C([0,T];\mathcal{H}^1)} &= \left\|u^3\right\|_{C([0,T];L^2)} = \|u\|^3_{C([0,T];L^6)} \leq C \|u\|^3_{C([0,T];H^1)} \\ &\leq C \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^3_{C([0,T];\mathcal{H}^1)}. \end{aligned}$$

Similar arguments lead to the estimates

$$\begin{aligned} \left\| \Gamma_T \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{C([0,T];\mathcal{H}^1)} &\leq C_T \left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_{\mathcal{H}^1} + TC_T \left\| \begin{pmatrix} 0 \\ u^3 \end{pmatrix} \right\|_{C([0,T];\mathcal{H}^1)} \\ &\leq C_T \left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_{\mathcal{H}^1} + TC_T \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^3_{C([0,T];\mathcal{H}^1)} \end{aligned}$$

$$
\begin{aligned}
&\left\| \Gamma_T \begin{pmatrix} u \\ v \end{pmatrix} - \Gamma_T \begin{pmatrix} u' \\ v' \end{pmatrix} \right\|_{C([0,T];\mathcal{H}^1)} \leq T C_T \left\| \begin{pmatrix} 0 \\ u^3 \end{pmatrix} - \begin{pmatrix} 0 \\ (u')^3 \end{pmatrix} \right\|_{C([0,T];\mathcal{H}^1)} \\
&\quad \leq T C'_T \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u' \\ v' \end{pmatrix} \right\|_{C([0,T];\mathcal{H}^1)} \left(\|u\|^2_{C([0,T];H^1)} + \|u'\|^2_{C([0,T];H^1)} \right)
\end{aligned}
$$

where the dependence on T in the constants C_T, C'_T come from the estimate of $\sup_{t\in[0,T]} \left\| e^{tA} \right\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^1)}$ and where we have used the inequality

$$
\begin{aligned}
\left\| u^3 - (u')^3 \right\|^2_{L^2} &= \int_{\mathbb{T}^3} \left(u^3(x) - (u')^3(x) \right)^2 dx \\
&= \int_{\mathbb{T}^3} (u - u')^2 \left(u^2 + u'u + (u')^2 \right)^2 dx \\
&\leq \left(\int_{\mathbb{T}^3} (u - u')^6 \, dx \right)^{1/3} \left(\int_{\mathbb{T}^3} \left(u^2 + u'u + (u')^2 \right)^3 dx \right)^{2/3} \\
&\leq C \left\| u - u' \right\|^2_{H^1} \left(\|u\|^4_{H^1} + \|u'\|^4_{H^1} \right).
\end{aligned}
$$

Taken $T_0 > 0$ and considering values $T \in [0, T_0]$, one has to restrict the action of Γ_T to a suitable ball of radius larger than $C_{T_0} \left\| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\|_{\mathcal{H}^1}$, where it is a contraction, both properties achievable by choosing T small enough; details of such argument are given in the proof of Proposition 6, which is more complete since more relevant for our purposes. ■

1.4 Global solution in $\mathcal{H}^1$, comments on extension to $s \neq 1$

We have seen above that

$$
\frac{d}{dt} \left(\frac{1}{2} \left\| (-\Delta)^{1/2} u \right\|^2_{L^2} + \frac{1}{2} \|v\|^2_{L^2} + \int_{\mathbb{T}^3} \frac{u^4}{4} dx \right) = 0
$$

for smooth solutions. With due care, one can show that the identity (consequence of the previous fact for smooth solutions)

$$
\left\| (-\Delta)^{1/2} u(t) \right\|^2_{L^2} + \|v(t)\|^2_{L^2} + 2 \int_{\mathbb{T}^3} \frac{u(t)^4}{4} dx = \left\| (-\Delta)^{1/2} u_0 \right\|^2_{L^2} + \|v_0\|^2_{L^2} + 2 \int_{\mathbb{T}^3} \frac{u_0^4}{4} dx \quad (1)
$$

holds for solutions $\begin{pmatrix} u \\ v \end{pmatrix}$ of class $C\left([0,T]; \mathcal{H}^1\right)$ (see the remark below). This implies:

Theorem 4 *For every* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{H}^1$, *there exists a unique solution* $\begin{pmatrix} u \\ v \end{pmatrix}$ *in* $C\left([0,T]; \mathcal{H}^1\right)$ *for every* $T > 0$.

The result extends to every space $\mathcal{H}^s$, with $s > 1$. The proof of local existence and uniqueness is the same; we address to [1] for propagation of higher regularity globally in time.

Remark 5 *Using contraction type arguments one can show that if* $\begin{pmatrix} u_0^n \\ v_0^n \end{pmatrix}$ *converges to* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ *in* $\mathcal{H}^1$*, the corresponding solutions converge in* $C\left([0,T]\,;\mathcal{H}^1\right)$*. If* $\begin{pmatrix} u_0^n \\ v_0^n \end{pmatrix}$ *belong to* $\mathcal{H}^s$ *for larger* $s > 1$*, solutions corresponding to* $\begin{pmatrix} u_0^n \\ v_0^n \end{pmatrix}$ *are more regular and for them classical calculus applies to prove the energy identity (1). Then the same identity for the solution, only of class* $C\left([0,T]\,;\mathcal{H}^1\right)$*, corresponding to* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$*, is obtained in the limit as* $n \to \infty$*. In principle this argument works only locally, but may be extended globally thanks to the estimate itself, with usual arguments of maximality that we do not report.*

The problem is now to study the equation in $\mathcal{H}^s$ for $s < 1$. Recall Sobolev embedding:

$$H^1 \subset L^6.$$

If we pretend to evaluate the nonlinearity u^3 in L^2, we need $s = 1$ ($L^2 = H^{s-1}$ for $s = 1$). Evaluating u^3 in H^{s-1} with $s < 1$ amount to look for an estimate of the form

$$\left| \int_{\mathbb{T}^3} u^3 \phi dx \right| \leq C \left\| \phi \right\|_{H^{1-s}}$$

knowing only $u \in H^s$. No matter how regular ϕ could be, we need $u \in L^3$ to have a well defined integral $\int_{\mathbb{T}^3} u^3 \phi dx$. Then, by Sobolev embedding $H^{\frac{1}{2}} \subset L^3$, we need

$$s \geq \frac{1}{2}.$$

This is not a proof that the theory can be extended to $s \geq \frac{1}{2}$. It is just a strong indication that without $s \geq \frac{1}{2}$ there is no hope, by the approach above (unless a new idea is developed, as described below).

It turns out that the thresold $s \geq \frac{1}{2}$ is a correct one: using the non-trivial tool of Strichartz estimates, one can prove local existence and uniqueness (and sometimes global solutions) for all $s \geq \frac{1}{2}$. A part from the nontrivial tool of Strichartz estimates that we shall not describe, the new idea is the same used to apply a probabilistic proof for all $s > 0$, hence we start explaining this idea.

2 Decomposition of solution

2.1 Introduction

The idea is particularly interesting today since it has something in common with the recent theories of Regularity Structures and Paracontrolled Distributions. One looks for a decomposition of the solution

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix} + \begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix}$$

(or more terms in more difficult problems) where $\begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix}$ contains the most singular part of the solution, most singular from a certain viewpont, and $\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix}$ is more regular and solves less difficult equation. In the cases when Probability enters the game, usually Probability is used to give a meaning to $\begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix}$; the regular part $\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix}$ is found by deterministic arguments.

In the framework of SPDEs this idea is very old, used in very first papers in the following wasy: $\begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix}$ is the noise (in the simplest form of the method) or the solution of the linear problem driven by noise, an Ornstein-Uhlenbeck process. Defining $\begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix}$ requires stochastic analysis. Then $\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix}$ solves a deterministic PDE depending on a random parameter, usually a PDE similar in structure to the original one, so that classical tools apply. This idea was not considered so important at that time because usually there were other strategies, more probabilistic, to solve the same problems; it was used mainly either to solve for the first time a new problem, or to solve it pathwise, useful for instance in the investigations of random dynamical systems. Only with the papers of Da Prato and Debussche and of Blomker and Romito this idea became a cornerstone to solve very difficult problems. And later on was extended enormously in the framework of Regularity Structures and Paracontrolled Distributions.

2.2 Decomposition for the wave equation

In the case of the nonlinear wave equation, both in the deterministic and probabilistic approach, $\begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix}$ is the solution of the homogeneous linear problem with the initial conditions:

$$\begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix} = e^{tA} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix} := \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} \overline{u}(t) \\ \overline{v}(t) \end{pmatrix}$$

solves

$$\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix} = \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ -[\widetilde{u}(s) + \overline{u}(s)]^3 \end{pmatrix} ds. \tag{2}$$

Where is the window for a gain, using such a simple idea? That the property required on $\overline{u}(s)$ to deal efficiently with the term $[\widetilde{u}(s) + \overline{u}(s)]^3$ are different from properties of the form $\overline{u} \in C([0,T]; H^s)$.

2.3 Condition on $\overline{u}$ to apply contraction principle

Proposition 6 *Assume that, for some $T_0 > 0$,*

$$\int_0^{T_0} \|\overline{u}(s)\|_{L^6}^3 \, ds < \infty.$$

Then equation (2) has a unique local solution in $C([0,T]; \mathcal{H}^1)$, for some $T \in (0, T_0]$.

As a consequence, the wave equation has a unique local solution in

$$L^3(0,T; L^6) \oplus C([0,T]; \mathcal{H}^1)$$

or in any other space $[Y \cap L^3(0,T; L^6)] \oplus C([0,T]; \mathcal{H}^1)$ such that $\overline{u} \in Y$.

Proof. Step 1. Set $\theta_{T_0}(\overline{u}) := \int_0^{T_0} \|\overline{u}(s)\|_{L^6}^3 \, ds < \infty$ and work for some $T \in (0, T_0]$. We introduce the map

$$\widetilde{\Gamma}_T \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}(t) = \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ -[\widetilde{u}(s) + \overline{u}(s)]^3 \end{pmatrix} ds.$$

Let us argue as in the proof of Proposition 3. If $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \in C([0,T]; \mathcal{H}^1)$, by Sobolev embedding theorem $H^1 \subset L^6$ we have $\widetilde{u} \in C([0,T]; L^6)$, hence $\widetilde{u}^3 \in C([0,T]; L^2)$. Then $\widetilde{\Gamma}_T \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \in C([0,T]; \mathcal{H}^1)$. We use several times the inequality

$$\left|(a+b)^3\right| \leq C\left(|a|^3 + |b|^3\right).$$

Moreover,

$$\begin{aligned} \left\|\widetilde{\Gamma}_T \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)} &\leq C_{T_0} \int_0^T \left\|\begin{pmatrix} 0 \\ -[\widetilde{u}(s) + \overline{u}(s)]^3 \end{pmatrix}\right\|_{\mathcal{H}^1} ds \\ &\leq TC_{T_0} \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}^3 + \theta_{T_0}(\overline{u}) \end{aligned}$$

$$\left\|\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}-\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}'\\ \widetilde{v}'\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)} \leq C_{T_0}\int_0^T\left\|\begin{pmatrix}0\\ [\widetilde{u}(s)+\overline{u}(s)]^3-[\widetilde{u}'(s)+\overline{u}(s)]^3\end{pmatrix}\right\|_{\mathcal{H}^1} ds$$

$$= C_{T_0}\int_0^T\left\|[\widetilde{u}(s)+\overline{u}(s)]^3-\left[\widetilde{u}'(s)+\overline{u}(s)\right]^3\right\|_{L^2} ds$$

$$\leq C'_{T_0}\left\|\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}-\begin{pmatrix}\widetilde{u}'\\ \widetilde{v}'\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}\left(T\|u\|^2_{C([0,T];H^1)}+T\|u'\|^2_{C([0,T];H^1)}+T^{1/3}\theta_{T_0}(\overline{u})\right)$$

where we have used the inequality

$$\begin{aligned}
\left\|[\widetilde{u}+\overline{u}]^3-\left[\widetilde{u}'+\overline{u}\right]^3\right\|^2_{L^2} &= \int_{\mathbb{T}^3}\left([\widetilde{u}+\overline{u}]^3-\left[\widetilde{u}'+\overline{u}\right]^3\right)^2 dx\\
&= \int_{\mathbb{T}^3}\left(\widetilde{u}-\widetilde{u}'\right)^2\left([\widetilde{u}+\overline{u}]^2+[\widetilde{u}+\overline{u}]\left[\widetilde{u}'+\overline{u}\right]+\left[\widetilde{u}'+\overline{u}\right]^2\right)^2 dx\\
&\leq \left(\int_{\mathbb{T}^3}\left(\widetilde{u}-\widetilde{u}'\right)^6 dx\right)^{1/3}\left(\int_{\mathbb{T}^3}\left([\widetilde{u}+\overline{u}]^2+[\widetilde{u}+\overline{u}]\left[\widetilde{u}'+\overline{u}\right]+\left[\widetilde{u}'+\overline{u}\right]^2\right)^3 dx\right)^{2/3}\\
&\leq C\left\|\widetilde{u}-\widetilde{u}'\right\|^2_{H^1}\left(\|\widetilde{u}\|^4_{H^1}+\left\|\widetilde{u}'\right\|^4_{H^1}+\|\overline{u}\|^4_{L^6}\right)
\end{aligned}$$

hence

$$\begin{aligned}
&\int_0^T\left\|[\widetilde{u}(s)+\overline{u}(s)]^3-\left[\widetilde{u}'(s)+\overline{u}(s)\right]^3\right\|_{L^2} ds\\
\leq\ & C'\int_0^T\left\|\widetilde{u}(s)-\widetilde{u}'(s)\right\|_{H^1}\left(\|\widetilde{u}(s)\|^2_{H^1}+\left\|\widetilde{u}'(s)\right\|^2_{H^1}+\|\overline{u}(s)\|^2_{L^6}\right) ds\\
\leq\ & C'\left\|\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}-\begin{pmatrix}\widetilde{u}'\\ \widetilde{v}'\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}\left(T\left(\|\widetilde{u}\|^2_{C([0,T];H^1)}+\left\|\widetilde{u}'\right\|^2_{C([0,T];H^1)}\right)+\int_0^T\|\overline{u}(s)\|^2_{L^6} ds\right).
\end{aligned}$$

Step 2. Chosen $R_0>\theta_{T_0}(\overline{u})$, let B_{T,R_0} be the closed ball in $C\left([0,T];\mathcal{H}^1\right)$ of center zero and radius R_0. Choose T such that

$$TC_{T_0}R_0^3+\theta_{T_0}(\overline{u})\leq R_0.$$

Then, if $\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}\in B_{T,R_0}$ we have

$$\left\|\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}\leq TC_{T_0}R_0^3+\theta_{T_0}(\overline{u})\leq R_0$$

namely $\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}\in$ is in B_{T,R_0}. The ball B_{T,R_0} is a complete metric space invariant under $\widetilde{\Gamma}_T$. In it, we have

$$\left\|\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}-\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}'\\ \widetilde{v}'\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}\leq C'_{T_0}\left(2TR_0^2+T^{1/3}\theta_{T_0}(\overline{u})\right)\left\|\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}-\begin{pmatrix}\widetilde{u}'\\ \widetilde{v}'\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}.$$

If we reduce T, in case it is necessary, in order to satisfy also the inequality

$$C'_{T_0}\left(2TR_0^2+T^{1/3}\theta_{T_0}\left(\overline{u}\right)\right)\leq\frac{1}{2}$$

then $\widetilde{\Gamma}_T$ is a contraction in B_{T,R_0}. Local existence and uniqueness follows, on the local time interval $[0,T]$. ■

Next proposition claims that an a priori bound on $\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}$ implies global solution.

Proposition 7 *Under the same assumptions, assume there is a constant $C_0>0$ such that any solution $\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}$ defined on some interval $[0,T']\subset[0,T]$ satisfies*

$$\sup_{t\in[0,T']}\left\|\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}\right\|_{\mathcal{H}^1}\leq C_0.$$

Then the solution is global in time (we may take $T'=T$).

Proof. Consider the modified problem

$$\widetilde{\Gamma}_{t_0,T}\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}(t)=e^{(t-t_0)A}\begin{pmatrix}\widetilde{u}_{t_0}\\ \widetilde{v}_{t_0}\end{pmatrix}+\int_{t_0}^{t}e^{(t-s)A}\begin{pmatrix}0\\ -\left[\widetilde{u}\left(s\right)+\overline{u}\left(s\right)\right]^3\end{pmatrix}ds.$$

It corresponds to the same equation but considered on the time interval $[t_0,t_0+T]$ with initial condition at time t_0 given by $\begin{pmatrix}\widetilde{u}_{t_0}\\ \widetilde{v}_{t_0}\end{pmatrix}$. Called c_{T_0} a constant bounding $\left\|e^{tA}\right\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^1)}$ on $[0,T_0]$, if $\begin{pmatrix}\widetilde{u}_{t_0}\\ \widetilde{v}_{t_0}\end{pmatrix}$ in the previous indentity is the value at time t_0 of a solution, taking into account the new assumption we have

$$\left\|e^{(t-t_0)A}\begin{pmatrix}\widetilde{u}_{t_0}\\ \widetilde{v}_{t_0}\end{pmatrix}\right\|_{\mathcal{H}^1}\leq c_{T_0}C_0.$$

The first main estimate of the previous proof modifies as

$$\left\|\widetilde{\Gamma}_T\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}\right\|_{C([0,T];\mathcal{H}^1)}\leq c_{T_0}C_0+TC_{T_0}R_0^3+\theta_{T_0}\left(\overline{u}\right).$$

Hence choose

$$R_0>c_{T_0}C_0+\theta_{T_0}\left(\overline{u}\right)$$

and T such that

$$c_{T_0}C_0+TC_{T_0}R_0^3+\theta_{T_0}\left(\overline{u}\right)\leq R_0$$

and the contraction property of the previous proof (not affected by the new term $e^{(t-t_0)A}\begin{pmatrix} \widetilde{u}_{t_0} \\ \widetilde{v}_{t_0} \end{pmatrix}$) hold.

Made these choices, $\widetilde{\Gamma}_{t_0,T}$ has a fixed point in $C\left([t_0,(t_0+T)\wedge T_0];\mathcal{H}^1\right)$. The choice of T is independent of t_0 and $\begin{pmatrix} \widetilde{u}_{t_0} \\ \widetilde{v}_{t_0} \end{pmatrix}$, hence the equation for $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$ can be solved first on $[0,T]$, then on $[T,2T\wedge T_0]$, and so on until $[0,T_0]$ is covered. ■

2.4 A result based on deterministic Strichartz estimates

We state without proof the following deep result.

Theorem 8 *Let (p,q) be such that*

$$\begin{aligned} 2 &< p \leq \infty \\ \frac{1}{p}+\frac{1}{q} &= \frac{1}{2}. \end{aligned}$$

Then

$$\|\overline{u}\|_{L^p(0,1;L^q)} \leq C\left(\|u_0\|_{H^{\frac{2}{p}}} + \|v_0\|_{H^{\frac{2}{p}-1}}\right).$$

For $p=3$, $q=6$ we get

$$\|\overline{u}\|_{L^3(0,1;L^6)} \leq C\left(\|u_0\|_{H^{\frac{2}{3}}} + \|v_0\|_{H^{\frac{2}{3}-1}}\right)$$

and therefore, based on Proposition 6 we have:

Corollary 9 *For every $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathcal{H}^{\frac{2}{3}}$, there exists has a unique local solution of the wave equation in the space*

$$\left[C\left([0,T];\mathcal{H}^{\frac{2}{3}}\right) \cap L^3\left(0,T;L^6\right)\right] \oplus C\left([0,T];\mathcal{H}^1\right).$$

With more refined arguments, always based on Strichartz tye estimates, one has local solutions for initial conditions in all $\mathcal{H}^s$ with $\frac{1}{2} \leq s < 1$.

3 Probabilistic results

3.1 Probabilistic Strichartz estimates

Recall that

$$\overline{u}(t) = \cos\left((-\Delta)^{1/2} t\right) u_0 + (-\Delta)^{-1/2} \sin\left((-\Delta)^{1/2} t\right) v_0.$$

Assume now that (u_0, v_0) is a Gaussian vector in $\mathcal{H}^s$. Then $\overline{u}(t)$ is Gaussian; certainly it is a Gaussian r.v. with values in $C([0,T]; H^s)$. Can we say more? Yes, following the quite general idea that Gaussianity improves L^2 regularity.

The Gaussian random field $\overline{u}(t,x)$, a priori has paths only of class $C\left([0,T]; L^2\right)$ (also $C([0,T]; H^s)$). Let us prove, under suitable assumptions, that it has a continuous version.

Theorem 10 *Given $s > 0$, assume that (u_0, v_0) is a mean zero centered Gaussian vector in $\mathcal{H}^s$ with the following properties:*

i) the r.v.'s $\left\{\widehat{u_0}(k), \widehat{v_0}(k), k \in \mathbb{Z}^3, k \neq 0\right\}$ are independent

ii) $\sum_k |k|^{2s} \mathbb{E}\left[|\widehat{u_0}(k)|^2\right] < \infty$, $\sum_k |k|^{2s+2} \mathbb{E}\left[|\widehat{v_0}(k)|^2\right] < \infty$.

Then $\overline{u}$ has a continuous version.

Proof. Let us estimate only the term

$$\overline{u}_1(t) := \cos\left((-\Delta)^{1/2} t\right) u_0 = \sum_k e^{2\pi i k\cdot x} \cos(|k|\,t)\, \widehat{u_0}(k).$$

The estimates for the other term are the same.

Step 1 (space regularity). We have

$$\begin{aligned}
\mathbb{E}\left[|\overline{u}_1(t,x) - \overline{u}_1(t,y)|^2\right] &= \mathbb{E}\left[\left|\sum_k \left(e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right)\cos(|k|\,t)\,\widehat{u_0}(k)\right|^2\right] \\
&= \sum_{k,h} \left(e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right)\overline{\left(e^{2\pi i h\cdot x} - e^{2\pi i h\cdot y}\right)}\cos(|k|\,t)\cos(|h|\,t)\,\mathbb{E}\left[\widehat{u_0}(k)\,\overline{\widehat{u_0}(h)}\right] \\
&= \sum_k \left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right|^2 \cos(|k|\,t)^2\, \mathbb{E}\left[|\widehat{u_0}(k)|^2\right] \\
&\le \sum_k \left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right|^2 \mathbb{E}\left[|\widehat{u_0}(k)|^2\right].
\end{aligned}$$

Now, since

$$\begin{aligned}
\left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right| &\le C\,|k|\,|x-y| \\
\left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right| &\le C
\end{aligned}$$

we get

$$\left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right| = \left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right|^{2s}\left|e^{2\pi i k\cdot x} - e^{2\pi i k\cdot y}\right|^{1-2s} \le C\,|k|^{2s}\,|x-y|^{2s}.$$

Hence

$$\begin{aligned}\mathbb{E}\left[\left|\overline{u}_1(t,x)-\overline{u}_1(t,y)\right|^2\right] &\leq C\left|x-y\right|^{2s}\sum_k |k|^{2s}\,\mathbb{E}\left[\left|\widehat{u_0}(k)\right|^2\right]\\ &\leq C\left|x-y\right|^{2s}.\end{aligned}$$

Step 2 (time regularity). Similarly,

$$\begin{aligned}\mathbb{E}\left[\left|\overline{u}_1(t,y)-\overline{u}_1(s,y)\right|^2\right] &= \mathbb{E}\left[\left|\sum_k e^{2\pi ik\cdot y}\left(\cos(|k|t)-\cos(|k|s)\right)\widehat{u_0}(k)\right|^2\right]\\ &= \sum_{k,h} e^{2\pi ik\cdot y}\overline{e^{2\pi ih\cdot y}}\left(\cos(|k|t)-\cos(|k|s)\right)\left(\cos(|h|t)-\cos(|h|s)\right)\mathbb{E}\left[\widehat{u_0}(k)\overline{\widehat{u_0}(h)}\right]\\ &= \sum_k \left|e^{2\pi ik\cdot y}\right|^2\left(\cos(|k|t)-\cos(|k|s)\right)^2\mathbb{E}\left[\left|\widehat{u_0}(k)\right|^2\right]\\ &\leq \sum_k \left(\cos(|k|t)-\cos(|k|s)\right)^2\mathbb{E}\left[\left|\widehat{u_0}(k)\right|^2\right].\end{aligned}$$

As above,

$$\left(\cos(|k|t)-\cos(|k|s)\right)^2 \leq C\left|k\right|^{2s}\left|t-s\right|^{2s}$$

hence we get

$$\mathbb{E}\left[\left|\overline{u}_1(t,y)-\overline{u}_1(s,y)\right|^2\right] \leq C\left|t-s\right|^{2s}.$$

Step 3 (space-time regularity). Finally,

$$\begin{aligned}\mathbb{E}\left[\left|\overline{u}_1(t,x)-\overline{u}_1(s,y)\right|^2\right] &\leq 2\mathbb{E}\left[\left|\overline{u}_1(t,x)-\overline{u}_1(t,y)\right|^2\right]+2\mathbb{E}\left[\left|\overline{u}_1(t,y)-\overline{u}_1(s,y)\right|^2\right]\\ &\leq C\left|x-y\right|^{2s}+C\left|t-s\right|^{2s}.\end{aligned}$$

Using Gaussianity, we get

$$\mathbb{E}\left[\left|\overline{u}_1(t,x)-\overline{u}_1(s,y)\right|^{2p}\right] \leq C_p\left|x-y\right|^{2sp}+C_p\left|t-s\right|^{2sp}$$

for every $p \geq 1$. Hence may apply Kolmogorov criterium with respect to the space-time variables together and find the final result. ■

Remark 11 *Apparently it seems there is a contradiction in the statement: how could it be that (u_0, v_0) is only of class $\mathcal{H}^s$ and $\overline{u}$ is continuous up to time $t = 0$? The reason is simply that (u_0, v_0) itself is much more than $\mathcal{H}^s$, with probability one, being Gaussian. It is not necessarily more in terms of Hilbertian Sobolev spaces, but it is much more when measured with other topologies, like the uniform one.*

Remark 12 *This is a side deterministic remark. Being $s>0$, in $\mathcal{H}^0$ the linear semigroup is a little bit Hölder continuous (this improvement is deterministic, not due to Gaussianity). Indeed, from an interpolation inequality between Sobolev spaces that, on the torus, can be proved by Fourier series in an elementary way (we have used it in the proof of compactness for the 2D Euler equations), we have*

$$\begin{aligned}\left\|e^{tA}z-e^{sA}z\right\|_{\mathcal{H}^0} &\leq \left\|e^{tA}z-e^{sA}z\right\|_{\mathcal{H}^s}^{1-s}\left\|e^{tA}z-e^{sA}z\right\|_{\mathcal{H}^{s-1}}^{s} \\ &\leq C_T\left\|e^{tA}z-e^{sA}z\right\|_{\mathcal{H}^{s-1}}^{s}.\end{aligned}$$

From the mean value theorem

$$\left\|e^{tA}z-e^{sA}z\right\|_{\mathcal{H}^{s-1}}^{s}\leq \sup_{\xi\in[0,T]}\left\|Ae^{\xi A}z\right\|_{\mathcal{H}^{s-1}}^{s}|t-s|^s$$

and, being A bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s-1}$,

$$\begin{aligned}\left\|Ae^{\xi A}z\right\|_{\mathcal{H}^{s-1}} &= \left\|e^{\xi A}Az\right\|_{\mathcal{H}^{s-1}}\leq\left\|e^{\xi A}\right\|_{\mathcal{L}(\mathcal{H}^{s-1},\mathcal{H}^{s-1})}\|Az\|_{\mathcal{H}^{s-1}} \\ &\leq C_T\|z\|_{\mathcal{H}^s}.\end{aligned}$$

Collecting these inequalities, we get

$$\left\|e^{tA}z-e^{sA}z\right\|_{\mathcal{H}^0}\leq C_T\|z\|_{\mathcal{H}^s}|t-s|^s.$$

Therefore

$$\|\overline{u}_1(t)-\overline{u}_1(s)\|_{L^2}\leq C_T\|(u_0,v_0)\|_{\mathcal{H}^s}|t-s|^s.$$

3.2 Local result

Theorem 13 *Given $s>0$, let μ_0 be a centered Gaussian measure on $\mathcal{H}^s$ such that the assumptions of Theorem 10 hold, for a r.v. (u_0,v_0) with law μ_0. Then, for μ_0-a.e. initial condition in $\mathcal{H}^s$, the cubic defocusing nonlinear wave equation has a local solution of class*

$$C\left([0,T]\times\mathbb{T}^3\right)\oplus C\left([0,T];\mathcal{H}^1\right).$$

Proof. From Theorem 10 we know that for μ_0-a.e. initial condition in $\mathcal{H}^s$, one has $\overline{u}$ continuous in space time, hence the assumption of Proposition 6 is fulfilled and we have a local solution in $C\left([0,T]\times\mathbb{T}^3\right)\oplus C\left([0,T];\mathcal{H}^1\right)$. ■

3.3 Global in time result

We want to apply Proposition 7; to this purpose the only problem is proving an a priori estimate for $\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix}$, because the regularity properties of $\overline{u}$ that allows one to apply Proposition 7 hold on every time interval, for μ_0-a.e. initial condition in $\mathcal{H}^s$.

In order to find an a priori estimate for $\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix}$, we use again the energy

$$\mathcal{E}\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} := \left(\frac{1}{2} \left\| (-\Delta)^{1/2} \widetilde{u} \right\|_{L^2}^2 + \frac{1}{2} \left\| \widetilde{v} \right\|_{L^2}^2 + \int_{\mathbb{T}^3} \frac{\widetilde{u}^4}{4} dx \right)$$

which now does not satisfy simply $\frac{d}{dt}\mathcal{E}\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = 0$. Let us compute the time derivative formally:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} &= -\langle \Delta\widetilde{u}, \widetilde{v} \rangle + \left\langle \widetilde{v}, \Delta\widetilde{u} - (\widetilde{u} + \overline{u})^3 \right\rangle + \langle \widetilde{v}, \widetilde{u}^3 \rangle \\
&= -\left\langle \widetilde{v}, (\widetilde{u} + \overline{u})^3 \right\rangle + \langle \widetilde{v}, \widetilde{u}^3 \rangle \\
&= -\left\langle \widetilde{v}, 3\widetilde{u}^2\overline{u} + 3\widetilde{u}\overline{u}^2 + \overline{u}^3 \right\rangle \\
&\leq 3 \left\| \overline{u} \right\|_\infty \left\| \widetilde{v} \right\|_{L^2} \left\| \widetilde{u}^2 \right\|_{L^2} + 3 \left\| \overline{u} \right\|_\infty^2 \left\| \widetilde{v} \right\|_{L^2} \left\| \widetilde{u} \right\|_{L^2} + \left\| \overline{u} \right\|_\infty^3 \left\| \widetilde{v} \right\|_{L^2} \\
&\leq C^* \left\| \widetilde{v} \right\|_{L^2}^2 + \left\| \widetilde{u} \right\|_{L^4}^4 + \left\| \widetilde{u} \right\|_{L^2}^2 + 1 \\
&\leq C^* \left\| \widetilde{v} \right\|_{L^2}^2 + 2 \left\| \widetilde{u} \right\|_{L^4}^4 + 2 \\
&\leq 8C^*\mathcal{E}\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} + 2
\end{aligned}$$

where C^* depends on $\left\| \overline{u} \right\|_\infty$. Hence, from Gronwall lemma, for $t \in [0, T_0]$,

$$\mathcal{E}\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix} \leq \mathcal{E}\begin{pmatrix} \widetilde{u}(0) \\ \widetilde{v}(0) \end{pmatrix} e^{8C^*T_0} + 2e^{8C^*T_0}.$$

Proving rigorously this inequality requires some work as explained in Remark 5; we omit the details. The bound on $\mathcal{E}\begin{pmatrix} \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix}$ obviously implies a similar bound on $\left\| \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \right\|_{\mathcal{H}^1}$. The final result is:

Theorem 14 *Given $s > 0$, let μ_0 be a centered Gaussian measure on $\mathcal{H}^s$ such that the assumptions of Theorem 10 hold, for a r.v. (u_0, v_0) with law μ_0. Then, for μ_0-a.e. initial condition in $\mathcal{H}^s$, the cubic defocusing nonlinear wave equation has a global solution of class*

$$C\left([0, T_0] \times \mathbb{T}^3\right) \oplus C\left([0, T_0]; \mathcal{H}^1\right)$$

for every $T_0 > 0$.

3.4 Remarks on the regularity of Gaussian measures

A doubt, already arosen in Remark 11, is that we are imposing additional regularity on initial conditions, by means of the probabilistic selection under μ_0, and thus we are not really solving the problem in $\mathcal{H}^s$ with small s. The doubt has a positive and a negative answer.

On one side, Step 1 of the proof of Theorem 10 is a computation performed at every time t, hence also at time $t = 0$, indicating that μ_0-a.s. the initial conditions we are dealing with are continuous. Hence they truly have a significant additional regularity compared to $\mathcal{H}^s$, in the case of small s.

On the other side, let us investigate more closely the mean zero centered Gaussian vectors (u_0, v_0) in $\mathcal{H}^{s_0}$ treated in Theorem 10. Let us restrict, for notational simplicity, the attention to u_0, mean zero centered Gaussian vector in H^{s_0}; we have denoted the regularity exponent by s_0 to stress below the change when we modify it. The random vector u_0 has the form

$$u_0 = \sum_k \widehat{u_0}(k)\, e^{2\pi i k \cdot x}$$

with

$$\sum_k |k|^{2s_0}\, \mathbb{E}\left[|\widehat{u_0}(k)|^2\right] < \infty.$$

Set

$$Z_k := \frac{\widehat{u_0}(k)}{\sqrt{\mathbb{E}\left[|\widehat{u_0}(k)|^2\right]}}.$$

They are independent standard Gaussian complex valued r.v.'s. Setting

$$\sigma_k := \sqrt{\mathbb{E}\left[|\widehat{u_0}(k)|^2\right]}$$

we have

$$u_0 = \sum_k \sigma_k Z_k e^{2\pi i k \cdot x}.$$

Recall, to compare, that $\sum_k Z_k e^{2\pi i k \cdot x}$ is white noise. Here we assume

$$\sum_k |k|^{2s_0}\, \sigma_k^2 < \infty.$$

Take $s_1 > s_0$. There exists a sequence $\{\sigma_k\}$, satisfying the previous property, such that

$$\sum_k |k|^{2s_1}\, \sigma_k^2 = \infty.$$

This implies that the law of u_0 gives measure zero to H^{s_1}.

Summarizing, given $s_1 > s_0 > 0$, we may construct a Gaussian measure μ_0 satisfying the assumption of Theorem 10 with $s = s_0$ such that $\mu_0(\mathcal{H}^{s_1}) = 0$, namely no element selected by μ_0 has regularity $\mathcal{H}^{s_1}$. From the viewpoint of the scale $\{\mathcal{H}^s\}_{s \geq 0}$ there is a genuine improvement of the deterministic results.

Remark 15 *Other measures can be used on initial conditions, for instance centered in points different from zero and even non Gaussian; see [1]. The principles of this approach are quite general. A surprising fact compared to the other examples in these notes is that no invariance (time or space) is needed. This seems to be related to the possibility of the decomposition outlined in Section 2.1, which does not look possible for Euler equations or particle systems.*

References

[1] N. Tzvetkov, Random data wave equations, lecture notes CIME 2016, to appear.