1 Preliminaries on deterministic ODEs

Consider the Cauchy problem in $\mathbb{R}^d$

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \in \mathbb{R}^d \end{cases}$$

or its integral version

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

Solutions are always supposed to be continuous, since they have to verify the integral identity; they are differentiable in a way dependent on the properties of f. The following results are well known.

Theorem 1 *If $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$\begin{aligned} |f(t,x) - f(t,y)| &\leq C|x-y| \\ |f(t,x)| &\leq C(1+|x|) \end{aligned}$$

then we have existence and uniqueness of solutions on $[0,T]$.

Theorem 2 *If f is only continuous and bounded, then we have at least existence of solutions on $[0,T]$.*

In dimension one, a key role is played by the following simple criterium.

Theorem 3 *If $d = 1$ and $f > 0$ in a neighbor $\mathcal{U}$ of x_0 (given for instance by $f(x_0) > 0$ and f continuous) then we have local uniqueness.*

Proof. Let $x(t)$ be a solution (continuous). On some interval $[0,\tau]$ we have $x(t) \in \mathcal{U}$ hence $f(x(t)) > 0$, hence

$$\frac{\dot{x}(t)}{f(x(t))} = 1$$

$$\int_{x_0}^{x(t)} \frac{ds}{f(s)} = t$$

$$x(t) = H^{-1}(t)$$

where $H(x) = \int_{x_0}^{x} \frac{ds}{f(s)}$ is locally invertible. ■

Therefore to construct examples of non-uniqueness in $d = 1$ we need $f(x_0) = 0$ (or f changing sign in any neighbour of x_0).

Example 4 *Assume*

$$f(x) = 1_{\{x>0\}} - 1_{\{x<0\}}.$$

Then, from $x_0 = 0$ *we have infinitely many solutions:* $x(t) = 0$, $x(t) = t$, $x(t) = -t$, *and any other that is equal to 0 for some time, then branches similarly to the previous ones.*

Example 5 *Assume*

$$f(x) = 1_{\{x>0\}} - 1_{\{x\leq 0\}}.$$

Then, from $x_0 = 0$ *we do not have solutions.*

Example 6

$$f(x) = 1_{\{x>0\}\cap\mathbb{Q}^c} - 1_{\{x<0\}\cap\mathbb{Q}^c}.$$

Suppose $x_0 \in \mathbb{Q}$. *Then* $x(t) = x_0$ *is a solution. But also* $x(t) = x_0 + t$ *is a solution, because* $f(x(t)) = f(x_0 + t) = 1$ *a.s., hence*

$$x_0 + \int_0^t f(x(s))\, ds = x_0 + t = x(t).$$

We may combine pieces of constant solution and linearly increasing ones as we like, filling-in densely the half-space $y \geq x_0$.

Example 7 *There exist continuous functions (indeed* γ*-Hölder continuous, with* $\gamma \leq \gamma_0 < 1$*) that are not Lipschitz on any interval and, at the same time, have uncountably many zeros; for instance the path of a Brownian motion* $x \mapsto B_x(\omega)$. *The function* $f(x) = |B_x(\omega)|$ *may be used to construct an example like the previous one.*

2 Perturbations (not necessarily white noise)

Let us add a piecewise constant random perturbation of the form ($h > 0$, $\alpha(h) > 0$)

$$g_h(t) = \alpha(h) \cdot X_k \qquad \text{on } [kh, (k+1)h[, \qquad k \in \mathbb{N}$$

where $X_k = \pm 1$ with equal probability (choose them independent, to have a natural intuition in view of white noise). Consider the equation

$$\begin{cases} \dot{x}(t) = f(x(t)) + \dot{g_h}(t) \\ x(0) = x_0 \end{cases}$$

or better

$$x(t) = x_0 + \int_0^t f(x(s))\, ds + g_h(t).$$

On each interval $[kh, (k+1)h[$ we solve an equation with the new drift

$$\widetilde{f}(x) = f(x) + \alpha(h) \cdot X_k.$$

If f is bounded and we choose $\alpha(h) > \|f\|_\infty$, we have $\left|\widetilde{f}(x)\right| > 0$ for all x, hence on $[kh, (k+1)h[$ we have uniqueness. Thus the addition of the highly fluctuation term $g_h(t)$ restores uniqueness.

Conjecture 8 *Uniqueness is maintained in the limit as $h \to 0$, $\alpha(h) \to \infty$.*

Problem 9 *We shall see that this is true, in dimension 1. In dimension $d > 1$, we shall see that it i still true, uniqueness, for the limit problem $h \to 0$, $\alpha(h) \to \infty$ (the case of whte noise). But it is open to prove it for approximations of white noise like the previous one. In dimension $d = 1$ it is open to prove it for general approximations.*

Maybe it is useful to compare with another proble when a different situation occurs: stabilization by noise. Consider the diagonal matrix (the final resuls is more general)

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & ... & \\ 0 & & \lambda_d \end{pmatrix}$$

with $\lambda_1 \geq ... \geq \lambda_d$. To fix the ideas, assume $\lambda_1 > 0$, $\lambda_1 + ... + \lambda_d < 0$. The linear system in $\mathbb{R}^d$

$$\dot{x}(t) = Ax(t)$$

is unstable. The Lyapunov exponent

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log |x(t)|$$

is positive and equal to λ_1, for most initial conditions. Let us perturb the equation as

$$\dot{x}(t) = Ax(t) + \sigma \sum_{k=1}^{N} B_k x(t) \dot{g}_h^k(t).$$

Assume $B_k^T = -B_k$, rotations, so that intuitively we do not directly affect the eigenvalues. Call λ_σ the corresponding top Lyapunov exponent. One can prove that there exist N, $B_1, ..., B_N$, such that

$$\lim_{\sigma \to \infty} \lambda_\sigma = \frac{TrA}{d}.$$

Thus the addition of the randon perturbations stabilizes the system. The result has been proved both for whte noise perturbations, by L. Arnold, H. Crauel and V. Wisthutz, and by periodic highly fluctuating perturbations by V. Arnold.

3 Brownian motion; Gaussian fields

Definition 10 *On a probability space $(\Omega, \mathcal{F}, P)$ we say that a stochastic process $(B_t)_{t\geq 0}$ is a Brownian motion if*

i) $B_0 = 0$

ii) for every $t \geq s \geq 0$, $B_t - B_s$ is $N(0, t-s)$

iii) for every $t_n \geq t_{n-1} \geq ... \geq t_1 \geq 0$, the increments $B_{t_n} - B_{t_{n-1}}, \dots, B_{t_1} - B_0$ are independent.

We usually will include also the condition:

iv) the trajectories $t \mapsto B_t(\omega)$ are continuous, for a.e. $\omega \in \Omega$.

A Brownian motion in $\mathbb{R}^d$ is just a vector valued stochastic process $B_t = \left(B_t^{(1)}, ...B_t^{(d)}\right)$ where the components are independent real valued Brownian motions.

Remark 11 *The GFF in dimension 1 without periodic boundary condition is a BM, with properties (i)-(iii). In order to have also (iv), ne can use Kolmogorov regularity criterium.*

Theorem 12 *In $D \subset \mathbb{R}^d$ bounded, a random field $\{X_x\}_{x\in D}$ satisfying for some constants $p, \alpha, C > 0$*

$$\mathbb{E}\left[|X_x - X_y|^p\right] \leq C\,|x-y|^{d+\alpha}$$

for all $x, y \in D$, has a continuous modification (there exists a random field $\left\{\widetilde{X}_x\right\}_{x\in D}$ susch that $\widetilde{X}_x = X_x$ a.s., for every $x \in D$; and $x \mapsto X_x(\omega)$ is continuous for a.e. $\omega \in \Omega$).

In the case of a Brownian motion satisfying (a priori) only properties (i)-(iii) we have

$$\mathbb{E}\left[|B_t - B_s|^p\right] = \mathbb{E}\left[\left|\frac{B_t - B_s}{\sqrt{t-s}}\right|^p\right](t-s)^{p/2} = C_p\,(t-s)^{p/2}$$

where $C_p = \mathbb{E}\left[|Z|^p\right]$ with Z being $N(0,1)$ (the r.v. $\frac{B_t - B_s}{\sqrt{t-s}}$ is $N(0,1)$). Hence for $p > 2$ we may apply Kolmogorov regularity criterium and get the existence of a Brownian motion satisfying also property (iv).

This argument is quite general for Gaussian random fields. Let $\{X_x\}_{x\in D}$ be a Gaussian random field, namely a family of r.v.'s such that for all $n \in \mathbb{N}$ and all $x_1, ..., x_n$ the vector $(X_{x_1}, ..., X_{x_n})$ is Gaussian. Assume it centered. Define

$$q(x,y) = \mathbb{E}\left[X_x X_y\right]$$

(the so called *covariance function*). In a sense, we have specified L^2 type properties only. But Gaussianity improves them:

Theorem 13 *If q is bounded, then a.s. the paths $x \mapsto X_x(\omega)$ are of class $\cap_{p\geq 1} L^p(D)$.*

If q is Hölder continuous, then there exists a continuos modification.

Proof. We only sketch the proof of the first claim; the second one is left as an exercise. We have

$$\mathbb{E}\left[\int_D |X_x|^p \, dx\right] = \int_D \mathbb{E}\left[|X_x|^p\right] dx = \int_D \mathbb{E}\left[\left|\frac{X_x}{\sqrt{q(x,x)}}\right|^p\right] q(x,x)^{p/2} \, dx = C_p \int_D q(x,x)^{p/2} \, dx$$

where $C_p = \mathbb{E}\left[|Z|^p\right]$ as above. ■

Remark 14 *This Gaussian improvement of regularity s the key ingredient in the theorems of solvability for a.e. initial conditions for certain dispersive equations, like the nonlinear wave equation* $\partial_{tt}^2 u = \Delta u - u^3$ *in dimension 3 (Burq-Tzvetkov). Solvability for such equations is based on Strickartz estimates, which are* $L_t^p(L_x^q)$ *estimates on the linear part of the equation. In a purely deterministic context, the proof of Strickartz estimates is difficult. In a Gaussian framework, the* $L_t^p(L_x^q)$ *regularity gain is given by Gaussianity; it is much easier and gives new results.*

Remark 15 *Going back to the theory of Abstract Wiener Spaces we could recognize that GFF, hence BM, lives in* $H^s = W^{s,2}$ *for all* $s < \frac{1}{2}$*, a space which however does not embed into continuous functions. The Gaussian improvent gives on* $W^{s,p}$ *for all* $s < \frac{1}{2}$ *and* $p \geq 2$*, which is embedded in the space of continuous functions.*

4 Definitions of uniqueness for ODEs perturbed by white noise

Let us finally arrive to ODEs perturbed by white noise $\dot{B}_t$

$$\begin{cases} \dot{X}_t = f(t, X_t) + \dot{B}_t \\ X_o = x_0 \end{cases}$$

or, in the more proper integral form

$$X_t = x_0 + \int_0^t f(s, X_s) \, ds + B_t$$

perturbed by the BM B_t.

Remark 16 *This is the limit problem as* $h \to 0$ *of the the problem above, with* $\alpha(h) = \frac{1}{\sqrt{h}}$*. Donsker theorem states that* g_h *converges in law to BM.*

Let $\omega \in \Omega$ be such that $t \mapsto B_t(\omega)$ is continuous. Consider the deterministic equation

$$X_t(\omega) = x_0 + \int_0^t f(s, X_s(\omega)) \, ds + B_t(\omega).$$

We call $\mathcal{C}(x_0, \omega)$ the set of all functions $X_t(\omega)$ of class $C\left([0,T];\mathbb{R}^d\right)$ that satisfy this identity.

Definition 17 *We say that path-by-path uniqueness holds when* $\mathcal{C}(x_0, \omega)$ *is a singleton, for a.e.* $\omega \in \Omega$.

Theorem 18 *If* f *satisfies the Lipschitz-type assumptions of Theorem 1 above, that* $\mathcal{C}(x_0, \omega)$ *is not empty and a singleton, for a.e.* $\omega \in \Omega$ *(hence path-by-path uniqueness holds).*

Theorem 19 *If* f *is continuous bounded, then* $\mathcal{C}(x_0, \omega)$ *is not empty, for a.e.* $\omega \in \Omega$.

We say that $(X_t)_{t\in[0,T]}$ is adapted to $(B_t)_{t\geq 0}$ if X_t is measurable with respect to the σ-algebra generated by all B_s for $s \leq t$. We say that two processes $X_t^{(i)}$, $i = 1, 2$ are indistinguishable if $P\left(X_\cdot^{(1)} = X_\cdot^{(2)}\right) = 1$.

Definition 20 *We say that pathwise uniqueness holds when the following is true. Given a probability space* $(\Omega, \mathcal{F}, P)$ *with a Brownian motion* B_t*, if* $X_t^{(i)}$*,* $i = 1, 2$ *are two stochastic processes on* $(\Omega, \mathcal{F}, P)$ *adapted to* B_t*, with* $X_\cdot^{(i)}(\omega) \in \mathcal{C}(x_0, \omega)$ *a.s., for both* $i = 1, 2$*, then* $X_t^{(i)}$ *are indistinguishable.*

Remark 21 *The previous definition is not entirely canonical; one should introduce filtrations* $(\mathcal{F}_t)_{t\geq 0}$*, define BM with respect to a filtration and procesess* $(X_t)_{t\in[0,T]}$ *adapted to a filtration; then pathwise uniqueness should be defined as indistinguishability between solutions adapted to a filtration coherent with the noise. We omit this level of generality and concentrate on the previous restricted definition.*

Remark 22 *The concept of pathwise uniqueness is more restrictive that path-by-path uniqueness, since it requires more structure of the objects. It corresponds, in a sense, to uniqueness of Lagrangian flows, opposite to uniqueness for a.e. initial conditions that corresponds to path-by-path uniqueness.*

Finally, there a third definition of uniqueness, so called "in law", that we do not discuss for the time being.