Random Initial Conditions and Noise in Ordinary and Partial Differential Equations

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2017-2018

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Chapter 1

Random Point Vortices

1.1 Introduction

These lectures are devoted to the following general questions: could Probability add something to the theory of differential equations, ordinary or partial? Could we prove stronger theorems using probability?

Two directions have been identified in recent years: put randomness in the initial conditions (deterministic equations with random initial conditions), add time-dependent randomness - *noise* - to the equations (stochastic differential equations). We shall explore both directions.

Concerning random initial conditions, we may distinguish between finite and infinite dimensional problems. Both in finite and infinite dimensions, we may classify problems as special examples or general theories. Let us say something more on each one of these four possibilities:

- finite dimensions, special problems: we shall describe the theory of Lanford on no-concentration of particles for interacting particle systems; and the theory of no-collision of point vortices of Marchioro-Pulvirenti;
- finite dimensions, general problems: starting from Di Perna-Lions, ODEs with only weakly differentiable coefficients (instead of Lipschitz ones) have been solved in some probabilistic sense, with several approaches;
- infinite dimensions, general problems: the previous ideas of Di Perna-Lions theory have been extended to infinite dimensional spaces;
- infinite dimensions, special problems: we shall describe elements of the theory of dispersive equations (wave, Schrödinger, KDV and others) and 2D Euler equations, with random initial conditions, where in both cases the effort is to solve the equations for less regular initial data than those allowed by deterministic tools.

Concerning noise perturbations of the equations, we shall also review several finite and infinite dimensional examples. In finite dimensions, the typical result is that very poor drift - leading to several pathologies in the deterministic case - is "regularized" by additive noise. In infinite dimensions there are similar abstract results as well as a few "regularization by noise" result for specific PDE examples.

1.2 Open problems

Concerning open problems, we shall concentrate mostly on the infinite dimensional case.

A general important question is to fill the gap between Di Perna-Lions theory and examples (from Mathematical Physics, so to speak). Applications to PDE meet some essential difficulties. However, maybe there is hope for certain dispersive equations. Application to infinite systems of interacting particles, however, could be possible and should be explorated; this creates a link with the first part of the course, on interacting particles, like Lanford result.

Concerning specific examples of PDEs, application of the ideas in fluid dynamics are perhaps just at the beginning. Here, both the case of random initial conditions and regularization by noise deserve further investigation.

1.3 Two dimensional fluids (short introduction)

The topics illustrated in this initial section are presented in a style between Mathematics and Physics: no rigour is pretended, but a simple illustration of ideas and objects. As a general reference, let us quote [43], see also [41]; but the literature on the subject is enormous.

We shall concentrate on the so called inviscid incompressible constant density fluids, described by the variables

$$\begin{aligned} u(t,x) &= \text{velocity} \\ p(t,x) &= \text{pressure} \end{aligned}$$

(the constant value of the density is taken equal to 1), $u : [0,T] \times D \to \mathbb{R}^2$, $p : [0,T] \times D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is the domain occupied by the fluid. We assume they satisfy (the so called *Euler equations*)

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

in a suitable sense. The first equations are Newton law along ideal particle trajectories: the acceleration $\frac{d}{dt}u$ is balanced by the force $-\nabla p$. The second equation encodes incompressibility.

The *vorticity* $\omega(t,x)$, $\omega : [0,T] \times D \to \mathbb{R}$, defined as

$$\omega = \partial_2 u_1 - \partial_1 u_2 = \nabla^{\perp} \cdot u$$

plays a central role. If (u,p) is a reasonable solution, one can check that

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

This is Euler equation in vorticity form, a nonlinear (because u and ω are related) transport equation. In the case of sufficiently regular solutions, ω is transported along ideal particle trajectories.

We mentioned ideal particle trajectories. We meant solutions of the equation

$$\frac{dx(t)}{dt} = u(t, x(t)).$$

We may think of infinitesimal portions of fluid (still macroscopic, not at the molecular level). The notation $\frac{d}{dt}u$ above stands for $\frac{d}{dt}u(t,x(t))$: one has (for sufficiently smooth solutions)

$$\frac{d}{dt}u(t,x(t)) = [\partial_t u + u \cdot \nabla u]_{(t,x(t))} = [-\nabla p]_{(t,x(t))}$$

$$\frac{d}{dt}\omega(t,x(t)) = [\partial_t \omega + u \cdot \nabla \omega]_{(t,x(t))} = 0$$

which explain some statements above.

Sometimes it is useful to keep in mind the differences with respect to 3D fluids. For them, Euler equations in the variables $u : [0,T] \times D \to \mathbb{R}^3$, $p : [0,T] \times D \to \mathbb{R}$ are the same as in the 2D case (here $D \subset \mathbb{R}^3$). But vorticity $\omega : [0,T] \times D \to \mathbb{R}^3$ is a vector field, it is defined as

$$\omega = \operatorname{curl} u$$

and it satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u.$$

The additional term $\omega \cdot \nabla u$ is the cause of many important (and difficult) facts; it is called vortex stretching (vorticity is not only transported, but also stretched). The 2D "projection" above simply means that the fluid $u : [0,T] \times D \to \mathbb{R}^3$ has a planar symmetry, namely it has the form

$$u(t,x) = (u_1(t,x_1,x_2), u_2(t,x_1,x_2), 0)$$

and therefore

$$\omega(t,x) = (0, 0, \omega_3(t,x_1,x_2)).$$

Namely, vorticity is always perpendicular to the plane of motion. Therefore it is sufficient to consider the scalar quantity $\omega_3(t,x_1,x_2)$, given by $\omega_3 = \partial_2 u_1 - \partial_1 u_2$. It is the quantity we called ω above, in the 2D case.

Kinetic energy

$$\frac{1}{2}\int_D |u(t,x)|^2 \, dx$$

is (with appropriate boundary conditions) an invariant quantity (for sufficiently smooth solutions), both in 2D and 3D. Let us check it formally in the case $D = \mathbb{R}^d$, $d = 2, 3$:

$$\frac{1}{2}\frac{d}{dt}\int_D |u(t,x)|^2 \, dx = \int_D u \cdot \partial_t u dx = -\int_D u \cdot (u \cdot \nabla u) \, dx - \int_D u \cdot \nabla p dx = 0$$

because

$$\int_D u \cdot \nabla p dx = -\int_D p \operatorname{div} u dx = 0$$

$$\int_D u \cdot (u \cdot \nabla u) \, dx = \frac{1}{2}\int_D u \cdot \nabla |u|^2 - dx = -\int_D |u|^2 \operatorname{div} u dx = 0.$$

Enstrophy

$$\int_D |\omega(t,x)|^2 \, dx$$

is an invariant quantity (with appropriate boundary conditions and for sufficiently smooth solutions) in 2D:

$$\frac{d}{dt}\int_D |\omega(t,x)|^2 \, dx = 2\int_D \omega \partial_t \omega dx = -2\int_D \omega (u \cdot \nabla \omega) \, dx = -\int_D u \cdot \nabla \omega^2 dx = \int_D \omega^2 \operatorname{div} u dx = 0.$$

In 3D this is not true, since the additional term

$$\int_D \omega (\omega \cdot \nabla u) \, dx$$

is not zero (in general) and quite relevant for the dynamics; in principle it is even possible that it leads to blow-up of ω (it is an open problem). In 2D, the previous computation can be repeated for any power $\int_D \omega(t,x)^n \, dx$, which are thus all invariants. As we mentioned above, also the pointwise value $\omega(t, x(t))$ itself is invariant, along particle trajectories, when suitably defined.

For the sequel, we need to invert the relation $\omega = \nabla^\perp \cdot u$. Call fluid *potential* a scalar field $\varphi : [0,T] \times D \to \mathbb{R}$ such that

$$\Delta \varphi = \omega$$

and take

$$u = \nabla^\perp \varphi.$$

We have (under suitable regularity) $\operatorname{div} u = 0$ and

$$\nabla^\perp \cdot u = \nabla^\perp \cdot \nabla^\perp \varphi = \partial_2^2 \varphi + \partial_1^2 \varphi = \omega.$$

Obviously it is necessary to be embdedded into a set-up (regularity and boundary conditions) where all these operations exist. If so, we have, with compact notations,

$$u = \nabla^{\perp}\Delta^{-1}\omega.$$

This is called *Biot-Savart law.* If there is a function $K(x, y)$ such that, for every $y \in D$, $x \mapsto K(x, y)$ solves

$$\Delta_x K(\cdot, y) = \delta_y$$

in the sense of distributions (over compact manifolds without boundary as in the case of the torus, the correct problem is $\Delta_x K(\cdot, y) = \delta_y - \frac{1}{|D|}$, see below), we set

$$u(t, x) = \int_{\mathbb{R}^2} \nabla^{\perp} K(x, y)\, \omega(t, y)\, dy.$$

For instance, in full space $D = \mathbb{R}^2$, under suitable regularity, we choose

$$\begin{aligned} \varphi(t, x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, \omega(t, y)\, dy \\ u(t, x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\omega(t, y)\, dy. \end{aligned}$$

The kernel

$$K(x) = \frac{1}{2\pi}\frac{x^{\perp}}{|x|^2}$$

emerges, that we shall meet several times below.

1.4 Point vortices, introduction

Again, we start with an heuristic introduction; see for instance [43], Chapter 4. With some degree of idealization, one can consider the previous 2D description in the case when

$$\omega(t, x) = \sum_{i=1}^{N} \xi_i \delta_{x_i(t)}$$

namely when the vorticity is concentrated in a finite number of points. Based on the Biot-Savart law, the associated velocity is

$$u(t, x) = \sum_{i=1}^{N} \xi_i K(x, x_i(t)).$$

The motion of vortex i should be described by the equation $\frac{dx_i(t)}{dt} = u(t, x_i(t))$. However, the function $K(x, y)$ is singular for $x = y$, hence $u(t, x)$ is well defined only at points

x different from $x_1(t)$, ..., $x_N(t)$. The "correct" choice is then to avoid self-interaction, namely to consider the following dynamics:

$$\frac{dx_i(t)}{dt} = \sum_{j\neq i} \xi_j K(x_i(t), x_j(t)).$$

There are at least two (similar) justifications for this choice (namely for neglecting self-interaction). One is a non-trivial theorem of [42] stating, in plain words, that smooth vortex patches concentrated around points have a dynamics close to the one of point vortices; the proof requires more ingredients than those we can use at this stage of the development of the theory. A second justification (slightly weaker) consists in investigating the motion of a fluid particle close to one point vortex, say vortex n. 1:

$$\frac{dx(t|x_0)}{dt} = u(t, x(t|x_0)) = \sum_{i=1}^{N} \xi_i K(x(t|x_0), x_i(t))$$

where the trajectory $x(t|x_0)$ has initial condition x_0 very close to $x_1(0)$ and the points $x_i(t)$, $i = 1, ..., N$, solve the previous point vortex system. One can show that $x(t|x_0)$ remains very close to $x_1(t)$, over finite time horizon, under the condition of no vortex collision. We develop the argument in the case $D = \mathbb{R}^2$ to fix the ideas (so that $K(x,y) = \frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2}$). See also Section 6.5 for a simlar but considerably more difficult result.

Proposition 1 *Let $[0,T]$ be an interval of existence and uniqueness of solution without collision, for the vortex dynamics. Then:*

i) $x(t|x_0)$, solution of $\frac{dx(t|x_0)}{dt} = u(t, x(t|x_0))$ with initial condition x_0, exists uniquely and it is different from the positions of vortices, on $[0,T]$;

ii) there are constants $\epsilon_0, C > 0$ (depending on T, on the minimal distance between vortices, on N and on $|\xi| = \max|\xi_i|$) such that for all $x_0 \in B(x_1(0), \epsilon_0) \setminus \{x_1(0)\}$ and all $t \in [0,T]$ one has

$$|x(t|x_0) - x_1(t)| \leq C|x_0 - x_1(0)|. \tag{1.1}$$

Proof. Let $r_0 > 0$ be the minimal distance between vortices on $[0,T]$. Choose a preliminary value of $0 < \epsilon_0 < \frac{r_0}{2}$ and assume $0 < |x_0 - x_1(0)| \leq \epsilon_0$; consider any interval $[0,\tau] \subset [0,T]$ [1] where $0 < |x(t|x_0) - x_1(t)| \leq \frac{r_0}{2}$. From

$$\frac{d}{dt}(x(t|x_0) - x_1(t)) = \sum_{i=2}^{N} \xi_i [K(x(t|x_0), x_i(t)) - K(x_1(t), x_i(t))] + \xi_1 K(x(t|x_0), x_1(t))$$

we deduce

$$\frac{1}{2}\frac{d}{dt}|x(t|x_0) - x_1(t)|^2 = \sum_{i=2}^{N} \xi_i [K(x(t|x_0), x_i(t)) - K(x_1(t), x_i(t))] \cdot (x(t|x_0) - x_1(t))$$

[1] There is at least one with $\tau > 0$ by continuity of trajectories

(since $\frac{(x(t|x_0)-x_1(t))^{\perp}}{|x(t|x_0)-x_1(t)|^2} \cdot (x(t|x_0) - x_1(t)) = 0$) and thus, for a suitable $L > 0$,

$$\leq N|\xi| L |x(t|x_0) - x_1(t)|^2 .$$

Such L exists because the function $x \mapsto K(x)$ is Lipschitz continuous with constant L for $|x| \geq \frac{r_0}{2}$; therefore

$$\begin{aligned} & |K(x(t|x_0), x_i(t)) - K(x_1(t), x_i(t))| \\ = & |K(x(t|x_0) - x_i(t)) - K(x_1(t) - x_i(t))| \\ \leq & L|(x(t|x_0) - x_i(t)) - (x_1(t) - x_i(t))| \\ \leq & L|x(t|x_0) - x_1(t)| \end{aligned}$$

(we use the fact that $|x(t|x_0) - x_i(t)|$ and $|x_1(t) - x_i(t)|$ are greater than $\frac{r_0}{2}$).

Hence

$$|x(t|x_0) - x_1(t)| \leq |x_0 - x_1(0)| e^{N|\xi|LT} \leq \epsilon_0 e^{N|\xi|LT}.$$

If we choose now $\epsilon_0 < \frac{r_0}{2}$ so that it satisfies also $\epsilon_0 \leq e^{-N|\xi|LT}/4$, we have $|x(t|x_0) - x_1(t)| \leq \frac{1}{4}$ for all $t \in [0, \tau]$.

Moreover, on $[0, \tau]$,

$$\begin{aligned} & -\frac{1}{2}\frac{d}{dt}\log|x(t|x_0) - x_1(t)|^2 \\ = & -\frac{1}{|x(t|x_0) - x_1(t)|^2}\sum_{i=2}^{N}\xi_i [K(x(t|x_0), x_i(t)) - K(x_1(t), x_i(t))] \cdot (x(t|x_0) - x_1(t)) \\ \leq & N|\xi| L. \end{aligned}$$

Hence

$$-\log|x(t|x_0) - x_1(t)| \leq -\log|x_0 - x_1(0)| + TN|\xi| L$$

$$|x(t|x_0) - x_1(t)| \geq \exp(\log|x_0 - x_1(0)| - TN|\xi| L).$$

By an easy argument by contradiction, one can take $\tau = T$, for this choice of ϵ_0. Thus we have also proved $0 < |x(t|x_0) - x_1(t)| \leq |x_0 - x_1(0)| e^{N|\xi|LT}$ for all $t \in [0, T]$, always for this choice of ϵ_0. Summarizing, we have found $\epsilon_0 > 0$ such that for $|x_0 - x_1(0)| \leq \epsilon_0$ we have $x(t|x_0)$ globally defined, always different from the positions of point vortices (including $x_1(t)$), and (1.1) holds. This proves part (ii). Part (i) is already proved for initial positions sufficiently close to those of the point vortices; a fortiori, it is true for the other initial positions, by an easy argument that we leave to the reader. ■

1.5 Invariants, Hamiltonian and Lyapunov functions

The presentation of this section is widely taken from [45]. Consider the case of N vortices in full space, hence $K(x,y) = \frac{1}{2\pi}\frac{(x-y)^{\perp}}{|x-y|^2}$,

$$\frac{dx_i(t)}{dt} = \frac{1}{2\pi}\sum_{j\neq i}\xi_j \frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2}.$$

Let $[0,T]$ be an interval of existence and uniqueness of solution without collision. Call

$$\Gamma = \sum_{i=1}^{N}\xi_i$$

the *global circulation* (which is obviously invariant). The quantity ($\frac{c(t)}{\Gamma}$ is called *center of vorticity or inertia*)

$$c(t) = \sum_{i=1}^{N}\xi_i x_i(t)$$

is invariant:

$$\frac{dc(t)}{dt} = \frac{1}{2\pi}\sum_{\substack{i,j=1,\dots,N\\ i\neq j}}\xi_i\xi_j \frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2} = 0$$

because each pair (i,j) with $i \neq j$ contributes with two terms, opposite one each other.

Also the quantity (called *moment of inertia*)

$$I(t) = \sum_{i=1}^{N}\xi_i |x_i(t)|^2$$

is invariant:

$$\frac{dI(t)}{dt} = \frac{1}{\pi}\sum_{\substack{i,j=1,\dots,N\\ i\neq j}}\xi_i\xi_j x_i(t)\frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2}$$

and now, for each pair (i,j) with $i \neq j$, notice that the sum of the two corresponding terms vanishes

$$\begin{aligned} & x_i(t)\frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2} - x_j(t)\frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2} \\ = \; & (x_i(t) - x_j(t))\frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2} = 0. \end{aligned}$$

Define

$$\begin{aligned} T(x_1,...,x_N) &= \sum_{i,j=1}^{N} \xi_i \xi_j |x_i - x_j|^2 \\ T(t) &= T(x_1(t),...,x_N(t)). \end{aligned}$$

Also $T(t)$ is invariant, because

$$T(t) = 2\left(\Gamma I(t) - |c(t)|^2\right).$$

Define Δ as the set in $\mathbb{R}^{2N}$ with at least two equal points. For a coordinate $x_i \in \mathbb{R}^2$ of the point $x = (x_1,...,x_N) \in \mathbb{R}^{2N}$ write q_i and p_i for its first and second coordinates.

The function $H : \mathbb{R}^{2N} \backslash \Delta \to \mathbb{R}$

$$H(x_1,...,x_N) = \sum_{\substack{i,j=1,...,N \\ i \neq j}} \xi_i \xi_j \log |x_i - x_j|$$

satisfies

$$\begin{aligned} \xi_i \frac{dq_i(t)}{dt} &= \partial_{p_i} H(x_1(t),...,x_N(t)) \\ \xi_i \frac{dp_i(t)}{dt} &= -\partial_{q_i} H(x_1(t),...,x_N(t)). \end{aligned}$$

This is the structure of an Hamiltonian system. Lebesgue measure is invariant (in a suitable sense; we shall come back to this issue). Also,

$$H(t) = H(x_1(t),...,x_N(t))$$

is invariant:

$$\frac{dH(t)}{dt} = 0.$$

The quantity

$$L(x_1,...,x_N) = -\sum_{\substack{i,j=1,...,N \\ i \neq j}} \log |x_i - x_j|$$

may be useful as a Lyapunov function in the attempt to prove no collision:

Exercise 2 *Given an initial condition* $(x_1(0),...,x_N(0)) \in \Delta^c$, *if we can prove that there exists* $C > 0$ *such that*

$$L(x_1(t),...,x_N(t)) \leq C$$

for all local solutions, then the solution is global, and unique.

Lemma 3 *If all $\xi_i > 0$, the function*

$$\widetilde{L}(x_1, ..., x_N) = T(x_1, ..., x_N) - H(x_1, ..., x_N)$$

has similar properties: given an initial condition $(x_1(0), ..., x_N(0)) \in \Delta^c$, if we can prove that there exists C such that

$$\widetilde{L}(x_1(t), ..., x_N(t)) \leq C$$

for all local solutions, then the solution is global, and unique.

Proof. It is sufficient to prove that there exist $a, b > 0$ such that

$$L(x_1, ..., x_N) \leq a\widetilde{L}(x_1, ..., x_N) + b$$

and then apply the previous exercise. ■

We may use the quantity $\widetilde{L}(x_1, ..., x_N)$ to prove a first global existence result.

Proposition 4 *If all $\xi_i > 0$ (or all negative) then solutions are global, without collision.*

Proof. Let $(x_1(t), ..., x_N(t))$ be a local solution. Since $\widetilde{L}$ is invariant,

$$\widetilde{L}(x_1(t), ..., x_N(t)) = \widetilde{L}(x_1(0), ..., x_N(0))$$

we apply the previous Lemma with $C = \widetilde{L}(x_1(0), ..., x_N(0))$. ■

When the signs of ξ_i are variable, we cannot use the invariants to prove no collision. In the attempt to use the function $L(x_1, ..., x_N)$ we have, with $L(t) = L(x_1(t), ..., x_N(t))$,

$$\begin{aligned} \frac{dL(t)}{dt} &= -\sum_{\substack{i,j=1,...,N \\ i \neq j}} \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^2} \left(\frac{dx_i(t)}{dt} - \frac{dx_j(t)}{dt} \right) \\ &= -\frac{1}{2\pi} \sum_{\substack{i,j=1,...,N \\ i \neq j}} \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^2} \left(\sum_{k \neq i} \xi_k \frac{(x_i(t) - x_k(t))^\perp}{|x_i(t) - x_k(t)|^2} - \sum_{k \neq j} \xi_k \frac{(x_j(t) - x_k(t))^\perp}{|x_j(t) - x_k(t)|^2} \right). \end{aligned}$$

The problem is to bound this sums. In the next section we show there are configurations of collapsing vortices. Therefore some special ingredient is needed to bound the previous terms.

1.6 Two and three vortices

Again, we base the content of this section on [45]. In order to get more involved in vortex dynamics, let us understand a few properties of the motion of two and three vortices.

In the case of two vortices (both ξ_1, ξ_2 not zero) we have

$$\frac{dx_1(t)}{dt} = \frac{1}{2\pi}\xi_2 \frac{(x_1(t) - x_2(t))^{\perp}}{|x_1(t) - x_2(t)|^2}$$

$$\frac{dx_2(t)}{dt} = -\frac{1}{2\pi}\xi_1 \frac{(x_1(t) - x_2(t))^{\perp}}{|x_1(t) - x_2(t)|^2}$$

$$c(t) = \xi_1 x_1(t) + \xi_2 x_2(t) = \text{constant}$$

$$I(t) = \xi_1 |x_1(t)|^2 + \xi_2 |x_2(t)|^2 = \text{constant}$$

$$T(t) = 2\xi_1\xi_2 |x_1(t) - x_2(t)|^2 = \text{constant}.$$

In particular we reduce the equations to

$$\frac{dx_1(t)}{dt} = \frac{1}{2\pi}\xi_2 \frac{(x_1(t) - x_2(t))^{\perp}}{|x_1(0) - x_2(0)|^2}$$

$$\frac{dx_2(t)}{dt} = -\frac{1}{2\pi}\xi_1 \frac{(x_1(t) - x_2(t))^{\perp}}{|x_1(0) - x_2(0)|^2}.$$

Case 1: $\xi_2 = -\xi_1$ (equal intensity counter-rotating vortices). In this case the two vortices move parallel with constant velocity. Indeed,

$$c(t) = \xi_1 (x_1(t) - x_2(t)) = \text{constant}$$

hence

$$\frac{dx_1(t)}{dt} = -\frac{1}{2\pi}\xi_1 \frac{(x_1(0) - x_2(0))^{\perp}}{|x_1(0) - x_2(0)|^2}$$

$$\frac{dx_2(t)}{dt} = -\frac{1}{2\pi}\xi_1 \frac{(x_1(0) - x_2(0))^{\perp}}{|x_1(0) - x_2(0)|^2}.$$

Case 2: $\xi_2 \neq -\xi_1$. In this case the two vortices rotate around their center $c = \frac{\xi_1 x_1(0) + \xi_2 x_2(0)}{\Gamma}$. Indeed,

$$x_2(t) = \frac{c\Gamma - \xi_1 x_1(t)}{\xi_2}$$

$$x_1(t) - x_2(t) = \frac{\xi_2 x_1(t) - c\Gamma + \xi_1 x_1(t)}{\xi_2} = \frac{\Gamma(x_1(t) - c)}{\xi_2}$$

and similarly

$$x_2(t) - x_1(t) = \frac{\Gamma(x_2(t) - c)}{\xi_1}$$

hence

$$\frac{dx_1(t)}{dt} = \frac{1}{2\pi}\Gamma\frac{(x_1(t) - c)^{\perp}}{|x_1(0) - x_2(0)|^2}$$

$$\frac{dx_2(t)}{dt} = \frac{1}{2\pi}\Gamma\frac{(x_2(t) - c)^{\perp}}{|x_1(0) - x_2(0)|^2}.$$

In the case of three vortices, we may have collision. The computations here are a little bit boring but the results are very interesting. The triangle formed by the points $(x_1(t), x_2(t), x_3(t))$ plays a central role. Introduce the lenghts of the sides: $l_{ij}(t) = |x_i(t) - x_j(t)|$. With due patience, one can write relations between $l_{ij}(t)$ themselves and with other quantities of interest. These are the results obtained by these relations:

Proposition 5 *Assume*

$$\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 = 0$$

and $(x_1(0), x_2(0), x_3(0))$ *satisfies*

$$T(0) = 0$$

(namely $\xi_1\xi_2|x_1(0) - x_2(0)|^2 + \xi_1\xi_3|x_1(0) - x_3(0)|^2 + \xi_2\xi_3|x_2(0) - x_3(0)|^2 = 0$*). The the solution* $(x_1(t), x_2(t), x_3(t))$*, on any interval before possible collision, forms an auto-similar triangle:*

$$\frac{l_{12}(t)}{l_{23}(t)} = \frac{l_{12}(0)}{l_{23}(0)}, \qquad \frac{l_{23}(t)}{l_{13}(t)} = \frac{l_{23}(0)}{l_{13}(0)}$$

(hence also $\frac{l_{12}(t)}{l_{13}(t)} = \frac{l_{12}(0)}{l_{13}(0)}$*) for all* t *in that interval.*

In addition, if the triangle $(x_1(0), x_2(0), x_3(0))$ *is not equilateral or degenerate (aligned points), then there exists* $T^* \neq 0$ *such that*

$$l_{ij}(t) = l_{ij}(0)\sqrt{1 - \frac{t}{T^*}}$$

and, if $T^* > 0$*, the three points collision in finite time, otherwise, if* $T^* < 0$*, the triangle increase to infinity.*

Example 6 *One can prove that the choice*

$$\xi_1 = \xi_2 = 2, \qquad \xi_3 = -1$$

$$x_1(0) = (-1, 0), \qquad x_2(0) = (1, 0), \qquad x_3(0) = \left(1, -\sqrt{2}\right)$$

leads to collision in finite time.

Remark 7 *Continuation after collision is an open problem. One could try to investigate it by zero-noise limit, as it was successfully done for the easier system of Vlasov-Poisson point charges.*

1.7 No collision for a.e. initial condition

We present now a famous result of solvability for a.e. initial conditions, taken from [30] (on the torus) and [43], Chapter 4 (in full space). It is our first example of solvability thanks to *random initial conditions*. In full space there are a few additional difficulties (the need to prove that particles cannot move too far from their initial configurations, a fact that is true only under appropriate conditions) that we prefer to avoid. Thus we work on the unitary torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, as in [30]; here Lebesgue measure is a probability, and particle displacement is controlled a priori by the compactness of the set.

The price to work on the torus $\mathbb{T}^2$ is the non-esplicit form of the Green kernel and Biot-Savart law. Let us spend a few preliminary remarks on this topic. Since the velocity field u has to be periodic, the vorticity is necessarily zero mean:

$$\int_{\mathbb{T}^2} \omega(x)\, dx = \int_{\mathbb{T}^2} \nabla^{\perp} \cdot u(x)\, dx = -\int_{\mathbb{T}^2} u(x) \cdot \nabla^{\perp} 1 dx = 0.$$

Writing expressions in Fourier form is not difficult (but not sufficient for our purposes). Given a zero-mean function $\omega \in L^2\left(\mathbb{T}^2\right)$, write its Fourier series

$$\omega(x) = \sum_{k \in \mathbb{Z}_0^2} \omega_k e^{2\pi i k \cdot x}$$

in the sense of $L^2\left(\mathbb{T}^2\right)$-convergence, where $\omega_k = \int_{\mathbb{T}^2} \omega(x)\, e^{-2\pi i k \cdot x} dx$. Define

$$\begin{aligned} \varphi(x|\omega) &= -\frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2} \omega_k e^{2\pi i k \cdot x} \\ u(x|\omega) &= -\frac{1}{2\pi} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2} k^{\perp} \omega_k e^{2\pi i k \cdot x}. \end{aligned}$$

We have

$$\begin{aligned} \Delta \varphi(x|\omega) &= \omega(x) \\ u(x|\omega) &= \nabla^{\perp} \varphi(x|\omega) \\ \operatorname{div} u(x|\omega) &= 0 \\ \nabla^{\perp} \cdot u(x|\omega) &= \omega(x). \end{aligned}$$

This defines, in Fourier, the velocity field associated to a vorticity field, the Biot-Savart law on $\mathbb{T}^2$. If ω is a distribution, we may extend the previous formulae; when $\omega = \delta_y - 1$

[2], for some $y \in \mathbb{T}^2$, $\omega_k = e^{-2\pi i k \cdot y}$, hence

$$\begin{aligned} G(x,y) &:= \varphi(x|\delta_y - 1) = -\frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2} e^{2\pi i k \cdot (x-y)} \\ K(x,y) &= \nabla_x^\perp G(x,y) = u(x|\delta_y - 1) = -\frac{1}{2\pi} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2} k^\perp e^{2\pi i k \cdot (x-y)}. \end{aligned}$$

One has $G(x,y) = G(x-y,0)$, $K(x,y) = K(x-y,0)$. We set

$$\begin{aligned} G(x) &= -\frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2} e^{2\pi i k \cdot x} \\ K(x) &= \nabla^\perp G(x) = -\frac{1}{2\pi} \sum_{k \in \mathbb{Z}_0^2} |k|^{-2} k^\perp e^{2\pi i k \cdot x}. \end{aligned}$$

These are periodic functions, with $G(-x) = G(x)$, $K(-x) = -K(x)$. The series defining $G(x)$ converges in $H^{1-\epsilon}(\mathbb{T}^2)$ for every $\epsilon > 0$; and in $W^{1,p}(\mathbb{T}^2)$ for every $p < 2$ (but not in $W^{1,2}(\mathbb{T}^2)$); and thus in $L^q(\mathbb{T}^2)$ for every $q < \infty$. The series defining $K(x)$ converges in $L^p(\mathbb{T}^2)$ for every $p < 2$ (but not in $L^2(\mathbb{T}^2)$). From the general theory of local regularity of elliptic equations, $G(x)$ (hence $K(x)$) is smooth outside $x = 0$ (and its periodic replicas). The behaviour at $x = 0$ is more difficult; one can prove that

$$r(x) := G(x) - \frac{1}{2\pi} \log|x|, \qquad |x|_\infty \le \frac{1}{2}$$

is smooth, because it solves in the sense of distributions, locally around $x = 0$, the equation $\Delta r = 0$. Moreover $r(-x) = r(x)$. It follows that, for

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} + R(x), \qquad |x|_\infty \le \frac{1}{2}$$

where $R(x)$ is smooth and $R(-x) = -R(x)$, which implies in particular that $R(0) = 0$.

Thus the dynamics of point vortices is given by

$$\frac{dx_i(t)}{dt} = \sum_{j \ne i} \xi_j K(x_i(t) - x_j(t)).$$

According to the ideas described in the previous sections (valid also on the torus), we want to prove

$$L(x_1(t), ..., x_N(t)) \le C$$

[2] It must be zero average; but this subtraction does not change the role of the Green kernel in representing solutions of Poisson equation $\Delta\varphi = \omega$ by convolution, for zero average fields ω

for certain initial conditions, where

$$L(x_1, ..., x_N) = -\sum_{\substack{i,j=1,...,N \\ i\neq j}} G(x_i - x_j).$$

And we have, with $L(t) = L(x_1(t), ..., x_N(t))$,

$$\begin{aligned}\frac{dL(t)}{dt} &= -\sum_{\substack{i,j=1,...,N \\ i\neq j}} \nabla G(x_i(t) - x_j(t)) \left(\frac{dx_i(t)}{dt} - \frac{dx_j(t)}{dt}\right) \\ &= -\sum_{\substack{i,j=1,...,N \\ i\neq j}} \nabla G(x_i(t) - x_j(t)) \left(\sum_{k\neq i} \xi_k \nabla^\perp G(x_i(t) - x_k(t)) - \sum_{k\neq j} \xi_k \nabla^\perp G(x_j(t) - x_k(t))\right).\end{aligned}$$

Here we see an important cancellation (its importance will be appreciated below): the term in the sum $\sum_{k\neq i}$ with $k = j$ and the term in the sum $\sum_{k\neq j}$ with $k = i$ do not contribute, because

$$\nabla G(x_i(t) - x_j(t)) \cdot \nabla^\perp G(x_i(t) - x_j(t)) = 0.$$

These are the most singular terms, since for small $|x_i(t) - x_j(t)|$ they behave like

$$\frac{1}{|x_i(t) - x_j(t)|^2}.$$

The other terms behave like

$$\frac{1}{|x_i(t) - x_j(t)|} \frac{1}{|x_i(t) - x_k(t)|}$$

with $j \neq k$, hence they are less singular when two particles approach each other. [At this stage, this explanation is not fully convincing, having in mind the triple collisions, but we shall clearly see the advantage soon.]

In order to make progresses, we need now to consider the flow map $x^0 \mapsto x(t|x^0)$. This is locally defined, when $x^0 \notin \Delta$. However, the time before collision depends on x^0 and complicate matters. To avoid these troubles, we mollify G in such a way that we have global solutions for all x^0, a smooth flow, but also equal to the original solutions if particles are not too close each other.

For every $\delta \in (0, 1)$, denote by $\log^{(\delta)}(r)$ a smooth function, defined for $r \geq 0$, such that

$$\begin{aligned}\log^{(\delta)}(r) &= \log(r) \text{ for } r \geq \delta \\ \left|\log^{(\delta)}(r)\right| &\leq C\log(r) \text{ for } r > 0 \\ \left|\frac{d}{dr}\log^{(\delta)}(r)\right| &\leq \frac{C}{r} \text{ for } r > 0\end{aligned}$$

for some constant $C > 0$. Set

$$\begin{aligned} G^{(\delta)}(x) &= \frac{1}{2\pi}\log^{(\delta)}|x| + r(x) \\ K^{(\delta)}(x) &= \nabla^{\perp}G^{(\delta)}(x) \end{aligned}$$

for $|x|_{\infty} \leq \frac{1}{2}$, periodically extended. Denote by $\left(x_1^{(\delta)}(t), ..., x_N^{(\delta)}(t)\right)$ the unique solution of

$$\frac{dx_i^{(\delta)}(t)}{dt} = \sum_{j\neq i}\xi_j K^{(\delta)}\left(x_i^{(\delta)}(t) - x_j^{(\delta)}(t)\right)$$

with given (arbitrary) initial condition.

Lemma 8 *Consider the smooth map $x^0 \mapsto x_i^{(\delta)}\left(t|x^0\right)$ in $\left(\mathbb{T}^2\right)^N$. The probability product measure Leb_{2N} on $\left(\mathbb{T}^2\right)^N$ is invariant for this map.*

Proof. It is a known fact for smooth flows that the determinant is given by the exponential of the divergence of the vector field, which here is zero, hence the determinand is identically equal to one. Hence the flow is Lebesgue measure preserving. Let us only check that the divergence is zero: it is the sum of divergences on each component $\mathbb{T}^2$, which are all equal to zero because the components have the form $\nabla^{\perp}\varphi^{(\delta)}(x)$ (apply Schwarz theorem on mixed second derivatives). ■

Similarly to above, let us introduce the function

$$L^{(\delta)}(x_1, ..., x_N) = -\sum_{\substack{i,j=1,...,N \\ i\neq j}}\left(G^{(\delta)}(x_i - x_j) - k\right)$$

where k is such that $-\left(G^{(\delta)}(x) - k\right) \geq 0$ for all $x \in \mathbb{T}^2$. Setting $L^{(\delta)}(t) = L^{(\delta)}\left(x_1^{(\delta)}(t), ..., x_N^{(\delta)}(t)\right)$, we have

$$\begin{aligned} \frac{dL^{(\delta)}(t)}{dt} &= -\sum_{\substack{i,j=1,...,N \\ i\neq j}}\nabla G^{(\delta)}\left(x_i^{(\delta)}(t) - x_j^{(\delta)}(t)\right)\cdot \\ &\quad\cdot\left(\sum_{k\neq i}\xi_k\nabla^{\perp}G^{(\delta)}\left(x_i^{(\delta)}(t) - x_k^{(\delta)}(t)\right) - \sum_{k\neq j}\xi_k\nabla^{\perp}G^{(\delta)}\left(x_j^{(\delta)}(t) - x_k^{(\delta)}(t)\right)\right). \end{aligned}$$

Again the terms in the last two sums of the form $\nabla^{\perp}G^{(\delta)}\left(x_i^{(\delta)}(t) - x_j^{(\delta)}(t)\right)$ cancel with $\nabla G^{(\delta)}\left(x_i^{(\delta)}(t) - x_j^{(\delta)}(t)\right)$. Using this estimate and the invariance of Lebesgue measure we can prove:

Lemma 9 *There exists a constant $C > 0$ such that, for all $\delta \in (0,1)$,*

$$-\int_{(\mathbb{T}^2)^N} \sup_{t\in[0,T]} \sum_{\substack{i,j=1,\dots,N \\ i\neq j}} \left(G^{(\delta)}\left(x_i^{(\delta)}\left(t|x^0\right) - x_j^{(\delta)}\left(t|x^0\right)\right) - k\right) dx^0 \le C.$$

Proof. We may summarize the last identity above in the form

$$\begin{aligned} \frac{dL^{(\delta)}(t)}{dt} &= \sum_{\substack{i,j.k=1,\dots,N \\ i\neq j, i\neq k, j\neq k}} a_{ijk} J_{i,jk}^{(\delta)}\left(x^{(\delta)}(t)\right) \\ J_{i,jk}^{(\delta)}(x) &: = \nabla G^{(\delta)}(x_i - x_j) \cdot \nabla^{\perp} G^{(\delta)}(x_i - x_k) \end{aligned}$$

for suitable coefficients a_{ijk}. Integrating in time, taking the supremum in time over $[0,T]$ and then integrating with respect to the initial condition x^0, we have

$$\int_{(\mathbb{T}^2)^N} \sup_{t\in[0,T]} L^{(\delta)}(t)\, dx^0 \le \int_{(\mathbb{T}^2)^N} \left| L^{(\delta)}(0)\right| dx^0 + C \sum_{\substack{i,j.k=1,\dots,N \\ i\neq j, i\neq k, j\neq k}} \int_0^T \left(\int_{(\mathbb{T}^2)^N} \left| J_{i,jk}^{(\delta)}\left(x^{(\delta)}\left(s|x^0\right)\right)\right| dx^0 \right) ds.$$

Now we use the most essential ingredient: the invariance of Lebesgue measure under the map $x^0 \mapsto x^{(\delta)}\left(s|x^0\right)$. This gives us

$$\int_{(\mathbb{T}^2)^N} \left| J_{i,jk}^{(\delta)}\left(x^{(\delta)}\left(s|x^0\right)\right)\right| dx^0 = \int_{(\mathbb{T}^2)^N} \left| J_{i,jk}^{(\delta)}\left(x^0\right)\right| dx^0.$$

From the properties imposed on $\log^{(\delta)}(r)$ we have

$$\left| J_{i,jk}^{(\delta)}\left(x^0\right)\right| \le \frac{C}{|x_i - x_j|\,|x_i - x_k|}$$

for some constant $C > 0$, hence

$$\int_{(\mathbb{T}^2)^N} \left| J_{i,jk}^{(\delta)}\left(x^0\right)\right| dx^0 \le C \int_{(\mathbb{T}^2)^2} \frac{1}{|x_i - x_k|} \left(\int_{\mathbb{T}^2} \frac{1}{|x_i - x_j|} dx_j \right) dx_i dx_k \le C'$$

for some constant $C' > 0$. Similarly, $\int_{(\mathbb{T}^2)^N} \left| L^{(\delta)}(0)\right| dx^0 \le C''$ for some constant $C' > 0$. In conclusion,

$$-\int_{(\mathbb{T}^2)^N} \sup_{t\in[0,T]} \sum_{\substack{i,j=1,\dots,N \\ i\neq j}} \left(G^{(\delta)}\left(x_i^{(\delta)}\left(t|x^0\right) - x_j^{(\delta)}\left(t|x^0\right)\right) - k\right) dx^0 \le C$$

for some constant $C > 0$. ■

Remark 10 *Without the cancellation of the most singular terms, we would have also*

$$\int_{(\mathbb{T}^2)^2} \frac{1}{|x_i - x_j|^2} dx_i dx_j = +\infty.$$

Theorem 11 *For Lebesgue a.e. initial condition $x^0 \in \Delta^c$, there is no collision and the solution is global and unique.*

Proof. Denote by $d_T^{(\delta)}(x^0)$ the minimal distance between vortices of the smoothed system, starting from x^0, over $[0,T]$. Then

$$\begin{aligned} d_T^{(\delta)}(x^0) &< \delta \Longleftrightarrow \exists t \in [0,T], \exists i \neq j : \left| x_i^{(\delta)}(t|x^0) - x_j^{(\delta)}(t|x^0) \right| < \delta \\ &\Longrightarrow - \sup_{t\in[0,T]} \sum_{\substack{i,j=1,\dots,N \\ i\neq j}} \left(G^{(\delta)}\left(x_i^{(\delta)}(t|x^0) - x_j^{(\delta)}(t|x^0) \right) - k \right) \\ &> -\left(G^{(\delta)}(\delta) - k \right) = -\frac{1}{2\pi}\log\delta - r(\delta) \end{aligned}$$

hence

$$\begin{aligned} & Leb_{2N}\left\{ x^0 \in (\mathbb{T}^2)^N : d_T^{(\delta)}(x^0) < \delta \right\} \\ \leq\ & Leb_{2N}\left\{ x^0 \in (\mathbb{T}^2)^N : - \sup_{t\in[0,T]} \sum_{\substack{i,j=1,\dots,N \\ i\neq j}} \left(G^{(\delta)}\left(x_i^{(\delta)}(t|x^0) - x_j^{(\delta)}(t|x^0) \right) - k \right) > -\frac{1}{2\pi}\log\delta - r(\delta) \right\} \\ \leq\ & \frac{C}{-\frac{1}{2\pi}\log\delta - r(\delta)} \end{aligned}$$

where we have used Chebyshev inequality and the lemma, and have assumed δ so small that $-\frac{1}{2\pi}\log\delta - r(\delta) > 0$. Thus, for a very large (in the sense of Lebesgue measure) set of initial conditions $d_T^{(\delta)}(x^0) \geq \delta$, which means that $x^{(\delta)}(t|x^0) = x(t|x^0)$ and no collision occurs. This property is true for a.e. initial conditions, by the arbitrariety of δ. ∎

Exercise 12 *Show that the map $x^0 \mapsto x(t|x^0)$ defined a.e. by the previous theorem is measurable and preserves Lebesgue measure.*

Remark 13 *There are other results in the literature similar to the theorem, for other systems, see for instance [1] that was previous to the investigations on vortex systems (but does not cover them). Celestial mechanics is also a subject where these tools have been considered.*

1.8 Weak form of Euler equations for point vortices

This section aims to answer the following natural question. Assume the point vortex dynamics is well posed; define the associated vorticity and velocity fields as

$$\begin{aligned} \omega_t(dx) &= \sum_{i=1}^{n} \xi_i \delta_{x_i(t)} \\ u_t(x) &= \sum_{i=1}^{n} \xi_i K(x, x_i(t)). \end{aligned}$$

Can we say that the pair (u, ω) satisfies the vorticity formulation

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

of the 2D Euler equations in some suitable weak sense?

The solution $\omega_t(dx)$ is a special example of measure-valued solutions and thus let us discuss the problem at this level of generality: we want to define vorticity-measure-valued solutions of the 2D Euler equation. Being $\omega_t(dy)$ a measure, the velocity is given by

$$u_t(x) = \int K(x, y)\, \omega_t(dy).$$

Since boundary conditions matter here, we simplify and discuss the case in full space; thus

$$K(x, y) = \frac{1}{2\pi} \frac{(x-y)^{\perp}}{|x-y|^2}.$$

Let $\phi \in C_c^{\infty}(\mathbb{R}^2)$ be a test function. If ω were a regular solution, still denoting the associated measure by $\omega_t(dx)$ we would have

$$\int \phi(x)\, \omega_t(dx) - \int \phi(x)\, \omega_0(dx) = -\int_0^t \int \phi(x)\, u_s(x) \cdot \nabla \omega_s(x)\, dx ds$$

where however the last term requires a definition. By Gauss-Green formulae (boundary terms do not appear because ϕ is compact support) the last term is equal to

$$= \int_0^t \int \omega_s(x)\, u_s(x) \cdot \nabla \phi(x)\, dx ds$$

which can be interpreted as

$$= \int_0^t \int u_s(x) \cdot \nabla \phi(x)\, \omega_s(dx)\, ds.$$

This expression is not well defined, in the direction of point particles that we are interested in: if $\omega_t(dx)$ contains a term of the form $\delta_{x_i(t)}$, then $u_s(x)$ contans a term of the form $K(x, x_i(t))$ which diverges precisely at $x_i(t)$.

We may further write

$$= \int_0^t \int \int K(x,y) \cdot \nabla \phi(x) \, \omega_s(dy) \, \omega_s(dx) \, ds.$$

Now comes the basic trick, common for instance to the so-called gradient systems in the theory of particle systems [37]. Just renaming x by y and y by x and using the property

$$K(y,x) = -K(x,y)$$

we have

$$\begin{aligned} \int \int K(x,y) \cdot \nabla \phi(x) \, \omega_s(dy) \, \omega_s(dx) &= \int \int K(y,x) \cdot \nabla \phi(y) \, \omega_s(dx) \, \omega_s(dy) \\ &= -\int \int K(x,y) \cdot \nabla \phi(y) \, \omega_s(dy) \, \omega_s(dx) \end{aligned}$$

which implies that

$$\int_0^t \int \int K(x,y) \cdot \nabla \phi(x) \, \omega_s(dy) \, \omega_s(dx) \, ds = \int_0^t \int \int K(x,y) \cdot \frac{\nabla \phi(x) - \nabla \phi(y)}{2} \omega_s(dy) \, \omega_s(dx) \, ds.$$

The advantage is that the function

$$H_\phi(x,y) := K(x,y) \cdot \frac{\nabla \phi(x) - \nabla \phi(y)}{2} \qquad \text{for } x \neq y$$

is bounded, because $|\nabla \phi(x) - \nabla \phi(y)| \sim |x-y|$ at small distances. We have got the indentity

$$\int \phi(x) \, \omega_t(dx) - \int \phi(x) \, \omega_0(dx) = \int_0^t \int \int H_\phi(x,y) \, \omega_s(dy) \, \omega_s(dx) \, ds$$

as a potentially interesting definition. When $\omega_t(dx)$ is a diffuse measure, the boundedness of $H_\phi(x,y)$ implies that the lat term is meanigful. However, when $\omega_t(dx)$ contains a term of the form $\delta_{x_i(t)}$, the product measure $\omega_s(dy) \, \omega_s(dx)$ contains a concentrated mass on the diagonal $x = y$, where $H_\phi(x,y)$ is not well defined. Let us make the following choice:

$$H_\phi(x,y) := \begin{cases} K(x,y) \cdot \frac{\nabla \phi(x) - \nabla \phi(y)}{2} & \text{for } \; x \neq y \\ 0 & \text{for } \; x = y \end{cases}.$$

The "zero" on the diagonal corresponds to the cancellation of the self-interacting term in the dynamics of point particles (naemly to the fact that we sum the contributions on particle i only from particles $j \neq i$).

Definition 14 *A time-dependent finite signed measure $\omega_t(dx)$, continuous in time in the weak topology, is a measure-valued solution of Euler equations if*

$$\int \phi(x)\,\omega_t(dx) - \int \phi(x)\,\omega_0(dx) = \int_0^t \int\int H_\phi(x,y)\,\omega_s(dy)\,\omega_s(dx)\,ds$$

for all $\phi \in C_c^\infty(\mathbb{R}^2)$, where $H_\phi(x,y)$ is defined above (with value zero on the diagonal).

Proposition 15 *Let $(x_1(t),...,x_n(t))$, $t \in [0,T]$ be a trajectory in $\mathbb{R}^{2n}\backslash\Delta$ and let $\xi_1,...,\xi_n$ being non-zero real numbers. It is a solution of the point vortex dynamics if and only if $\omega_t(dx) = \sum_{i=1}^n \xi_i \delta_{x_i(t)}$ is a measure-valued solution of Euler equations.*

Proof. If $(x_1(t),...,x_n(t))$, $t \in [0,T]$ is a solution outside the diagonal Δ for the point vortex dynamics and $\phi \in C_c^\infty(\mathbb{R}^2)$, then

$$\sum_{i=1}^n \xi_i \phi(x_i(t)) - \sum_{i=1}^n \xi_i \phi(x_i(0)) = \sum_{i=1}^n \xi_i \int_0^t \left(\nabla\phi(x_i(s)) \sum_{j\neq i}^n \xi_j K(x_i(s), x_j(s)) \right) ds.$$

Defined $\omega_t(dx) = \sum_{i=1}^n \xi_i \delta_{x_i(t)}$, the term $\sum_{i=1}^n \xi_i \phi(x_i(t))$ is $\int \phi(x)\,\omega_t(dx)$, also at time $t=0$. Moreover, since (by $K(y,x) = -K(x,y)$)

$$\begin{aligned}\sum_{i=1}^n \sum_{j\neq i}^n \nabla\phi(x_i(s))\,\xi_i\xi_j K(x_i(s),x_j(s)) &= \sum_{j=1}^n \sum_{i\neq j}^n \nabla\phi(x_j(s))\,\xi_j\xi_i K(x_j(s),x_i(s)) \\ &= -\sum_{i=1}^n \sum_{j\neq i}^n \nabla\phi(x_j(s))\,\xi_i\xi_j K(x_i(s),x_j(s))\end{aligned}$$

one has

$$\begin{aligned}\sum_{i=1}^n \sum_{j\neq i}^n \nabla\phi(x_i(s))\,\xi_i\xi_j K(x_i(s),x_j(s)) &= \sum_{i=1}^n \sum_{j\neq i}^n \frac{\nabla\phi(x_i(s)) - \nabla\phi(x_j(s))}{2}\xi_i\xi_j K(x_i(s),x_j(s)) \\ &= \sum_{i=1}^n \sum_{j\neq i}^n \xi_i\xi_j H_\phi(x_i(s),x_j(s)) \\ &= \sum_{i,j=1}^n \xi_i\xi_j H_\phi(x_i(s),x_j(s))\end{aligned}$$

because H_ϕ is zero on the diagonal

$$= \int\int H_\phi(x,y)\,\omega_s(dy)\,\omega_s(dx).$$

Thus we have proved that $\omega_t(dx) = \sum_{i=1}^{n} \xi_i \delta_{x_i(t)}$ is a measure-valued solution of Euler equations.

Conversely, from the property that $\omega_t(dx) = \sum_{i=1}^{n} \xi_i \delta_{x_i(t)}$ is a measure-valued solution of Euler equations, one gets that

$$\sum_{i=1}^{n} \xi_i \phi(x_i(t)) - \sum_{i=1}^{n} \xi_i \phi(x_i(t_0)) = \sum_{i=1}^{n} \xi_i \int_{t_0}^{t} \left(\nabla \phi(x_i(s)) \sum_{j \neq i}^{n} \xi_j K(x_i(s), x_j(s)) \right) ds$$

hods true for all $\phi \in C_c^\infty(\mathbb{R}^2)$ and all $t, t_0 \in [0, T]$. Let us prove that $x_1(t)$ satisfies the first equation of the point vortex system; the proof for the others is the same. Given $t_0 \in [0, T]$, there is a small neighbour Υ of t_0 in $[0, T]$ and a radius $r > 0$ with the properties that $x_1(t) \in B(x_1(t_0), r)$ for $t \in \Upsilon$ and $x_i(t) \notin B(x_1(t_0), 2r)$ for $t \in \Upsilon$ and $i \neq 1$. And, given any coordinate $k = 1, 2$ there is $\phi \in C_c^\infty(\mathbb{R}^2)$ such that $\phi(x) = x^{(k)}$ (the $k-$th coordinate of $x \in \mathbb{R}^2$) in $B(x_1(t_0), r)$ and $\phi(x) = 0$ outside $B(x_1(t_0), 2r)$. With this test function we get, for $t \in \Upsilon$, in the identity above,

$$\xi_1 x_1^{(k)}(t) - \xi_1 x_1^{(k)}(t_0) = \xi_1 \int_{t_0}^{t} \left(e_k \cdot \sum_{j \neq i}^{n} \xi_j K(x_i(s), x_j(s)) \right) ds$$

where e_k is the $k-$th vector of the canonical basis of $\mathbb{R}^2$. Then

$$x_1(t) - x_1(t_0) = \int_{t_0}^{t} \left(\sum_{j \neq i}^{n} \xi_j K(x_i(s), x_j(s)) \right) ds$$

which implies, by the arbitrariety of t_0, that $x_1(t)$ satisfies the first equation of the point vortex system. ■

Chapter 2

Gaussian Measures

2.1 Introduction

We move not to investigate deterministic Partial Differential Equations (PDEs) with random initial conditions. In the case of a finite number of point vortices, the space of configurations is finite dimensional and Lebesgue measure is sufficient to develop the theory. For PDEs, configurations are fields, no more particles, and thus we need measures on spaces of fields. For the time being we limit ourselves to Gaussian measures in Hilbert spaces, the most natural surrogate of Lebesgue measure. Strictly speaking, these Gaussian measures do not even have invariance by rotation (opposite to the standard Gaussian vector in $\mathbb{R}^d$), but we shall introduce particular measures - especially the white noise - that are rotation invariant in a suitable weak sense.

Strange enough at first sight, these measure are central in the investigation of 2D fluids, although these are nonlinear dynamics (Gaussianity is preserved by linear transformations, hence it is usually associated to linear problems). Experiments of turbulence in 2D revealed that deviation from Gaussianity is extremely small, if any, hence Gaussian statistics are relevant. A part from our central topic of 2D Euler equations, they are also relevant for dispersive problems and Burgers equations. Also in the abstract theory of generalized flows in Hilbert spaces, they play a basic role as a reference measure.

2.2 Gaussian measures

2.2.1 Definition and properties

Generalities on Gaussian measures in Banach and Hilbert spaces cn be found in several books, see for instance [11], [21], [24], [39].

Let H be a separable Hilbert space, with inner product $\langle .,. \rangle$ and norm $\|\cdot\|$, endowed with the Borel σ-field $\mathcal{B}(H)$. Denote by π_h the projector $x \mapsto \left\langle x, \frac{h}{|h|} \right\rangle$ in H.

Definition 16 *We say that a probability measure μ on $(H, \mathcal{B}(H))$ is Gaussian if*

$$(\pi_h)_\sharp \mu$$

is a Gaussian measure on $\mathbb{R}$ for every $h \in H \backslash \{0\}$. A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in H is called a Gaussian vector in H if its law on H is a Gaussian probability measure, or equivalently if the real-valued random variable

$$\langle X, h \rangle$$

is Gaussian for every $h \in H$.

Definition 17 *If μ is a Gaussian probability measure on $(H, \mathcal{B}(H))$, law of a Gaussian vector X,*

i) the element $m \in H$ defined by the identity

$$\langle m, h \rangle = \int_H \langle x, h \rangle \mu(dx) = \mathbb{E}[\langle X, h \rangle]$$

is called mean of μ

ii) the linear operator Q in H defined by the identity

$$\langle Qh, k \rangle = \int_H \langle x, h \rangle \langle x, k \rangle \mu(dx) = \mathbb{E}[\langle X, h \rangle \langle X, k \rangle]$$

is called covariance operator of μ; the same terminology will be applied to Gaussian random variables.

The definition is meaningful because one can prove that, given a Gaussian probability measure μ on $(H, \mathcal{B}(H))$, $m \in H$ and a linear bounded operator $Q : H \to H$ exist and are unique, with the previous properties. We do not prove this claim, as well as the following one (since we usually construct the measures, see below):

Proposition 18 *Q is non-negative selfadjoint; and trace class, namely*

$$\sum_{i=1}^{\infty} \langle Qe_i, e_i \rangle < \infty$$

for every complete orthonormal system $\{e_i\}$ of H.

Remark 19 *As a consequence, one can see that Q is compact. If $\{e_i\}$ is made of eigenvectors of Q, with eigenvalues $\{\sigma_i^2\}$ (such a basis exists because Q is compact self-adjoint), then*

$$\sum_{i=1}^{\infty} \langle Qe_i, e_i \rangle = \sum_{i=1}^{\infty} \sigma_i^2$$

(called trace of Q).

Definition 20 *Given a Gaussian r.v. X from $(\Omega, \mathcal{F}, \mathbb{P})$ to H, called Q the covariance of the law of X, taken a complete orthonormal system $\{e_i\}$ of H made of eigenvectors of Q, with eigenvalues $\{\sigma_i^2\}$, we call Karhunen–Loève expansion of X the formula*

$$X = \sum_{i=1}^{\infty} \langle X, e_i \rangle e_i = \sum_{i=1}^{\infty} \sigma_i G_i e_i$$

where $G_i = \langle X, e_i \rangle / \sigma_i$ when $\sigma_i \neq 0$ (otherwise set $Y_i = 0$).

The definition is meaningul and interesting because:

Proposition 21 *The real valued Gaussian variables $\langle X, e_i \rangle$ (resp. Y_i), $i \in \mathbb{N}$ are independent, with variance σ_i^2 (resp. one) and the series converges in $L^2(\Omega; H)$.*

Proof. We just sketch the convergence in $L^2(\Omega; H)$: since $\mathbb{E}[G_i G_j] = \mathbb{E}[G_i]\mathbb{E}[G_j] = 0$ when $i \neq j$,

$$\begin{aligned} \mathbb{E}\left[\left\|\sum_{i=n+1}^{m} \sigma_i G_i e_i\right\|^2\right] &= \sum_{i,j=n+1}^{m} \sigma_i \sigma_j \mathbb{E}[G_i G_j] \langle e_i, e_j \rangle \\ &= \sum_{i=n+1}^{m} \sigma_i^2 \langle e_i, e_i \rangle = \sum_{i=n+1}^{m} \sigma_i^2 < \infty. \end{aligned}$$

Hence $\sum_{i=1}^{n} \sigma_i G_i e_i$ is a Cauchy sequence in $L^2(\Omega; H)$, that is complete. ■

The precise definition of Karhunen–Loève expansion is not essential in itself later on but it expresses a crucial decomposition that will be used below also in reverse order, namely to *construct* Gaussian measures in Hilbert spaces starting from a basis and a sequence of independent standard varieblеs.

2.2.2 Construction of Gaussian measures

We prove the converse of the facts stated above:

Theorem 22 *Given $m \in H$ and a non-negative selfadjoint and trace class linear operator $Q : H \to H$, there exists a (unique) Gaussian probability measure on $(H, \mathcal{B}(H))$ with m and Q as mean and covariance operator.*

The measure can be constructed as follows: taken a complete orthonormal system $\{e_i\}$ of H made of eigenvectors of Q, with eigenvalues $\{\sigma_i^2\}$, so that $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, taken a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a sequence of real valued standard Gaussian random variables $\{G_i\}$ (see the Appendix), the series

$$X = m + \sum_{i=1}^{\infty} \sigma_i G_i e_i$$

converges in $L^2(\Omega; H)$ to a Gaussian vector X with mean m and covariance operator Q.

Proof. We limit the details to the case $m = 0$, without restriction of the difficulties. Let $X_n = \sum_{i=1}^{n} \sigma_i G_i e_i$; it is very easy to check that X_n are Gaussian vectors. Let us prove that $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega; H)$. For $m > n$ we have

$$\begin{aligned} \mathbb{E}\left[\|X_m - X_n\|_H^2\right] &= \mathbb{E}\left[\left\|\sum_{i=n+1}^{m} \sigma_i G_i e_i\right\|^2\right] = \sum_{i,j=n+1}^{m} \sigma_i \sigma_j \mathbb{E}\left[G_i G_j\right] \langle e_i, e_j\rangle \\ &= \sum_{i=n+1}^{m} \sigma_i^2 \langle e_i, e_i\rangle = \sum_{i=n+1}^{m} \sigma_i^2 < \infty. \end{aligned}$$

Hence, since $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$, we deduce that $\{X_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(\Omega; H)$ and thus converges to some random vector $X \in L^2(\Omega; H)$. For every $h \in H$, we have as a consequence

$$\langle X, h\rangle = L^2(\Omega)\text{-}\lim_{n\to\infty} \langle X_n, h\rangle .$$

Hence we deduce that $\langle X, h\rangle$ is Gaussian, for every $h \in H$ (hence X is Gaussian), mean zero, and also that

$$\begin{aligned} \mathbb{E}\left[\langle X, h\rangle \langle X, k\rangle\right] &= \lim_{n\to\infty} \mathbb{E}\left[\langle X_n, h\rangle \langle X_n, k\rangle\right] = \lim_{n\to\infty} \sum_{i,j=1}^{n} \sigma_i \sigma_j \mathbb{E}\left[G_i G_j\right] \langle e_i, h\rangle \langle e_j, k\rangle \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sigma_i^2 \langle e_i, h\rangle \langle e_i, k\rangle . \end{aligned}$$

Now, it is a simple exercise to show that this limit is equal to

$$\langle Qh, k\rangle$$

where Q can be expressed as $Qh = \sum_{i=1}^{\infty} \sigma_i^2 \langle e_i, h\rangle e_i$. This proves that Q is the covariance operator. ■

Remark 23 *Strange enough, we had two ways to prove that the mixed terms were equal to zero: if $i \neq j$*

$$\mathbb{E}\left[G_i G_j\right] = \mathbb{E}\left[G_i\right] \mathbb{E}\left[G_j\right] = 0$$

but also

$$\langle e_i, e_j\rangle = 0.$$

This abundance of cancellations is suspicious. The reason is that Gaussian meausures have much stronger convergence properties; just the convergence in $L^2(\Omega; H)$ is, in a sense, too easy and follows from more than one argument. We shall see the additional properties much later in the lectures, when dealing with dispersive equations.

Remark 24 *In the computation of* $\mathbb{E}\left[\langle X,h\rangle\langle X,k\rangle\right]$ *the coefficients* σ_i^2 *are not needed anymore to make the series convergent. Even without them, the limit* $\lim_{n\to\infty}\sum_{i=1}^n\langle e_i,h\rangle\langle e_i,k\rangle$ *exists, equal to* $\langle h,k\rangle$*. This fact is also very strange, a priori: the coefficients* σ_i^2 *are essential sometimes, no more sometime else. We shall see below soon a formalization of this fact.*

2.3 General Gaussian measures on the torus

2.3.1 Elements of Fourier analysis on the torus

Consider the torus $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$. Although not always strictly necessary, in the sequel we consider only zero-average functions on $\mathbb{T}^d$, to avoid troubles sometimes (for instance when we solve $\Delta f=g$, g has to be zero average; when we define Sobolev spaces, we want to use the multiplier $|k|^{2\alpha}$ also for negative α without restrictions). When we give a name to a space, like $L^2\left(\mathbb{T}^d;\mathbb{C}\right)$, we tacitly assume it is restricted to zero-average functions. The wave numbest then will be restricted to

$$\mathbb{Z}_0^d:=\mathbb{Z}^d\backslash\{0\}.$$

Recall that a complete orthonormal system in the complex Hilbert space $L^2\left(\mathbb{T}^d;\mathbb{C}\right)$ (with the inner product $\langle f,g\rangle_{L^2\left(\mathbb{T}^d;\mathbb{C}\right)}=\int_{\mathbb{T}^d}f(x)\overline{g(x)}dx$) is given by the functions $e_k(x)=e^{2\mathbb{Z}_0^d\pi ik\cdot x}$, $k\in$.

A complex-valued function $f\in L^2\left(\mathbb{T}^d;\mathbb{C}\right)$, developed in series $f(x)=\sum_{k\in\mathbb{Z}_0^d}\widehat{f}(k)e_k(x)$, $\widehat{f}(k)=\langle f,e_k\rangle_{L^2\left(\mathbb{T}^d;\mathbb{C}\right)}$ is real valued if and only if $\widehat{f}(-k)=\overline{\widehat{f}(k)}$. The space of such functions is the Hilbert space $L^2\left(\mathbb{T}^d\right)$, with the inner product $\langle f,g\rangle=\int_{\mathbb{T}^d}f(x)g(x)dx$. We may split $\mathbb{Z}_0^d$ in two parts

$$\mathbb{Z}_0^d=\Lambda\cup(-\Lambda)$$

where $\Lambda\subset\mathbb{Z}_0^d$ and $-\Lambda$ are disjoint. Then, using $\widehat{f}(-k)=\overline{\widehat{f}(k)}$, we have

$$\begin{aligned}\widehat{f}(k)e_k(x)+\widehat{f}(-k)e_{-k}(x) &= \widehat{f}(k)e_k(x)+\overline{\widehat{f}(k)e_k(x)}=2\operatorname{Re}\left(\widehat{f}(k)e_k(x)\right)\\ &= 2\operatorname{Re}\widehat{f}(k)\cos(2\pi ik\cdot x)-2\operatorname{Im}\widehat{f}(k)\sin(2\pi ik\cdot x)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\widehat{f}(k) &= \int_{\mathbb{T}^d}f(x)\cos(2\pi ik\cdot x)\,dx\\ \operatorname{Im}\widehat{f}(k) &= \int_{\mathbb{T}^d}f(x)\sin(2\pi ik\cdot x)\,dx.\end{aligned}$$

Using these facs and a few additional computations one can check that a complete orthonormal system of the real space $L^2\left(\mathbb{T}^d\right)$ is

$$e_k(x) = \begin{cases} \cos(2\pi i k \cdot x) & \text{for} \quad k \in \Lambda \\ \sin(2\pi i k \cdot x) & \text{for} \quad k \in -\Lambda \end{cases}$$

with $e_0 = 1$.

We may introduce Hilbert subspaces of H as follows: for every $\alpha > 0$, we set

$$H^\alpha := W^{\alpha,2}\left(\mathbb{T}^d\right) := \left\{ f \in L^2\left(\mathbb{T}^d\right) : \sum_{k\in\mathbb{Z}_0^d} |k|^{2\alpha} \left|\widehat{f}(k)\right|^2 < \infty \right\}$$

with inner product

$$\langle f, g \rangle_{H^\alpha} = \sum_{k\in\mathbb{Z}_0^d} |k|^{2\alpha} \widehat{f}(k) \overline{\widehat{g}(k)}.$$

Lemma 25 *The linear operator Λ^α defined as*

$$\Lambda^\alpha f = \sum_{k\in\mathbb{Z}_0^d} |k|^\alpha \widehat{f}(k) e_k$$

is an isomorphism between $W^{\alpha,2}\left(\mathbb{T}^d\right)$ and $L^2\left(\mathbb{T}^d\right)$; and more generally [1] *between*

$$\Lambda^\alpha : W^{\alpha+\beta,2}\left(\mathbb{T}^d\right) \to W^{\beta,2}\left(\mathbb{T}^d\right).$$

In particular

$$\langle f, g \rangle_{H^\alpha} = \langle \Lambda^\alpha f, \Lambda^\alpha g \rangle \tag{2.1}$$

Let us identify a function $f \in L^2\left(\mathbb{T}^d\right)$ with the sequence of its Fourier coefficients $\left(\widehat{f}(k)\right) \in \mathbb{C}^{\mathbb{Z}^d}$. Under this identification, we can write

$$H^\alpha := W^{\alpha,2}\left(\mathbb{T}^d\right) := \left\{ \{\xi(k)\}_{k\in\mathbb{Z}_0^d} \in \mathbb{C}^{\mathbb{Z}^d} : \xi(-k) = \overline{\xi(k)} \text{ and } \sum_{k\in\mathbb{Z}_0^d} |k|^{2\alpha} |\xi(k)|^2 < \infty \right\}.$$

This definition is meaningful also for negative α, so we adopt it for all

$$\alpha \in \mathbb{R}.$$

Introducing distributions on $\mathbb{T}^d$, performing Fourier analysis on distributions, identifying $L^2\left(\mathbb{T}^d\right)$ with a certain natural subset of distributions, one can define spaces of distributions

[1] By little abuse of notation, we denote in the same way the operator independently of the spaces between it acts.

which correspond to $W^{\alpha,2}\left(\mathbb{T}^d\right)$ with negative α, under the correspondence between spaces of functions and spaces of sequences described above. We write $L^2\left(\mathbb{T}^d\right)$ for $W^{0,2}\left(\mathbb{T}^d\right)$, in the spaces of sequences.

The operator Λ^α may be defined, for every $\alpha \in \mathbb{R}$, as acting on any space of sequences $W^{\alpha+\beta,2}\left(\mathbb{T}^d\right)$. Simply, it maps a sequence $\{\xi(k)\}_{k\in\mathbb{Z}_0^d} \in W^{\alpha+\beta,2}\left(\mathbb{T}^d\right)$ into the sequence $\{|k|^\alpha \xi(k)\}_{k\in\mathbb{Z}_0^d} \in W^{\beta,2}\left(\mathbb{T}^d\right)$.

Lemma 26 *All the results of Lemma 25 remain true for every $\alpha, \beta \in \mathbb{R}$.*

As an exercise one can also prove:

Theorem 27 *For every $\beta \in \mathbb{R}$ and $\alpha > 0$, the embedding $W^{\alpha+\beta,2}\left(\mathbb{T}^d\right) \subset W^{\beta,2}\left(\mathbb{T}^d\right)$ is compact. Moreover, the operator $\Lambda^{-\alpha}$, considered as an operator in $W^{\beta,2}\left(\mathbb{T}^d\right)$, is compact.*

For every $\alpha \in \mathbb{R}$, let us describe a complete orthonormal system of $W^{\alpha,2}\left(\mathbb{T}^d\right)$.

Lemma 28 *Given $\alpha \in \mathbb{R}$, the sequence*

$$\{f_k\} = \left\{|k|^{-\alpha} e_k\right\} = \left\{\Lambda^{-\alpha} e_k\right\}$$

is a complete orthonormal system in $W^{\alpha,2}\left(\mathbb{T}^d\right)$.

Proof. We limit ourselves to notice that

$$\left\langle f_\eta, f_{\eta'}\right\rangle_{W^{\alpha,2}\left(\mathbb{T}^d\right)} = \left\langle \Lambda^\alpha f_\eta, \Lambda^\alpha f_{\eta'}\right\rangle = \left\langle e_\eta, e_{\eta'}\right\rangle = \delta_{\eta,\eta'}.$$

■

Remark 29 *Let us explain better some notations used above. For negative α, e_k corresponds to a distribution (in the identification of $L^2\left(\mathbb{T}^d\right)$ with a certain natural subset of distributions) and thus to an element of the space of sequences $W^{\alpha,2}\left(\mathbb{T}^d\right)$, the sequence equal to zero except for the k-position, where it is equal to 1. That sequence in not $W^{\alpha,2}\left(\mathbb{T}^d\right)$-norm one. We set again $f_k := |k|^{-\alpha} e_k$, also for negative α, understanding with f_k the sequence equal to zero except for the k-position, where it is equal to $|k|^{-\alpha}$. As above, one can check that $\{f_k\}$ is a complete orthonormal system in $W^{\alpha,2}\left(\mathbb{T}^d\right)$.*

2.3.2 Examples of Gaussian measures

After these definitions, let us define a Gaussian measure in $L^2\left(\mathbb{T}^d\right)$. In the sequel we mix the general notations above about Gaussian measures with those used here for the Fourier decomposition on the torus; in particular we replace the indexes in $\mathbb{N}$ by indexes in $\mathbb{Z}^d$; nothing changes, except notations, because $\mathbb{Z}^d$ is countable.

If $\alpha > \frac{d}{2}$, then $\sum_{k\in\mathbb{Z}_0^d} |k|^{-2\alpha} < \infty$. From Theorem 22 we have:

Proposition 30 *Let $\{G_k\}_{k\in\mathbb{Z}^d}$ be a sequence of i.i.d. standard Gaussian variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume*

$$\alpha > \frac{d}{2}$$

and take

$$\sigma_k = |k|^{-\alpha}$$

The series

$$X = \sum_{k\in\mathbb{Z}_0^d} \sigma_k G_k(\omega) e_k(x)$$

converges in $L^2\left(\Omega; L^2\left(\mathbb{T}^d\right)\right)$ to a Gaussian random variable with values in $L^2\left(\mathbb{T}^d\right)$. The law of X is a Gaussian probability measure on $L^2\left(\mathbb{T}^d\right)$, with mean zero and covariance operator

$$Qh = \sum_{k\in\mathbb{Z}_0^d} \sigma_k^2 \langle h, e_k\rangle e_k \qquad h \in L^2\left(\mathbb{T}^d\right).$$

It is useful to generalize this example to any space $W^{\alpha,2}\left(\mathbb{T}^d\right)$ in place of $L^2\left(\mathbb{T}^d\right)$ (especially with negative exponent α). Take any $\alpha \in \mathbb{R}$. Recall that $\{f_k\} = \left\{|k|^{-\alpha} e_k\right\}$ is a complete orthonormal system of $W^{\alpha,2}\left(\mathbb{T}^d\right)$.

Proposition 31 *Given any $\alpha \in \mathbb{R}$, defined $\{f_k\} = \left\{|k|^{-\alpha} e_k\right\}$, taken a sequence $\{G_k\}_{k\in\mathbb{Z}_0^d}$ of i.i.d. standard Gaussian variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the numbers*

$$\sigma_k = |k|^{-\alpha}$$

for some

$$\alpha > \frac{d}{2}$$

the series

$$X = \sum_{k\in\mathbb{Z}_0^d} \sigma_k G_k(\omega) f_k$$

converges in $L^2\left(\Omega; W^{\alpha,2}\left(\mathbb{T}^d\right)\right)$ and defines a Gaussian random variable in $W^{\alpha,2}\left(\mathbb{T}^d\right)$. The law of X is a Gaussian probability measure on $W^{\alpha,2}\left(\mathbb{T}^d\right)$, with mean zero and covariance operator $Q : W^{\alpha,2}\left(\mathbb{T}^d\right) \to W^{\alpha,2}\left(\mathbb{T}^d\right)$ given by

$$Qv = \sum_{k\in\mathbb{Z}_0^d} \sigma_k^2 \langle v, f_k\rangle_{W^{\alpha,2}(\mathbb{T}^d)} f_k \qquad v \in W^{\alpha,2}\left(\mathbb{T}^d\right).$$

2.4 White noise and Wiener integral on $\mathbb{T}^d$

Central to our discussion are two Gaussian measures on certain Sobolev spaces $W^{\alpha,2}\left(\mathbb{T}^d\right)$. In this section we extensively use the abbreviation H^α for $W^{\alpha,2}\left(\mathbb{T}^d\right)$.

Let us start with *white noise* on $\mathbb{T}^d$. The idea is to take the Fourier basis (e_k) described above, a sequence $\{G_k\}_{k\in\mathbb{Z}_0^d}$ of i.i.d. standard Gaussian variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and try to understand the convergence of the series

$$X = \sum_{k\in\mathbb{Z}_0^d} G_k(\omega)\, e_k.$$

Convergence of this series is understood here as the convergence in some topology of the finite sums

$$X_N = \sum_{k\in\mathbb{Z}_0^d, |k|\leq N} G_k(\omega)\, e_k.$$

We have

$$X_N = \sum_{k\in\mathbb{Z}_0^d, |k|\leq N} \sigma_k G_k(\omega)\, f_k$$

where

$$f_k = |k|^\alpha\, e_k, \qquad \sigma_k = |k|^{-\alpha}.$$

If we choose any $\alpha > \frac{d}{2}$, the condition $\sum_{k\in\mathbb{Z}_0^d}\sigma_k^2 < \infty$ is fulfilled and we have convergence of X_N to a well defined Gaussian random variable X in $L^2\left(\Omega; H^{-\alpha}\right)$. One can realize that the random variable X and its law do not depend (the random variable in the sense of equivalence class) on the value of $\alpha > \frac{d}{2}$. Setting

$$H^{-\frac{d}{2}-} = \cap_{\alpha>\frac{d}{2}} H^{-\alpha}$$

with Fréchet topology $\left(d(f,g) = \sum_{n=1}^\infty 2^{-n}\left(\|f-g\|_{H^{-\frac{d}{2}-\frac{1}{n}}} \wedge 1\right)\right)$, we may consider X as a random variable in $H^{-\frac{d}{2}-}$ and its law as a probability measure on this space.

Definition 32 *The random variable $X = \sum_{k\in\mathbb{Z}_0^d} G_k(\omega)\, e_k$ defined above, which takes values in $H^{-\frac{d}{2}-}$, is called white noise on $\mathbb{T}^d$, and its law is called white noise measure on $\mathbb{T}^d$.*

In principle, we cannot write $\langle X, h\rangle$ when $h \in L^2\left(\mathbb{T}^d\right)$, because X takes values only in $H^{-\frac{d}{2}-}$. We could introduce the duality between $H^{-\frac{d}{2}-\epsilon}$ and $H^{\frac{d}{2}+\epsilon}$ (for any $\epsilon > 0$), extend the inner product $\langle\cdot,\cdot\rangle$ to this duality and define $\langle X, h\rangle$ when $h \in H^{\frac{d}{2}+\epsilon}$. However, in a suitable sense, $\langle X, h\rangle$ is well defined for every $h \in L^2\left(\mathbb{T}^d\right)$, as we now explain.

Take $h \in L^2(\mathbb{T}^d)$. The sequence of random variables

$$\langle X_N, h \rangle = \sum_{k \in \mathbb{Z}_0^d, |k| \leq N} G_k \langle e_k, h \rangle$$

is obviously well defined. It is Cauchy in $L^2(\Omega)$: for $M > N$

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{N<|k|\leq M} G_k \langle e_k, h \rangle\right|^2\right] &= \sum_{N<|k|,|k'|\leq M} \mathbb{E}[G_k G_{k'}] \langle e_k, h \rangle \langle e_{k'}, h \rangle \\ &= \sum_{N<|k|\leq M} \langle e_k, h \rangle^2 \end{aligned}$$

and the series $\sum_{k \in \mathbb{Z}_0^d} \langle e_k, h \rangle^2$ converges to $\|h\|^2$. Thus we define $\langle X, h \rangle$ as the $L^2(\Omega)$-limit of $\langle X_N, h \rangle$.

Definition 33 *Given $h \in L^2(\mathbb{T}^d)$, the $L^2(\Omega)$-limit of $\langle X_N, h \rangle$*

$$\langle X, h \rangle := L^2(\Omega) \text{ - } \lim_{n \to \infty} \langle X_N, h \rangle$$

is called Wiener integral of h. It is a centered Gaussian random variable with variance $\|h\|^2$.

Proposition 34 *For every $h, k \in L^2(\mathbb{T}^d)$*

$$\mathbb{E}[\langle X, h \rangle \langle X, k \rangle] = \langle h, k \rangle .$$

Proof.

$$\begin{aligned} \mathbb{E}[\langle X, h \rangle \langle X, k \rangle] &= \lim_{N\to\infty} \mathbb{E}[\langle X_N, h \rangle \langle X_N, k \rangle] = \lim_{N\to\infty} \sum_{|k|,|k'|\leq N} \mathbb{E}[G_k G_{k'}] \langle e_k, h \rangle \langle e_{k'}, k \rangle \\ &= \lim_{N\to\infty} \sum_{|k|\leq N} \langle e_k, h \rangle \langle e_k, k \rangle = \langle h, k \rangle . \end{aligned}$$

■

Remark 35 *The previous proposition tells us that, in a sense, the covariance of white noise in $L^2(\mathbb{T}^d)$ is identity. However, white noise is not a random variable in $L^2(\mathbb{T}^d)$.*

Remark 36 *Especially in Physics, one loosely think to X as a random function $X(x)$ of $x \in \mathbb{T}^d$ and introduce the "function"*

$$q(x, y) = \mathbb{E}[X(x) X(y)]$$

called covariance function. Continuing this series of heuristic computations we have

$$\mathbb{E}\left[\int_{\mathbb{T}^d} X(x)\,h(x)\,dx \int_{\mathbb{T}^d} X(y)\,k(y)\,dy\right] = \int_{\mathbb{T}^d} h(z)\,k(z)\,dz$$

namely

$$q(x,y) = \delta(x-y)\,.$$

This is the famous "definition" that can be found in some literature: white noise is a delta-correlated centered random field.

Remark 37 *Although in the previous example we cannot talk of covariance in $L^2(\mathbb{T}^d)$ in a strick sense, the message is also that the notion of covariance depends on the inner product, namely from the space where we look to the measure.*

2.4.1 White noise and Brownian motion

We may repeat the present theory for the Hilbert space $L^2(0,1)$ instead of $L^2(\mathbb{T}^d)$ (no periodic conditions), without any essential change. Define

$$B_t := \left\langle X, 1_{[0,t]}\right\rangle \qquad t\in[0,1]\,.$$

One can prove that $(B_t)_{t\in[0,1]}$ is a Brownian motion (not necessarily continuous). Formally $B_t = \int_0^t X(s)\,ds$, hence X may be formally thought as the derivative of Brownian motion. These remarks correspond to a well known construction of Brownian motion based on the formula

$$B_t := \sum_{n=1}^{\infty} G_n g_n(t)$$

where $g_n(t) = \int_0^t e_n(s)\,ds$ and $\{e_n\}_{n\in\mathbb{N}}$ is any complete orthonormal system in $L^2(0,1)$.

Moreover, heuristically,

$$\langle X,h\rangle = \int_0^1 h(t)\,X(t)\,dt = \int_0^1 h(t)\,\frac{dB(t)}{dt}dt = \int_0^1 h(t)\,dB(t)\,.$$

In other words, $\langle X,h\rangle$ corresponds to a stochastic integral with respect to the process $B(t)$. Since h is deterministic, these integrals are usually called *Wiener integrals* instead of Itô integrals.

2.5 Gaussian Free field on $\mathbb{T}^d$

The second central measure for our interests is the so called *Gaussian Free field* (GFF) on $\mathbb{T}^d$. In a sentence, it is white noise in $W^{1,2}(\mathbb{T}^d)$ instead of $L^2(\mathbb{T}^d)$, hence it is one degree more regular. Two rough motivations are: it corresponds to Brownian motion instead of white noise; its covariance is the Poisson kernel. Let us see the precise definition. See other details on [10], [47], [51].

Taken a sequence $\{G_k\}_{k\in\mathbb{Z}_0^d}$ of i.i.d. standard Gaussian variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, consider the series

$$F=\sum_{k\in\mathbb{Z}_0^d}\frac{1}{|k|}G_k e_k.$$

We have to investigate the convergence of the finite sums

$$F_N=\sum_{k\in\mathbb{Z}_0^d,|k|\leq N}\frac{1}{|k|}G_k(\omega)\,e_k=\sum_{k\in\mathbb{Z}_0^d,|k|\leq N}\sigma_k G_k(\omega)\,\widetilde{f}_k$$

where

$$\widetilde{f}_k=|k|^{\alpha-1}\,e_k,\qquad \sigma_k=|k|^{-\alpha}\,.$$

If we choose any $\alpha>\frac{d}{2}$, the condition $\sum_{k\in\mathbb{Z}_0^d}\sigma_k^2<\infty$ is fulfilled and we have convergence of F_N to a well defined Gaussian random variable F in $L^2\left(\Omega;H^{-\alpha+1}\right)$, being $H^{-\alpha+1}$ the space where $\left(\widetilde{f}_k\right)$ is a complete orthonormal system. As above, we may consider F as a random variable in $H^{-\frac{d}{2}+1-}$ and its law as a probability measure on this space.

Definition 38 *The random variable $F=\sum_{k\in\mathbb{Z}_0^d}\frac{1}{|k|}G_k e_k$ defined above, which takes values in $H^{-\frac{d}{2}+1-}$, is called Gaussian Free field (GFF) on $\mathbb{T}^d$, and its law is called GFF measure on $\mathbb{T}^d$.*

Remark 39 *In dimension 1, it is analogous to Wiener measure.*

Given $h\in W^{-1,2}\left(\mathbb{T}^d\right)$, we may define $\langle F_N,h\rangle$ as

$$\langle F_N,h\rangle=\sum_{|k|\leq N}\frac{1}{|k|}G_k\langle e_k,h\rangle$$

where we set (recall that $\Lambda^{-1}:W^{-1,2}\left(\mathbb{T}^d\right)\to L^2\left(\mathbb{T}^d\right)$)

$$\langle e_k,h\rangle:=\left\langle \Lambda e_k,\Lambda^{-1}h\right\rangle=|k|\left\langle e_k,\Lambda^{-1}h\right\rangle.$$

Lemma 40 *For every $h\in W^{-1,2}\left(\mathbb{T}^d\right)$,*

$$\langle F_N,h\rangle=\sum_{|k|\leq N}G_k\langle f_k,h\rangle_{W^{-1,2}\left(\mathbb{T}^d\right)}$$

where recall (from Lemma 28) that $\{f_k\}=\{|k|\,e_k\}$ is a complete orthonormal system in $W^{-1,2}\left(\mathbb{T}^d\right)$.

Proof. The term $\langle e_k, h\rangle$ is also equal also to $|k|^2 \left\langle \Lambda^{-1} e_k, \Lambda^{-1} h\right\rangle$ and we have relation (2.1) between inner products, so in particular

$$\left\langle \Lambda^{-1} e_k, \Lambda^{-1} h\right\rangle = \langle e_k, h\rangle_{W^{-1,2}\left(\mathbb{T}^d\right)}$$

which implies

$$\langle F_N, h\rangle = \sum_{N<|k|\leq M} G_k\,|k|\,\langle e_k, h\rangle_{W^{-1,2}\left(\mathbb{T}^d\right)} = \sum_{N<|k|\leq M} G_k\,\langle f_k, h\rangle_{W^{-1,2}\left(\mathbb{T}^d\right)}\,.$$

■

Based on this lemma, entirely analogous to the previous case of white noise are the following facts.

Proposition 41 *Given $h \in W^{-1,2}\left(\mathbb{T}^d\right)$, the $L^2(\Omega)$-limit*

$$\langle F, h\rangle$$

of $\langle F_N, h\rangle$ exists; it is a centered Gaussian random variable with variance $\|h\|^2_{W^{-1,2}\left(\mathbb{T}^d\right)}$. For every $h, k \in W^{-1,2}\left(\mathbb{T}^d\right)$

$$\mathbb{E}\left[\langle F, h\rangle\langle F, k\rangle\right] = \langle h, k\rangle_{W^{-1,2}\left(\mathbb{T}^d\right)} = \left\langle \Lambda^{-1} h, \Lambda^{-1} k\right\rangle.$$

Remark 42 *Rewriting $\left\langle \Lambda^{-1} h, \Lambda^{-1} k\right\rangle$ as $\left\langle \Lambda^{-2} h, k\right\rangle$, in a sense, the covariance in $L^2\left(\mathbb{T}^d\right)$ of GFF is Λ^{-2}, the inverse of the Laplacian (this is why we said above that the covariance is the Poisson kernel). However, GFF is not a random variable in $L^2\left(\mathbb{T}^d\right)$. See also next remark.*

Remark 43 *Similarly to Remark 36, we may write*

$$\mathbb{E}\left[\int_{\mathbb{T}^d} X(x)\,h(x)\,dx \int_{\mathbb{T}^d} X(y)\,k(y)\,dy\right] = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} G(x-y)\,h(x)\,k(y)\,dz$$

hence the covariance function of GFF is

$$q(x,y) = G(x-y)$$

where $G(x)$ is the Poisson kernel on $\mathbb{T}^d$. Namely, up to a smooth remainder, we have

$$q(x,y) \sim \log|x-y|$$

when x, y are close.

2.6 Gauss measure and abstract Wiener space

The previous description is self-contained and sufficient for our purposes but embedding it into the framework of the so called *abstract Wiener space* may help the understanding and the connection with the literature. Many texts include elements on the theory of abstract Wiener spaces, like [11], [38].

Let H be a Hilbert space. Given $m = 0$ and $Q = Id$, the identity operator, we would like to define a Gaussian measure in H with covariance Q, because of its property of invariance by rotation. But it is impossible, we know that Q has to be trace class. Let us describe the surrogate of this concept.

Let $\{e_i\}$ be a complete orthonormal system of H. The Borel σ-field $\mathcal{B}(H)$ is generated by the family $\mathcal{C}$ of sets of the form

$$\pi_{e_1}^{-1}(A_1) \cap \cdots \cap \pi_{e_1}^{-1}(A_n)$$

when n varies in $\mathbb{N}$ and $A_1, ..., A_n$ in the Borel sets of $\mathbb{R}$. We call these sets cylindrical rectangles. The algebra $\mathcal{A}$ generated by $\mathcal{C}$ is made of the sets of the form $\pi_n^{-1}(A)$ when n varies in $\mathbb{N}$ and A in the Borel sets of $\mathbb{R}^n$, where $\pi_n := (\pi_{e_1}, ..., \pi_{e_n})$. In the following definition, notice the term "Gauss" measure, opposite to Gaussian measure.

Definition 44 *We call Gauss measure the finite additive measure μ_G^0 on $(H, \mathcal{A})$ such that all $(\pi_{e_i})_\sharp \mu$ are independent standard Gaussian measures.*

This measure exists (uniquely): on a cylindrical rectangle its value is equal to

$$\mu_G^0\left(\pi_{e_1}^{-1}(A_1) \cap \cdots \cap \pi_{e_1}^{-1}(A_n)\right) = \prod_{i=1}^{n} \int_{A_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \tag{2.2}$$

In a sense, this measure has covariance equal to the identity: if h, k are elments of H in the span of finitely many elements of $\{e_i\}$, we have

$$\begin{aligned}
\int_H \langle x, h\rangle \langle x, k\rangle \mu_G^0(dx) &= \sum_{i,j=1}^{N} h_i k_j \int_H \langle x, e_i\rangle \langle x, e_j\rangle \mu(dx) \\
&= \sum_{i=1}^{N} h_i k_i = \langle h, k\rangle .
\end{aligned}$$

However, it turns out that the Gauss measure is not σ-additive on $\mathcal{A}$ (we do not prove this). In order to make a progress, let us introduce a very important assumption.

Assume we have a Hilbert space B [2] with inner product $\langle ., .\rangle_B$ and norm $\|\cdot\|_B$, such that:

[2] The notation B alludes to the fact that, in a more general theory of abstract Wiener spaces, Banach spaces B are used; here we restrict ourselves to a simpler set-up.

i) $H \subset B$ with continuous dense embedding

ii) there is a complete orthonormal system $\{f_i\}$ of B, with $\{f_i\} \subset H$, and a sequence of (strictly) positive real numbers $\{\sigma_i\}$ satisfying

$$\sum_{i=1}^{\infty} \sigma_i^2 < \infty$$

such that the sequence of vectors

$$\{e_i\} = \{\sigma_i f_i\}$$

is a complete orthonormal system of H

iii) defined the linear bounded operator $\sqrt{Q} : B \to H$ as

$$\sqrt{Q}h = \sum_{i=1}^{\infty} \langle h, f_i \rangle_B \, \sigma_i f_i$$

one has

$$\langle h, k \rangle_B = \left\langle \sqrt{Q}h, \sqrt{Q}k \right\rangle.$$

Remark 45 *From (i)-(ii) it is true that* $\sqrt{Q} : B \to H$ *is well defined and bounded, because*

$$\sqrt{Q}h = \sum_{i=1}^{\infty} \langle h, f_i \rangle_B \, e_i$$

and $\sum_{i=1}^{\infty} \langle h, f_i \rangle_B^2 = \|h\|_B^2$.

Remark 46 *Recall Lemma 28 to help the intuition: H could be $L^2\left(\mathbb{T}^d\right)$, B could be $W^{-s,2}\left(\mathbb{T}^d\right)$ for suitable positive s, hence basis $\{|k|^s e_k\}$ of $L^2\left(\mathbb{T}^d\right)$ and $\{f_k\}$ of $W^{-s,2}\left(\mathbb{T}^d\right)$ correspond each other by the relation $f_k = |k|^s e_k$. And one has*

$$\langle f, g \rangle_{W^{-s,2}\left(\mathbb{T}^d\right)} = \left\langle \Lambda^{-s} f, \Lambda^{-s} g \right\rangle.$$

Example 47 *From the facts recalled in the previous remark, the pair of spaces*

$$\begin{aligned} H &= L^2\left(\mathbb{T}^d\right) \\ B &= W^{-s,2}\left(\mathbb{T}^d\right) \end{aligned}$$

satisfy the previous assumptions (i)-(ii)-(iii) when

$$s > \frac{d}{2}$$

and the operator $\sqrt{Q}$ is Λ^{-s}.

Example 48 *But also the pair of spaces*

$$\begin{aligned} H &= W^{1,2}\left(\mathbb{T}^d\right) \\ B &= W^{-s+1,2}\left(\mathbb{T}^d\right) \end{aligned}$$

satisfy the previous assumptions (i)-(ii)-(iii) when $s > \frac{d}{2}$ and the operator $\sqrt{Q}$ is again Λ^{-s}.

Theorem 49 *Under assumptions (i)-(ii), the operator $Q : B \to B$ defined as*

$$Qh = \sqrt{Q}\sqrt{Q} = \sum_{i=1}^{\infty} \langle h, f_i \rangle_B \sigma_i^2 f_i$$

is trace class in B and defines a centered Gaussian measure μ_G on B with the property

$$\int_B \langle x, h \rangle_B \langle x, k \rangle_B \mu_G(dx) = \langle Qh, k \rangle_B \qquad h, k \in B$$

and also the property

$$\int_B \langle x, h \rangle \langle x, k \rangle \mu_G(dx) = \langle h, k \rangle, \qquad h, k \in H. \tag{2.3}$$

The measure μ_G extends μ_G^0 when it is considered as a measure on cylinder sets of B.

Proof. For the first part, just notice that

$$\sum_{i=1}^{\infty} \langle Qf_i, f_i \rangle_B = \sum_{i=1}^{\infty} \sigma_i^2 < \infty$$

and apply Theorem 22. Identity (2.3) is proved as follows. Set $A = \left(\sqrt{Q}\right)^{-1}$. Then we have

$$\begin{aligned} \int_B \langle x, h \rangle \langle x, k \rangle \mu_G(dx) &= \int_B \langle Ax, Ah \rangle_B \langle Ax, Ak \rangle_B \mu_G(dx) \\ &= \left\langle QA^2 h, A^2 k \right\rangle_B = \langle Ah, Ak \rangle_B = \langle h, k \rangle \end{aligned}$$

where we leave to the reader to justify some intermediate steps. Finally, the last claim requires to extend the projections π_{e_i} to B, consider the sets $\pi_{e_1}^{-1}(A_1) \cap \cdots \cap \pi_{e_1}^{-1}(A_n)$ as subsets of B and consider definition (2.2) as the definition of a measure μ_G^0 on the generators of an algebra of events on B; then one can check that μ_G extends μ_G^0 (this fact is essentially clear from (2.3)). ■

In a sense, with property (2.3) and the fact that μ_G extends μ_G^0 we have realized our program of defining a centered Gaussian measure associated to the identity operator, but we had to enlarge the original Hilbert space.

Definition 50 *The triple (H, B, μ_G) will be called an abstract Wiener space.*

Example 51 *White noise on the torus is the abstract Wiener space of Example 47 above.*

Example 52 *The GFF on the torus is the abstract Wiener space of Example 48 above.*

2.7 Random fields

Very often all the previous concepts are introduced, in the literature, using the language of random fields, instead of random variables taking values in Hilbert spaces or probability measures on Hilbert spaces. Let us briefly introduce this language, without pretending to be exhaustive.

To avoid abstract sentences, consider again the example of the torus and set $H = W^{\alpha,2}\left(\mathbb{T}^d\right)$ for a certain given $\alpha \geq 0$ (so it is a function space).

Definition 53 *A random field (in the strict sense) $(X_x)_{x\in\mathbb{T}^d}$ is a family of random varables indexed by $x \in \mathbb{T}^d$. A random field "in the broad sense" is an element of $L^2\left(\mathbb{T}^d; L^2(\Omega)\right)$, namely a collection of equivalence classes of random variables, indexed by a.e. $x \in \mathbb{T}^d$ (and taken the equivalence classes also in x).*

Let X be a Gaussian random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $L^2\left(\mathbb{T}^d\right)$, as in all the sections above. Let us explain two equivalent ways we may think to X as a random field in the broad sense. We know that $X \in L^2\left(\Omega; L^2\left(\mathbb{T}^d\right)\right)$. By Fubini-Tonelli theorem, we may see X as an element of $L^2\left(\Omega \times \mathbb{T}^d; \mathbb{R}\right)$ and also as an element of $L^2\left(\mathbb{T}^d; L^2(\Omega)\right)$. Therefore, it defines a random field in the broad sense.

Let us give an alternative more concrete construction, not based on Fubini-Tonelli theorem. Let X be a centered Gaussian random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $L^2\left(\mathbb{T}^d\right)$ and let Q be the covariance of the associated Gaussian measure on $L^2\left(\mathbb{T}^d\right)$. Take a complete orthonormal system $\{e_k\}$ of $L^2\left(\mathbb{T}^d\right)$ made of eigenvectors of Q, with eigenvalues $\left\{\sigma_k^2\right\}$. With this choice, we know that the Gaussian random variables $\langle X, e_k\rangle$, $k \in \mathbb{Z}^d$ are independent, centered with variance σ_k^2. Consider now the finite sums

$$\sum_{|k|\leq N} \langle X(\omega), e_k\rangle e_k(x).$$

These are elements of $L^2\left(\mathbb{T}^d; L^2(\Omega)\right)$. The series is Cauchy in $L^2\left(\mathbb{T}^d; L^2(\Omega)\right)$ because, for

$M > N$,

$$\begin{aligned}\int_{\mathbb{T}^d} \mathbb{E}\left[\left|\sum_{N<|k|\leq M} \langle X, e_k\rangle e_k(x)\right|^2\right] dx &= \sum_{N<|k|,|k'|\leq M} \int_{\mathbb{T}^d} \mathbb{E}\left[\langle X, e_k\rangle \langle X, e_{k'}\rangle\right] |e_k(x)| |e_{k'}(x)| dx \\ &= \sum_{N<|k|\leq M} \int_{\mathbb{T}^d} \mathbb{E}\left[\langle X, e_k\rangle^2\right] |e_k(x)|^2 dx \\ &= \sum_{N<|k|\leq M} \sigma_k^2 \int_{\mathbb{T}^d} |e_k(x)|^2 dx = \sum_{N<|k|\leq M} \sigma_k^2\end{aligned}$$

whence it follows the Cauchy property, because $\sum_{k\in\mathbb{Z}_0^d} \sigma_k^2 < \infty$. The limit

$$\sum_{k\in\mathbb{Z}_0^d} \langle X, e_k\rangle e_k$$

in $L^2\left(\mathbb{T}^d; L^2(\Omega)\right)$ is the random field in the broad sense described above.

Recall Sobolev embedding theorem: if $\alpha > \frac{d}{2}$, then $W^{\alpha,2}\left(\mathbb{T}^d\right)$ is continuously embedded into $C\left(\mathbb{T}^d\right)$. A centered Gaussian random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $W^{\alpha,2}\left(\mathbb{T}^d\right)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $C\left(\mathbb{T}^d\right)$. Composing this Banach-valued random variable with the pointwise evaluation map at any given point $x \in \mathbb{T}^d$, a real-valued random variable X_x is well defined. The family $(X_x)_{x\in\mathbb{T}^d}$ is a random field (in the strict sense).

Remark 54 *When $d = 1$, usually we call stochastic processes these random fields.*

Both constructions apply to the case when we start with a Gaussian measure μ on $(H, \mathcal{B}(H))$, instead of a Hilbert space valued random variable. In this case we simply define the *canonical process* X: we introduce the canonical space $(\Omega, \mathcal{F}, \mathbb{P}) = (H, \mathcal{B}(H), \mu)$ and we define X to be the identity, $X(h) = h$ for every $h \in H$. This is a random variable with law μ and we may associate to it random fields as above.

2.7.1 Distributional random fields (random distributions)

In the case when X is a centered Gaussian random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $W^{\alpha,2}\left(\mathbb{T}^d\right)$ but with negative α, it is not possible to associate (classes of equivalence of) functions to X. We simply say that X is a *random distribution* of class $W^{\alpha,2}\left(\mathbb{T}^d\right)$.

An nteresting fact, however, is the possibility to define suitable averages of certain random distributions. We describe this in the following section.

2.8 Averages of the GFF on $\mathbb{T}^2$

Let F be the GFF on $\mathbb{T}^2$, shortly defined as

$$F = \sum_{k \in \mathbb{Z}_0^d} \frac{1}{|k|} G_k e_k$$

where $\{G_k\}_{k \in \mathbb{Z}_0^2}$ is a sequence of i.i.d. standard Gaussian variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{e_k\}_{k \in \mathbb{Z}_0^2}$ is the usual basis of $L^2(\mathbb{T}^2)$. Taken a Borel set $A \subset \mathbb{T}^2$, we want to define $\frac{1}{|A|} \int_A F(x)\, dx$, but obviously F is not a function. A "solution" however is easy, because the indicator function 1_A belongs to $L^2(\mathbb{T}^2)$, hence to $W^{-1,2}(\mathbb{T}^2)$, and we know from Proposition 41 that a Gaussian random variable denoted by $\langle F, h \rangle$ is well defined for every $h \in W^{-1,2}(\mathbb{T}^2)$. Thus we simply set

$$\frac{1}{|A|} \int_A F(x)\, dx := \left\langle F, \frac{1}{|A|} 1_A \right\rangle .$$

Remark 55 *Check that the same can be done in any dimension and also for the white noise in place of the GFF.*

This definition was easy and very general. More specific of the GFF (and other random distributions but not all) is the possibility to define integrals on 1-dimensional sets. We use here the language of distributions and the duality between $W^{1,2}(\mathbb{T}^2)$ and $W^{-1,2}(\mathbb{T}^2)$.

Lemma 56 *Let $\gamma : [0,1] \to \mathbb{T}^2$ be a closed simple C^1 curve. Consider the map*

$$f \mapsto \int_\gamma f(\sigma)\, d\sigma := \int_0^1 f(\gamma(s)) \left|\dot{\gamma}(s)\right| ds$$

from $W^{1,2}(\mathbb{T}^2)$ to $\mathbb{R}$. The map is well defined and continuous, hence it defines an element of $W^{-1,2}(\mathbb{T}^2)$, that we denote by Γ:

$$\Gamma(f) := \int_0^1 f(\gamma(s)) \left|\dot{\gamma}(s)\right| ds.$$

Proof. First, recall that

$$\int_0^1 f(\gamma(s)) \left|\dot{\gamma}(s)\right| ds = \int_0^L f(l(\sigma))\, d\sigma$$

where L is the length $\int_0^1 \left|\dot{\gamma}(s)\right| ds$ of the curve and $l : [0, L] \to \mathbb{T}^2$ is the reparametrization of the curve by arc length. Hence it is sufficient to prove that there exists a constant $C > 0$ such that

$$\left| \int_0^L f(l(\sigma))\, d\sigma \right| \leq C \|f\|_{W^{1,2}(\mathbb{T}^2)}$$

for all f smooth (hence all $f \in W^{1,2}\left(\mathbb{T}^2\right)$, by density). Now we restrict ourselves to prove the claim in the simple case when γ is the boundary of a ball $B(0,r) \subset \mathbb{T}^2$ with small $r > 0$. The proof in the general case is not so different but requires some additional argument.

Thus consider $B(0,r)$ and its boundary described by the curve $\gamma : [0, 2\pi r] \to \mathbb{T}^2$ given by $\gamma(s) = r(\cos t/r, \sin t/r)$. Define a smooth vector field $v : \mathbb{T}^2 \to \mathbb{R}^2$ such that $v(x) = \frac{x}{|x|}$ on $\partial B(0,r)$ (one can take v with compact support around $\partial B(0,r)$, of the form $v(x) = g(|x|)\, x$ with suitable g). By Gauss-Green formula, we have

$$\int_{B(0,r)} \nabla f(x) \cdot v(x)\, dx = -\int_{B(0,r)} f(x) \operatorname{div} v(x)\, dx + \int_{\partial B(0,r)} f(\sigma)\, v(\sigma) \cdot n(\sigma)\, d\sigma$$

where $n(\sigma)$ is the outer normal to $\partial B(0,r)$. Hence, since $v(\sigma) \cdot n(\sigma) = 1$,

$$\begin{aligned}
\int_{\partial B(0,r)} f(\sigma)\, d\sigma &= \int_{B(0,r)} \nabla f(x) \cdot v(x)\, dx + \int_{B(0,r)} f(x) \operatorname{div} v(x)\, dx \\
&\leq \|v\|_\infty \int_{\mathbb{T}^2} |\nabla f(x)|\, dx + \|\operatorname{div}\|_\infty \int_{\mathbb{T}^2} |f(x)|\, dx \\
&\leq C \|f\|_{W^{1,2}(\mathbb{T}^2)}
\end{aligned}$$

by Hölder inequality. ■

Based on the previous lemma and Proposition 41, we give the following definition. Denote by $|\gamma|$ the length of γ, $\int_0^1 \left|\dot{\gamma}(s)\right| ds$.

Definition 57 *Let $\gamma : [0,1] \to \mathbb{T}^2$ be a closed simple C^1 curve and let $\Gamma \in W^{-1,2}\left(\mathbb{T}^2\right)$ be the associated distribution, given by Lemma 56. Let F be the GFF on $\mathbb{T}^2$. We set*

$$\frac{1}{|\gamma|} \int_\gamma F(\sigma)\, d\sigma := \left\langle F, \frac{1}{|\gamma|}\Gamma \right\rangle$$

called average of F on γ.

In particular, denoting by $B(x,r)$ the ball in $\mathbb{T}^2$ of center x and radius r, and by $\partial B(x,r)$ the counter-clockwise curve at the boundary of $B(x,r)$, we set

$$F(x,r) := \frac{1}{2\pi r} \int_{\partial B(x,r)} F(\sigma)\, d\sigma.$$

This is a well defined random variable, for every $(x,r) \in \mathbb{T}^2 \times (0,\infty)$. One can prove that the random field $F(x,r)$ admits a continous version.

Theorem 58 *On $(x,r) \in \mathbb{T}^2 \times (0,\infty)$, there exists a Hölder continuous version of the random field $F(x,r)$.*

Proof. We only give the idea. One can apply Kolmogorov regularity theorem in each set of the form $(x,r) \in \mathbb{T}^2 \times [a,b]$, with $0 < a < b < \infty$. One has to prove that

$$\mathbb{E}\left[\left|F(x,r) - F\left(x',r'\right)\right|^p\right] \leq C\left(\left|x-x'\right|^{3+\alpha} + \left|r-r'\right|^{3+\alpha}\right)$$

for some $p, \alpha > 0$. Due to Gaussianity of $F(x,r) - F(x',r')$, it is sufficient to prove

$$\mathbb{E}\left[\left|F(x,r) - F\left(x',r'\right)\right|^2\right] \leq C\left(\left|x-x'\right|^{\epsilon} + \left|r-r'\right|^{\epsilon}\right)$$

for some $\epsilon > 0$. It is also sufficient to prove separately

$$\mathbb{E}\left[\left|F(0,r) - F\left(0,r'\right)\right|^2\right] \leq C\left|r-r'\right|^{\epsilon}$$

$$\mathbb{E}\left[\left|F(x,r) - F\left(x',r\right)\right|^2\right] \leq C\left|x-x'\right|^{\epsilon}.$$

Now one has to perform suitalbe lengthy computations, based on the next lemma. ■

Lemma 59

$$Var\left[\int_\gamma F(\sigma)\,d\sigma\right] = \left\|\Gamma_\gamma\right\|_{H^{-1}}^2 = \int_0^1\int_0^1 G(\gamma(t) - \gamma(s))\left|\dot{\gamma}(s)\right|\left|\dot{\gamma}(t)\right| dsdt.$$

More generally,

$$\mathbb{E}\left[\int_\gamma F(\sigma)\,d\sigma \int_{\gamma'} F(\sigma)\,d\sigma\right] = \int_0^1\int_0^1 G\left(\gamma(t) - \gamma'(s)\right)\left|\dot{\gamma}(s)\right|\left|\dot{\gamma}'(t)\right| dsdt.$$

Proof. The following proof is a little bit formal, just to sketch the idea. We have

$$\left\|\Gamma_\gamma\right\|_{H^{-1}}^2 = \left\langle \Delta^{-1}\Gamma_\gamma, \Gamma_\gamma \right\rangle = \int_0^1 \left(\Delta^{-1}\Gamma_\gamma\right)(\gamma(s))\left|\dot{\gamma}(s)\right| ds$$

where

$$\left(\Delta^{-1}\Gamma_\gamma\right)(x) = \int_0^1 G(x - \gamma(t))\left|\dot{\gamma}(t)\right| ds$$

and G is Green kernel on $\mathbb{T}^2$, $G(x) = \log|x| + r(x)$ close to $x = 0$. Hence

$$\left\|\Gamma_\gamma\right\|_{H^{-1}}^2 = \int_0^1\int_0^1 G(\gamma(t) - \gamma(s))\left|\dot{\gamma}(s)\right|\left|\dot{\gamma}(t)\right| dsdt.$$

We leave the second identity as an exercise. ■

2.8.1 Special properties of the GFF

With little effort, from the previous lemma one can show that

$$Var\left[F\left(x,r\right)\right] \sim \log\frac{1}{r}$$

for small values of r. Thus, in a sense, $\left|F\left(x,r\right)\right| \sim \sqrt{\log\frac{1}{r}}$. This is not strictly correct, but gives a general idea. However, due to flucturations, there are points x where the size of $\left|F\left(x,r\right)\right|$ is much bigger. One can prove that, for a.e. realization of the continuous field $F\left(x,r\right)$, the set of points x such that

$$\lim_{r\to 0}\frac{\left|F\left(x,r\right)\right|}{\log\frac{1}{r}} = a > 0$$

is non empty and has Hausdorff dimension equal to a certain value in $(0,2)$, depending on a.

Thus in some vague sense $\left|F\left(x,r\right)\right|$ looks like a huge family of point vortices, with exceptionally high values localized in small sets.

For reasons related to conformal field theory, it is interesting trying to introduce on $\mathbb{T}^2$ a measure of the form

$$e^{\gamma F(x)}dx$$

where $F\left(x\right)$ is a non-rigorous notation for the GFF. But F is a random distribution of class H^{-}, thus $F\left(x\right)$ has no meaning. Then one considers the approximations

$$C_\epsilon e^{\gamma F(x,\epsilon)}dx.$$

The choice of the normalizing constant is made by a simple Gaussian computations. Let us look for C_ϵ such that $\mathbb{E}\left[C_\epsilon e^{\gamma F(x,\epsilon)}\right]$ is constant. From a known formula on the moment generating function of a Gaussian r.v., we have

$$\mathbb{E}\left[e^{\gamma F(x,\epsilon)}\right] = e^{\frac{\gamma^2}{2}Var[F(x,\epsilon)]} = e^{\frac{\gamma^2}{2}\log\frac{1}{\epsilon}} = \frac{1}{\epsilon^{\frac{\gamma^2}{2}}}$$

hence $C_\epsilon = \epsilon^{\frac{\gamma^2}{2}}$. One can prove the following result.

Theorem 60 *For every $\gamma < 2$, chosen $\epsilon_n = 2^{-n}$, the random measures $\epsilon_n^{\frac{\gamma^2}{2}} e^{\gamma F(x,\epsilon_n)}$ weakly converge, a.s., to a random measure μ on $\mathbb{T}^2$, which has no atoms and is positive on positive Lebesgue measure sets.*

The intuition is that Lebesgue measure on $\mathbb{T}^2$ is made much larger where $F\left(x\right)$ has exceptionally large positive values (recall the result above on their existence) and almost zero where $F\left(x\right)$ has exceptionally large negative values. Another intuition is that there is a metric behind, which makes distant points that are close under Euclidean metric, and viceversa, with the result that certain points of $\mathbb{T}^2$ are crossed nearby by a huge amount of geodesics. See [10] for further informations.

Chapter 3

From Random Particles to Measures on Fields

3.1 Point vortices and white noise

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent r.v., on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{T}^2$; assume they are all uniformly distributed. On the same probability space, let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of centered independent r.v. with the same distribution, say $N(0,1)$. Assume that $(X_i)_{i\in\mathbb{N}}$ and $(\xi_i)_{i\in\mathbb{N}}$ are independent.

For every $N \in \mathbb{N}$, consider the random signed measure on $\mathbb{T}^2$

$$\omega_N(dx) = \sum_{i=1}^{N} \frac{\xi_i}{\sqrt{N}} \delta_{X_i}(dx).$$

It is a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to the space $W^{-1-,2}(\mathbb{T}^2)$. Indeed, $W^{1+\epsilon,2}(\mathbb{T}^2)$ is continuously embedded into $C(\mathbb{T}^2)$ by Sobolev embedding theorem and that the delta Dirac δ_x is a continuous linear functional on each $W^{1+\epsilon,2}(\mathbb{T}^2)$.

Recall the concept of white noise on $\mathbb{T}^2$: it is a Gaussian measure on H^{-1-}. Both the law of the *empirical measure* S_N of the point vortices and white noise are probability measures on H^{-1-}.

Theorem 61 *The law of ω_N on H^{-1-} converges weakly to the law of white noise (weakly in the probabilistic sense, in the topology of H^{-1-}).*

Proof. Step 1. If we give for granted the CLT of Section 6.3, it is a simple application of that theorem. Let us prove this claim. In Step 3 we prove, in our particular case, the only detail about the CLT that was left unproved in Section 6.3.

Given $\epsilon > 0$, consider the separable Hilbert space $H = W^{-1-\epsilon,2}(\mathbb{T}^2)$ and the random vectors $\xi_i \delta_{X_i}$, which take values in H. They are independent and equally distributed. If we prove that $\xi_1 \delta_{X_1}$ has finite second moment in H, and centered, then the CLT applies and the limit Gaussian measure has the same covarance of $\xi_1 \delta_{X_1}$. We have

$$\mathbb{E}\left[\|\xi_1 \delta_{X_1}\|_H^2\right] = \mathbb{E}\left[\xi_1^2\right] \mathbb{E}\left[\|\delta_{X_1}\|_H^2\right] = \mathbb{E}\left[\|\delta_{X_1}\|_H^2\right].$$

But

$$\begin{aligned} \|\delta_{X_1}\|_H &= \sup_{\|\phi\|_{W^{1+\epsilon,2}} \leq 1} |\delta_{X_1}(\phi)| = \sup_{\|\phi\|_{W^{1+\epsilon,2}} \leq 1} |\phi(X_1)| \leq \sup_{\|\phi\|_{W^{1+\epsilon,2}} \leq 1} \|\phi\|_\infty \\ &\leq C \sup_{\|\phi\|_{W^{1+\epsilon,2}} \leq 1} \|\phi\|_{W^{1+\epsilon,2}} = C \end{aligned}$$

where we have used Sobolev embedding theorem $W^{1+\epsilon,2}(\mathbb{T}^2) \subset C(\mathbb{T}^2)$. Hence $\mathbb{E}\left[\|\xi_1 \delta_{X_1}\|_H^2\right] \leq C^2$. Moreover,

$$\mathbb{E}\left[\langle \xi_1 \delta_{X_1}, h \rangle_H\right] = \mathbb{E}\left[\xi_1\right] \mathbb{E}\left[\langle \delta_{X_1}, h \rangle_H\right] = 0$$

for every $h \in H$. Hence CLT applies. The "covariance in $L^2(\mathbb{T}^2)$" of $\xi_1 \delta_{X_1}$ is, for $h, k \in W^{1+\epsilon,2}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$

$$\mathbb{E}\left[\langle \xi_1 \delta_{X_1}, h \rangle \langle \xi_1 \delta_{X_1}, k \rangle\right] = \mathbb{E}\left[\xi_1^2\right] \mathbb{E}\left[\langle \delta_{X_1}, h \rangle \langle \delta_{X_1}, k \rangle\right] = \mathbb{E}\left[h(X_1) k(X_1)\right] = \int_{\mathbb{T}^2} h(x) k(x)\, dx$$

hence it is the same as the "covariance in $L^2(\mathbb{T}^2)$" of white noise. Inspection in the relations between covarances in different spaces proves that the two measures have the same covarances also in $W^{-1-\epsilon,2}(\mathbb{T}^2)$.

Thus we have proved the theorem, except that we have replaced H^{-1-} by $W^{-1-\epsilon,2}(\mathbb{T}^2)$. Thinking to the Fréchet topology of H^{-1-}, it is not difficult to see that weak convergence in $W^{-1-\epsilon,2}(\mathbb{T}^2)$ for every $\epsilon > 0$ implies weak convergence in H^{-1-}; however, we write the details in the next step.

Step 2. Let us prove convergence in H^{-1-}. Denote by μ_N the law of ω_N and by μ the law of White Noise. The first fact is that $\{\mu_n\}$ is tight in H^{-1-}. Indeed, given $\epsilon > 0$ and $j \in N$, there is a compact set $K_{\epsilon,j} \subset H^{-1-\frac{1}{j}}$ such that

$$\mu_n\left(K_{\epsilon,j}^c\right) < \frac{\epsilon}{2^j} \qquad \text{for all } n \in N.$$

Set $K_\epsilon = \cap_j K_{\epsilon,j}$. We have

$$\mu_n\left(K_\epsilon^c\right) = \mu_n\left(\cup_j K_{\epsilon,j}^c\right) \leq \sum_j \mu_n\left(K_{\epsilon,j}^c\right) < \epsilon \qquad \text{for all } n \in N.$$

But the set K_ϵ is relatively compact in the topology of H^{-1-}. Hence $\{\mu_n\}$ is tight in H^{-1-}.

From Prohorov theorem we may find a subsequence $\{\mu_{n_k}\}$ converging weakly to some ν, in H^{-1-}. Then $\nu = \mu$. Indeed, since continuous bounded functions on $H^{-1-\delta}$ are also continuous bounded on H^{-1-}, we deduce that $\{\mu_{n_k}\}$ converges weakly to ν in every $H^{-1-\delta}$. Hence $\nu = \mu$. It finally follows that the full sequence $\{\mu_n\}$ converges weakly to μ in H^{-1-}.

Step 3. In this step we prove that the family $\{\omega_N\}$ is tight in $W^{-1-\epsilon,2}(\mathbb{T}^2)$. This is the missing part in the proof of Section 6.3. The space $W^{-1-\frac{\epsilon}{2},2}(\mathbb{T}^2)$ is compactly embedded into $W^{-1-\epsilon,2}(\mathbb{T}^2)$. Thus it is sufficient to prove that for every $\delta > 0$ there is $R_\delta > 0$ such that

$$P\left(\|\omega_N\|_{W^{-1-\frac{\epsilon}{2},2}} > R_\delta\right) \leq \delta$$

(the ball $B(0, R_\delta)$ of $W^{-1-\frac{\epsilon}{2},2}(\mathbb{T}^2)$ is precompact in $W^{-1-\epsilon,2}(\mathbb{T}^2)$). But

$$P\left(\|\omega_N\|_{W^{-1-\frac{\epsilon}{2},2}} > R_\delta\right) \leq \frac{\mathbb{E}\left[\|\omega_N\|^2_{W^{-1-\frac{\epsilon}{2},2}}\right]}{R_\delta^2}$$

hence it is sufficient to bound $\mathbb{E}\left[\|\omega_N\|^2_{W^{-1-\frac{\epsilon}{2},2}}\right]$ independently of N. By usual arguments on the non-diagonal terms based on independece and mean zero, we find

$$\mathbb{E}\left[\|\omega_N\|^2_{W^{-1-\frac{\epsilon}{2},2}}\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\|\xi_i\delta_{X_i}\|^2_{W^{-1-\frac{\epsilon}{2},2}}\right]$$

and this is equal to $\mathbb{E}\left[\|\xi_1\delta_{X_1}\|^2_{W^{-1-\frac{\epsilon}{2},2}}\right]$, which is finite. The proof is complete. ■

Remark 62 *Since $\omega_N(dx)$ is invariant for the dynamics of point vortices, we presume that white noise could be invariant for Euler dynamics. In the next chapter we shall see that this is true, in a suitable sense.*

3.2 Renormalized energies

Consider N random point vortices of the form

$$\omega_N(dx) = \sum_{i=1}^{N}\frac{\xi_i}{\sqrt{N}}\delta_{X^i}$$

with X^i uniformly distributed on $\mathbb{T}^2$, $\xi_i \sim N(0,1)$, all indepedent random variables. At the end of the last chapter we have seen that the random measure $\omega_N(dx)$ converges to white noise, as $N \to \infty$, in the topology of H^{-1-} and in the sense of convergence in law.

We begin this section with a remark about energies. We investigate the kinetic energy of the point vortex system and of white noise and argue about their relations. Strictly speaking both energies are infinite, hence we need to subtract an infinite contribution (a "renormalization").

The random Hamiltonian associated to point vortices is

$$\mathcal{H}_N = \frac{1}{2}\frac{1}{N}\sum_{i\neq j}\xi_i\xi_j G\left(X^i - X^j\right).$$

where the Green function $G(x)$ behaves like $-\log|x|$ for small x, up to a smooth function.

It gives the mutual kinetic energy. Indeed, formally speaking, the total kinetic energy should be given by

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{T}^2}|u(x)|^2\,dx &= \frac{1}{2}\int_{\mathbb{T}^2}\left|\sum_{i=1}^N \frac{\xi_i}{\sqrt{N}}K\left(x-X^i\right)\right|^2 dx \\
&= \frac{1}{2}\frac{1}{N}\sum_{i,j=1}^N \xi_i\xi_j\int_{\mathbb{T}^2}K\left(x-X^i\right)\cdot K\left(x-X^j\right)dx \\
&= \mathcal{E}_{self}+\mathcal{E}_{int}
\end{aligned}$$

where

$$\mathcal{E}_{self} := \frac{1}{2}\frac{1}{N}\sum_{i=1}^N \xi_i^2\int_{\mathbb{T}^2}\left|K\left(x-X^i\right)\right|^2 dx$$

is infinite, because each integral $\int_{\mathbb{T}^2}\left|K\left(x-X^i\right)\right|^2 dx$ diverges. But

$$\mathcal{E}_{int} := \frac{1}{2}\frac{1}{N}\sum_{i\neq j}\xi_i\xi_j\int_{\mathbb{T}^2}K\left(x-X^i\right)\cdot K\left(x-X^j\right)dx$$

is a.s. finite, since a.s. $X^i \neq X^i$ and the improper integral $\int_{\mathbb{T}^2}K\left(x-X^i\right)\cdot K\left(x-X^j\right)dx$ converges. Moreover,

$$\begin{aligned}
\mathcal{E}_{int} \; : &= \frac{1}{2}\frac{1}{N}\sum_{i\neq j}\xi_i\xi_j\int_{\mathbb{T}^2}\nabla^\perp G\left(x-X^i\right)\cdot\nabla^\perp G\left(x-X^j\right)dx \\
&= \frac{1}{2}\frac{1}{N}\sum_{i\neq j}\xi_i\xi_j\int_{\mathbb{T}^2}G\left(x-X^i\right)\Delta G\left(x-X^j\right)dx \\
&= \frac{1}{2}\frac{1}{N}\sum_{i\neq j}\xi_i\xi_j G\left(X^i-X^j\right)
\end{aligned}$$

namely

$$\mathcal{E}_{int} = \mathcal{H}_N.$$

Proposition 63

$$\mathbb{E}\left[\left|\mathcal{H}_N\right|^2\right] = \frac{N-1}{N}\int_{\mathbb{T}^2}G(x)^2\,dx.$$

Proof. Preliminary, we have

$$\mathbb{E}\left[|\mathcal{H}_N|^2\right] = \frac{1}{4}\frac{1}{N^2}\sum_{i\neq j}\sum_{i'\neq j'}\mathbb{E}\left[\xi_i\xi_j\xi_{i'}\xi_{j'}\right]\mathbb{E}\left[G\left(X^i - X^j\right)G\left(X^{i'} - X^{j'}\right)\right].$$

Consider the term $\mathbb{E}\left[\xi_i\xi_j\xi_{i'}\xi_{j'}\right]$. As soon as one index is different from the others, it is zero, by independence and zero average of the ξ_i's. Thus ony terms made of two pairs with different indexes or all four equal indexes survive; equal indexes is impossible because of the constraint $i \neq j$, hence only pairs with different indexes survive, which means (due to $i \neq j$) that either $i' = i$ and $j' = j$, or $i' = j$ and $j' = i$. Hence, rewriting first $\mathbb{E}\left[|\mathcal{H}_N|^2\right]$ as $\frac{1}{N^2}\sum_{i<j}\sum_{i'<j'}$ so that only $i' = i$ and $j' = j$ survives, we get

$$\mathbb{E}\left[|\mathcal{H}_N|^2\right] = \frac{1}{N^2}\sum_{i<j}\mathbb{E}\left[\xi_i^2\xi_j^2\right]\mathbb{E}\left[G\left(X^i - X^j\right)^2\right] = \frac{1}{N^2}\sum_{i<j}G_0 = \frac{N-1}{N}G_0.$$

where for shortness we have denoted $\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}G\left(x-y\right)^2 dxdy$ by the constant G_0. With a simple change of variables (using the perodicity of G one proves that $G_0 = \int_{\mathbb{T}^2}G\left(x\right)^2 dx$. ■

Therefore we see that the interaction energy is bounded in mean square, with respect to N.

Consider now white noise on $\mathbb{T}^2$, namely the Gaussian vector

$$\omega = \sum_{k\in Z_0^2} G_k e_k$$

which we know to converge in H^{-1-}. As usual, G_k are independent $N\left(0,1\right)$; we choose

$$e_k\left(x\right) = e^{2\pi i k\cdot x}.$$

The associated velocity $u = K * \omega$ belongs only to H^{-}, hence we cannot compute the energy $\frac{1}{2}\int_{\mathbb{T}^2}\left|u\left(x\right)\right|^2 dx$. But we may introduce the approximation

$$\mathcal{E}_N := \frac{1}{2}\int_{\mathbb{T}^2}\left|u_N\left(x\right)\right|^2 dx$$

where

$$u_N = K * \omega_N, \qquad \omega_N = \sum_{|k|\leq N} G_k e_k$$

and investigate

$$\widetilde{\mathcal{E}}_N := \mathcal{E}_N - \mathbb{E}\left[\mathcal{E}_N\right].$$

The next proposition shows that $\mathbb{E}\left[\mathcal{E}_N\right]$ has a logaritmic diverence as $N \to \infty$; subtracting the logaritmic divergence to $\mathcal{E}_N$, the renormalized energy $\widetilde{\mathcal{E}}_N$ is bounded in mean square.

Proposition 64

$$\mathbb{E}\left[\mathcal{E}_N\right] = \frac{1}{8\pi^2} \sum_{|k| \leq N} \frac{1}{|k|^2}$$

$$\mathbb{E}\left[\left|\widetilde{\mathcal{E}}_N\right|^2\right] = \frac{2}{(8\pi^2)^2} \sum_{|k| \leq N} \frac{1}{|k|^4}.$$

Moreover, the sequence $\widetilde{\mathcal{E}}_N$ is Cauchy in in $L^2(\Omega)$; denote its limit by $: \mathcal{E} :$ and call it renormalized energy. One has

$$\mathbb{E}\left[|: \mathcal{E} :|^2\right] = \frac{2}{(8\pi^2)^2} \sum_{k \in Z_0^2} \frac{1}{|k|^4} = C \int_{\mathbb{T}^2} G(x)^2 \, dx.$$

Proof. Let us first rewrite $\mathcal{E}_N$ as

$$\mathcal{E}_N = \frac{1}{8\pi^2} \sum_{|k| \leq N} \frac{G_k^2}{|k|^2}.$$

This simple formula comes from the following computations:

$$(K * e_k)(x) = \frac{1}{2\pi} \frac{k^\perp}{|k|^2} e_k(x)$$

(simply check that $\nabla^\perp \cdot \left(\frac{1}{2\pi} \frac{k^\perp}{|k|^2} e_k\right) = e_k$)

$$\begin{aligned}
\mathcal{E}_N \quad : \quad &= \frac{1}{2} \int_{\mathbb{T}^2} \left| \left(K * \sum_{|k| \leq N} G_k e_k \right)(x) \right|^2 dx \\
&= \frac{1}{2} \sum_{|k|,|k'| \leq N} G_k G_{k'} \int_{\mathbb{T}^2} (K * e_k)(x) \cdot (K * e_{k'})(x) \, dx \\
&= \frac{1}{8\pi^2} \sum_{|k|,|k'| \leq N} G_k G_{k'} \frac{k^\perp}{|k|^2} \cdot \frac{k'^\perp}{|k'|^2} \int_{\mathbb{T}^2} e_k(x) \, e_{k'}(x) \, dx \\
&= \frac{1}{8\pi^2} \sum_{|k| \leq N} \frac{G_k^2}{|k|^2}
\end{aligned}$$

becuase $\int_{\mathbb{T}^2} e_k(x) \, e_{k'}(x) \, dx = \delta_{kk'}$. Therefore

$$\mathbb{E}\left[\mathcal{E}_N\right] = \frac{1}{8\pi^2} \sum_{|k| \leq N} \frac{1}{|k|^2}.$$

Moreover,

$$\widetilde{\mathcal{E}}_N := \frac{1}{8\pi^2} \sum_{|k| \le N} \frac{G_k^2 - 1}{|k|^2}$$

and since $\mathbb{E}\left[\left(G_k^2 - 1\right)\left(G_{k'}^2 - 1\right)\right] = 2\delta_{k,k'}$ (because $\mathbb{E}\left[\left(G_k^2 - 1\right)^2\right] = 3 - 2 + 1 = 2$)

$$\begin{aligned} \mathbb{E}\left[\left|\widetilde{\mathcal{E}}_N\right|^2\right] &= \frac{1}{(8\pi^2)^2} \sum_{|k|,|k'| \le N} \frac{1}{|k|^2} \frac{1}{|k'|^2} \mathbb{E}\left[\left(G_k^2 - 1\right)\left(G_{k'}^2 - 1\right)\right] \\ &= \frac{2}{(8\pi^2)^2} \sum_{|k| \le N} \frac{1}{|k|^4}. \end{aligned}$$

Exactly in the same way one can prove that $\widetilde{\mathcal{E}}_N$ is a Cauchy sequence in $L^2(\Omega)$. The limit, called $: \mathcal{E} :$, has variance $\sum_{k \in Z_0^2} \frac{1}{|k|^4}$. Finally, solving Poisson equation in Fourier space, it is clear that the transform of $G(x)$ is $\frac{1}{|k|^2}$ (up to a constant) and thus $\sum_{k \in Z_0^2} \frac{1}{|k|^4}$ is equal to $\int_{\mathbb{T}^2} G(x)^2 \, dx$. ■

Remark 65 *A purpose of this section is to show the parallel between combinatorial computations (for point vortices) and Gaussian computations (for white noise).*

3.3 PDE viewpoint on Energy

Let us finally describe a less combinatorical approach to the energy of White Noise, more inspired to a trick already seen for the weak vorticity formulation of Euler equations. For a smooth vorticity field ω, with $u = K * \omega$, $\omega = \nabla^\perp \cdot u$, $\varphi = -G * \omega$ the stream function, such that $\nabla^\perp \varphi = u$, we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{T}^2} |u(x)|^2 \, dx &= \frac{1}{2} \int_{\mathbb{T}^2} \nabla^\perp \varphi(x) \cdot u(x) \, dx = -\frac{1}{2} \int_{\mathbb{T}^2} \varphi(x) \nabla^\perp \cdot u(x) \, dx \\ &= -\frac{1}{2} \int_{\mathbb{T}^2} \varphi(x) \omega(x) \, dx = \frac{1}{2} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} G(x - y) \omega(x) \omega(y) \, dx dy. \end{aligned}$$

This is a well known reformulation of energy in terms of vorticity. Now, assume ω is White Noise. The question is: can we define the previous expression, or at least a renormalized form of it?

Let us first discuss the following question: if ω is a distribution on $\mathbb{T}^2$ of class $H^{-\alpha}$, can we define a distribution on $\mathbb{T}^2 \times \mathbb{T}^2$ formally corresponding to $\omega(x)\omega(y)$, distribution that we shall denote by $\omega \otimes \omega$? On test functions of the form $f \otimes g : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ the definition must be

$$(\omega \otimes \omega)(f \otimes g) = \omega(f)\omega(g)$$

definition that extends to linear combinations of such test functions. One can show that $\omega \otimes \omega$ is a well defined element of $W^{-2\alpha,2}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$, but in general it is not better (in particular cases it is better, like in the case of measures). We do not prove this result since we do not need it. We just use the fact that $\omega \otimes \omega$ is a well defined on smooth functions $f \in C^\infty\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$. The definition is by iteration: for every $y \in \mathbb{T}^2$ the number $\omega\left(f\left(\cdot, y\right)\right)$ is well defined and, as a function of y, it is smooth; call $g_f\left(y\right) = \omega\left(f\left(\cdot, y\right)\right)$ and set

$$\left(\omega \otimes \omega\right)\left(f\right) = \omega\left(g_f\right).$$

Recalling that White Noise is a random distribution ω with values in $H^{-1-\epsilon}$, if $(x, y) \overset{f_G}{\mapsto} G\left(x-y\right)$ would be a function of class $H^{2+2\epsilon}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ we could define $\left(\omega \otimes \omega\right)\left(f_G\right)$. But G behaves like $-\log|x|$ for small x, hence is not of class $H^{2+2\epsilon}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ (by Sobolev embedding theorem, these functions have to be continuous). We cannot define $\left(\omega \otimes \omega\right)\left(f_G\right)$. Obviously we are not surprised, because we have seen above that the energy is not well defined, without a renormalization.

Let us introduce a smooth function $\log^\delta |x|$ as the one used in the Chapter on point vortices, equal to $\log^\delta |x|$ for $|x| \geq \delta$, just more precise to have $G^\delta\left(0\right) = 1 - \log\delta$; and the associated Green kernel G^δ. Now $\left(\omega \otimes \omega\right)\left(f_{G^\delta}\right)$ is well defined. Let us define the random variables

$$\mathcal{E}^\delta := \left(\omega \otimes \omega\right)\left(f_{G^\delta}\right)$$

and

$$\widetilde{\mathcal{E}}^\delta := \mathcal{E}^\delta - \mathbb{E}\left[\mathcal{E}^\delta\right].$$

We have:

Proposition 66

$$\mathbb{E}\left[\left(\omega \otimes \omega\right)\left(f_{G^\delta}\right)\right] = \frac{1 - \log\delta}{2}$$

$$\mathbb{E}\left[\left|\widetilde{\mathcal{E}}^\delta\right|^2\right] = \frac{1}{2}\int_{\mathbb{T}^2} G^\delta\left(x\right)^2 dx.$$

Proof. We give only a formal proof using the "rule"

$$\mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\right] = \delta\left(x-y\right).$$

We have

$$\begin{aligned} \mathbb{E}\left[\left(\omega \otimes \omega\right)\left(f_{G^\delta}\right)\right] &= \mathbb{E}\left[\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} G^\delta\left(x-y\right)\omega\left(x\right)\omega\left(y\right)dxdy\right] \\ &= \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} G^\delta\left(x-y\right)\mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\right]dxdy \\ &= \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} G^\delta\left(x-y\right)\delta\left(x-y\right)dxdy = \frac{1}{2}G^\delta\left(0\right) \end{aligned}$$

$$\mathbb{E}\left[\left|\mathcal{E}^{\delta}\right|^{2}\right]=\frac{1}{4}\int_{(\mathbb{T}^{2})^{4}}G^{\delta}\left(x-y\right)G^{\delta}\left(x'-y'\right)\mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\omega\left(x'\right)\omega\left(y'\right)\right]dxdydx'dy'.$$

Now we have to use the following general "rule" of Gaussian fields (called Isserlis-Wick formula), that we shall discuss ad due time:

$$\begin{aligned}\mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\omega\left(x'\right)\omega\left(y'\right)\right] &= \mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\right]\mathbb{E}\left[\omega\left(x'\right)\omega\left(y'\right)\right]\\ &+\mathbb{E}\left[\omega\left(x\right)\omega\left(x'\right)\right]\mathbb{E}\left[\omega\left(y\right)\omega\left(y'\right)\right]\\ &+\mathbb{E}\left[\omega\left(x\right)\omega\left(y'\right)\right]\mathbb{E}\left[\omega\left(y\right)\omega\left(x'\right)\right].\end{aligned}$$

In our case,

$$\begin{aligned}\mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\omega\left(x'\right)\omega\left(y'\right)\right] &= \delta\left(x-y\right)\delta\left(x'-y'\right)\\ &+\delta\left(x-x'\right)\delta\left(y-y'\right)\\ &+\delta\left(x-y'\right)\delta\left(x'-y\right).\end{aligned}$$

It follows

$$\begin{aligned}\mathbb{E}\left[\left|\mathcal{E}^{\delta}\right|^{2}\right] &= \frac{1}{4}\int_{(\mathbb{T}^{2})^{4}}G^{\delta}\left(x-y\right)G^{\delta}\left(x'-y'\right)\delta\left(x-y\right)\delta\left(x'-y'\right)dxdydx'dy'\\ &+\frac{1}{4}\int_{(\mathbb{T}^{2})^{4}}G^{\delta}\left(x-y\right)G^{\delta}\left(x'-y'\right)\delta\left(x-x'\right)\delta\left(y-y'\right)dxdydx'dy'\\ &+\frac{1}{4}\int_{(\mathbb{T}^{2})^{4}}G^{\delta}\left(x-y\right)G^{\delta}\left(x'-y'\right)\delta\left(x-y'\right)\delta\left(x'-y\right)dxdydx'dy'\end{aligned}$$

$$=\frac{1}{4}G^{\delta}\left(0\right)^{2}+\frac{1}{4}\int_{(\mathbb{T}^{2})^{2}}G^{\delta}\left(x-y\right)^{2}dxdy+\frac{1}{4}\int_{(\mathbb{T}^{2})^{2}}G^{\delta}\left(x-y\right)^{2}dxdy$$

hence

$$\begin{aligned}\mathbb{E}\left[\left|\widetilde{\mathcal{E}}^{\delta}\right|^{2}\right] &= \mathbb{E}\left[\left|\mathcal{E}^{\delta}\right|^{2}\right]-\mathbb{E}\left[\left(\omega\otimes\omega\right)\left(f_{G^{\delta}}\right)\right]^{2}\\ &= \frac{1}{2}\int_{(\mathbb{T}^{2})^{2}}G^{\delta}\left(x-y\right)^{2}dxdy=\frac{1}{2}\int_{\mathbb{T}^{2}}G^{\delta}\left(x\right)^{2}dx.\end{aligned}$$

■

3.4 Energy conditional measures and 2D turbulence

First of all, let us notice an interesting fact that was under our eyes: the configuration space of N point vortices is "foliated" by its invariants, for instance by $\mathcal{H}_N$, in the sense that the dynamics lives in a "shell" of constant mutual energy $\{\mathcal{H}_N = c\}$, with c given by the mutual energy of the initial configuration. At the same time, we have discovered that Lebesgue measure on positions is invariant; or, if we randomize also the intensities, a measure like

$$\rho_N\left(dx_1...dx_N, d\xi_1, ..., d\xi_N\right) = dx_1...dx_N \mathcal{N}\left(d\xi_1, ..., d\xi_N\right)$$

where $\mathcal{N}$ is the standard multidimensional Normal distribution is invariant. These two aspects, invarance of a measure and foliation by deterministic invariant quantities are not in contradiction: they would be in contradiction with the wrong belief that the dynamic is ergodic on the full configuration space, hence should visit all regions. Point vortex dynamic is not ergodic on the full configuration space.

Therefore one could consider a subregion of positive ρ_N mass of configuration space, for instance

$$\{\mathcal{H}_N \in [a, b]\}$$

with $b > a$, and condition ρ_N to this region, namely consider the probability distribution defined on Borel sets A as

$$\rho_{N,a,b}\left(A\right) = \frac{\rho_N\left(A \cap \{\mathcal{H}_N \in [a, b]\}\right)}{\rho_N\left(\{\mathcal{H}_N \in [a, b]\}\right)}.$$

A natural conjecture is that this family of measures, when N varies, is tight and has a unique limit point, given by the analogous conditional measure for White Noise μ:

$$\mu_{a,b}\left(A\right) = \frac{\mu\left(A \cap \{: \mathcal{E} :\in [a, b]\}\right)}{\mu\left(\{: \mathcal{E} :\in [a, b]\}\right)}$$

for all Borel sets A of H^{-1-}.

This generalization of our discussion may be very relevant for comparison with experiments of 2D turbulence. Let us say, first of all, that turbulent namely very complex) motions in 2D fluids is a field of intense research from the nuemrical-physical community; turbulence in 3D fluids remain one of the most important partially open topic in classical physics and its 2D version, although different, is also very important.

One of the typical experiments performed in laboratories consists in producing several small vortices of different sign in a 2D fluid and see what happens when time evolves. Gaussian statistics of the fundamental quantities are observed, with a reasonable degree of precision, although small deviations exists. Thus, a Gaussian mathematical model is not unreasonable, although some kind of correction should be included. Well, although this is no yet clear at all, a possibility is to condition the Gaussian measure to a range of energy. The reason could be that the mechanism used in the laboratory produces configurations with a limited range of energy.

The measure $\mu_{a,b}$ above is obviously not Gaussian. However, consider a typical realization of velocity field u, which we know to be of class H^{-}. We would like to compute quantities like $u\left(x + \Delta x\right) - u\left(x\right)$; we may compute $\langle u, h\rangle$ when $h \in H^{-1}$ (to compute $u\left(x + \Delta x\right) - u\left(x\right)$ we need h of delta Dirac class, which is only H^{-1-}). We take h with very localized support. To simplify the exposition, think to h as a smooth function with localized support around a point x_0. Under the measure μ, we know that $\langle u, h\rangle$ is Gaussian. Under $\mu_{a,b}$, can we say that $\langle u, h\rangle$ is still close to Gaussian? Can we say that

$$\mu\left(\langle u, h\rangle \in I \text{ and } : \mathcal{E} :\in [a, b]\right) \sim \mu\left(\langle u, h\rangle \in I\right) \mu\left(: \mathcal{E} :\in [a, b]\right)$$

when h is a smooth function with localized support around a point x_0? Comparing with statements in the physical literature, the equivalence should be good when I does not contain tails; namely, deviations from Gaussianity could appear for extreme values.

Chapter 4

2D Euler Equations with Random Initial Conditions

4.1 Introduction

In this Chapter we investigate distributional random solutions of the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, formulated in terms of the vorticity ω

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{4.1}$$

where u is the velocity, divergence free vector field such that $\omega = \partial_2 u_1 - \partial_1 u_2$. See Section 1.3 for some additional details.

In order to understand the role of the results on distributional solutions in the framework of solvability of 2D Euler equations, let us recall some elements of the classical theory (see for instance [18], [40], [41], [?], [?]).

1. Given ω_0 of class $L^\infty\left(\mathbb{T}^2\right)$, existence and uniqueness has been proved of weak solutions of class $L^\infty\left([0,T]\times\mathbb{T}^2\right) \cap C\left([0,T]; L^p\left(\mathbb{T}^2\right)\right)$ for every $p \in [1,\infty)$, satisfying
$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s, u_s \cdot \nabla \phi \rangle \, ds \tag{4.2}$$
for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$ ([60], [61], [?]);

2. Given ω_0 of class $L^p\left(\mathbb{T}^2\right)$ for some $p \in [1,\infty)$, existence has been proved of weak solutions of class $C\left([0,T]; L^p\left(\mathbb{T}^2\right)\right)$, satisfying (4.2).

3. Existence of measure-valued solutions $\omega_t(dx)$ has been proved, of class $L^\infty\left(0,T;\mathcal{M}\left(\mathbb{T}^2\right)\cap H^{-1}\left(\mathbb{T}^2\right)\right)$, satisfying for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$ the so called weak vorticity formulation
$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_\phi(x,y)\, \omega_s(dx)\, \omega_s(dy)\, ds \tag{4.3}$$

where

$$H_\phi(x,y) := \frac{1}{2} K(x-y)(\nabla\phi(x) - \nabla\phi(y))$$

and $K(x)$ is Biot-Savart kernel on $\mathbb{T}^2$, when the initial condition is a measure of class $H^{-1}(\mathbb{T}^2)$ with a certain condition of preference for a single sign, see [27], [54], [29]; here we have denoted by $\mathcal{M}(\mathbb{T}^2)$ the space of finite signed measures and by $H^\alpha(\mathbb{T}^2)$ the classical Sobolev spaces of order $\alpha \in \mathbb{R}$ defined in Section **??**.

4. Existence and uniqueness has been proved of a measure-valued solution of the form $\omega_t(dx) = \sum_{i=1}^N \xi_i \delta_{X_t^i}$, fulfilling (4.3), when the initial condition has the form $\omega_0(dx) = \sum_{i=1}^N \xi_i \delta_{X_0^i}$, with real valued intensities $\xi_1, ..., \xi_N$, and $(X_0^1, ..., X_0^N)$ belonging to a set of full Lebesgue measure in $(\mathbb{T}^2)^N$, see [**?**], [**?**].

Obviously there are many other results, reported in the references above and other works, including counterexamples to uniqueness like [52], [26]. The previous choice has been made to illustrate the attempt to include weaker and weaker concepts of solutions. Very important for result n. 3 has been the symmetrization step from (4.2) to (4.3): the kernel $H_\phi(x,y)$ is bounded, smooth outside the diagonal, discontinuous along the diagonal; hence a fine analysis of the concentration of $\omega_t(dx)$ around the diagonal is important but at least the singularity of order $\frac{1}{|x|}$ of Biot-Savart kernel $K(x)$ has been removed.

The purpose of this chapter is to continue the list above in the direction of distributional vorticity fields, using probability. Although the main result is probabilistic in nature, we like to state it first in purely deterministic terms, for the sake of comparison with the "scale" of results above. Denote by $H^{-1-}(\mathbb{T}^2)$ the space $\bigcap_{\epsilon>0} H^{-1-\epsilon}(\mathbb{T}^2)$, with the topology described in Section **??** and notice that $\mathcal{M}(\mathbb{T}^2) \subset H^{-1-}(\mathbb{T}^2)$, because by Sobolev embedding $H^{1+\epsilon}(\mathbb{T}^2) \subset C(\mathbb{T}^2)$. Moreover, denote by K_ϵ the smooth approximations of K given by (4.6) below and, given a sequence $\epsilon_n \to 0$, set $H_\phi^n(x,y) := \frac{1}{2} K_{\epsilon_n}(x-y)(\nabla\phi(x) - \nabla\phi(y))$; by classical distribution theory, $s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle$ is well defined and continuous when $\omega \in C([0,T]; H^{-1-}(\mathbb{T}^2))$.

Theorem 67 *There exist $\epsilon_n \to 0$ and a large (in particular dense) set*

$$\mathcal{IC}_0 \subset H^{-1-}(\mathbb{T}^2) \setminus (H^{-1}(\mathbb{T}^2) \cup \mathcal{M}(\mathbb{T}^2))$$

of initial conditions such that for all $\omega_0 \in \mathcal{IC}_0$ the following properties hold.

i) there exists $\omega \in C([0,T]; H^{-1-}(\mathbb{T}^2))$ such that, for every $\phi \in C^\infty(\mathbb{T}^2)$, the sequence of functions $s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle$ is a Cauchy sequence in $L^2(0,T)$ and, denoted by $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle$ its limit, one has the analog of (4.3), namely

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds \tag{4.4}$$

ii) there is a sequence $\{\omega^{(n)}\}$ *of solutions of Euler equations of class* $L^\infty\left([0,T]\times\mathbb{T}^2\right)\cap C\left([0,T];L^p\left(\mathbb{T}^2\right)\right)$ *for every* $p\in[1,\infty)$ *(those of point 1 above) such that* $\omega^{(n)}\to\omega$ *in* $C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$.

Remark 68 *How large is the set of initial conditions, it is clarified below in Section 4.6. It is a full measure set with respect to the Gaussian measure* μ *introduced in Section* **??**. *In particular, it is a dense set.*

Remark 69 *In fact the set of initial conditions given by this theorem is included in a more regular space* $H^{-1-,\infty}\left(\mathbb{T}^2\right)$, *where also the solutions live, defined in Section* **??** *below. We have not used* $H^{-1-,\infty}\left(\mathbb{T}^2\right)$ *in place of* $H^{-1-}\left(\mathbb{T}^2\right)$ *because* $\mathcal{M}\left(\mathbb{T}^2\right)\not\subseteq H^{-1-,\infty}\left(\mathbb{T}^2\right)$ *and thus the statement would be less clear. Moreover* $H^{-1}\left(\mathbb{T}^2\right)$ *and* $\mathcal{M}\left(\mathbb{T}^2\right)$ *are not included one in the other, which again explains the statement.*

The deterministic theorem 67 is a consequence, as explained in Section 4.6, of the probabilistic Theorem 90 below, which states that Euler equations, interpreted in the form (4.4), has a stochastic solution, a stationary stochastic process with time marginal given by the so called *white noise* on $\mathbb{T}^2$, defined in Section **??** below; and that this solution is limit of random point vortices and of random L^∞ solutions, suitable random versions of points 1 and 4 above.

It is interesting to compare the previous list with a similar one known for the nonlinear wave equation

$$\partial_{tt}^2 u=\Delta u-u^3 \qquad \text{in } \mathbb{T}^3$$

(variants are known for several other dispersive equations). Consider this equation in the spaces

$$(u,\partial_t u)\in\mathcal{H}^s=H^s\times H^{s-1}.$$

Wery shortly:

- for $s\geq 1$: well posedness is known by relatively classical and not so difficult energy methods;
- for $s\in\left(\frac{1}{2},1\right)$: (sometimes only local in time) well posedness has been proved by more refined tools, precisely by Strichartz estimates;
- for $s\in\left[0,\frac{1}{2}\right]$ there are counterexamples to well posedness, but the equation is well posed at least for a.e. initial condition with respect to certain Gaussian measures (globally in time).

The picture for 2D Euler equations is weaker since we lack uniqueness results in distributional spaces.

4.2 Weak vorticity formulation

4.2.1 Colored noise

In Chapter 2 we have introduced White Noise on $\mathbb{T}^2$, that will be used here. For technical reasons, sometimes it is convenient to consider a smooth approximation of white noise. A simple one is $\omega_N(\theta, x) = \mathrm{Re} \sum_{|n| \leq N} G_n(\theta) e_n(x)$ but, although the difference is really minor, for the PDE approach followed here the use of mollifiers looks a bit more natural. We set, for $\epsilon > 0$,

$$\omega_\epsilon(x) = \langle \omega, \theta_\epsilon(x - \cdot) \rangle$$

formally written also as $(\theta_\epsilon * \omega)(x) = \int_{\mathbb{T}^2} \theta_\epsilon(x-y) \omega(y) dy$, where $\theta_\epsilon(x) = \epsilon^{-2} \theta(\epsilon^{-1} x)$, and θ is a smooth probability density on $\mathbb{T}^2$ with a small support around $x = 0$. Assume θ symmetric. We have

$$\mathbb{E}[\langle \omega_\epsilon, \phi \rangle \langle \omega_\epsilon, \psi \rangle] = \mathbb{E}[\langle \omega, \theta_\epsilon * \phi \rangle \langle \omega, \theta_\epsilon * \psi \rangle] = \langle \theta_\epsilon * \phi, \theta_\epsilon * \psi \rangle$$

$$\begin{aligned} \mathbb{E}[\omega_\epsilon(x) \omega_\epsilon(y)] &= \mathbb{E}[\langle \omega, \theta_\epsilon(x - \cdot) \rangle \langle \omega, \theta_\epsilon(y - \cdot) \rangle] = \langle \theta_\epsilon(x - \cdot), \theta_\epsilon(y - \cdot) \rangle \\ &= \int_{\mathbb{T}^2} \theta_\epsilon(x - y - z) \theta_\epsilon(z) dz = (\theta_\epsilon * \theta_\epsilon)(x - y) =: \delta^\epsilon_{x-y} \end{aligned}$$

where we have used the notation δ^ϵ_a to denote $(\theta_\epsilon * \theta_\epsilon)(a)$ because it is an approximation of the Dirac delta distribution.

Notice that $\omega_\epsilon \in C^\infty(\mathbb{T}^2)$ with probability one. Moreover, since $\langle \omega_\epsilon, \phi \rangle = \langle \omega, \theta_\epsilon * \phi \rangle$ and $\theta_\epsilon * \phi \to \phi$ in $H^{1+\gamma}(\mathbb{T}^2)$ for every $\phi \in H^{1+\gamma}(\mathbb{T}^2)$ and given $\gamma > 0$, we have the following statement:

Lemma 70 *$\mathbb{P}$-almost surely, for every $\phi \in H^{1+\gamma}(\mathbb{T}^2)$ we have*

$$\lim_{\epsilon \to 0} \langle \omega_\epsilon, \phi \rangle = \langle \omega, \phi \rangle .$$

4.2.2 Preliminaries

Let us first recall the weak vorticity formulation in the case of measure-valued vorticities. First, one rewrites equation (4.1) against test functions $\phi \in C^\infty(\mathbb{T}^2)$, using $\mathrm{div}\, u = 0$:

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s, u_s \cdot \nabla \phi \rangle ds.$$

Then recall that Biot-Savart law gives us

$$u_t(x) = \int_{\mathbb{T}^2} K(x - y) \omega_t(dy)$$

where $K(x,y)$ is the Biot-Savart kernel; in full space it is given by $K(x-y) = \frac{1}{2\pi}\frac{(x-y)^{\perp}}{|x-y|^2}$; on the torus its form is less simple but we still have K smooth for $x \neq y$, $K(y-x) = -K(x-y)$,

$$|K(x-y)| \leq \frac{C}{|x-y|}$$

for small values of $|x-y|$. See for instance [53], [14] for details and the Appendix for additional informations. Thus we write the weak formulation in the more explicit form

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(x-y) \nabla \phi(x) \, \omega_s(dx) \, \omega_s(dy) \, ds.$$

Since the double space integral, when we rename x by y and y by x, is the same (the renaming doesn't affect the value), and $K(y-x) = -K(x-y)$, we get (4.3). Identity (4.3) is the weak vorticity formulation of Euler equations. Depending on the assumptions on the measures ω_s (whether or not they have concentrated masses), one has to specify the value of $K(0)$, which is not given a priori, and thus the value of $H_\phi(x,x)$; in the analysis of point vortices, for instance, it is usually set equal to zero, to avoid self-interaction. The weak vorticity formulation of Euler equations proved to be a fundamental tool in the investigation of limits of solutions, especially in the context of measures. Below we shall follow a similar path in the case of white noise distributional solutions.

4.2.3 The nonlinear term for white noise vorticity

Our purpose now is to define

$$\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} H_\phi(x,y) \, \omega(x) \, \omega(y) \, dx dy$$

when $\omega : \Xi \to C^\infty(\mathbb{T}^2)'$ is a white noise.

Preliminarily, notice that if $\omega \in C^\infty(\mathbb{T}^2)'$ is a distribution, we can define a distribution $\omega \otimes \omega \in C^\infty(\mathbb{T}^2 \times \mathbb{T}^2)'$ which satisfies

$$\langle \omega \otimes \omega, \phi \otimes \psi \rangle = \langle \omega, \phi \rangle \langle \omega, \psi \rangle$$

for all $\phi, \psi \in C^\infty(\mathbb{T}^2)$, where $\phi \otimes \psi$ denotes the function $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$. The definition of $\omega \otimes \omega$ can be based on limits of test functions of the form $\sum_{i=1}^n \phi_i(x) \psi_i(y)$, or more directly on the following argument. Given $f \in C^\infty(\mathbb{T}^2 \times \mathbb{T}^2)$, for each $x \in \mathbb{T}^2$ we have $f(x,\cdot) \in C^\infty(\mathbb{T}^2)$, hence $\langle \omega, f(x,\cdot) \rangle$ is well defined. The function $g(x) = \langle \omega, f(x,\cdot) \rangle$ belongs to $C^\infty(\mathbb{T}^2)$, as one can verify using the continuity properties of distributions on test functions. Then we can set

$$\langle \omega \otimes \omega, f \rangle = \langle \omega, g \rangle, \qquad \text{where } g(x) = \langle \omega, f(x,\cdot) \rangle. \tag{4.5}$$

If $\omega \in H^{-s}\left(\mathbb{T}^2\right)$ for some $s>0$, one can check that $\omega \otimes \omega \in H^{-2s}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$.

Let us go back to white noise. First notice that, being $\omega \in H^{-1-}\left(\mathbb{T}^2\right)$ with probability one, we have at least

$$\omega \otimes \omega \in H^{-2-}\left(\mathbb{T}^2 \times \mathbb{T}^2\right) \text{ with probability one.}$$

Hence $\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)\,\omega(x)\,\omega(y)\,dxdy$, or more properly the duality

$$\langle \omega \otimes \omega, f\rangle$$

is well defined when $f \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$. The question is: can we define

$$\langle \omega \otimes \omega, H_\phi\rangle$$

for the function H_ϕ, which is smooth outside the diagonal, and bounded, but discontinuous along the diagonal and thus not of class H^{2+}? We have the following results, over which all our analysis is based. The first result is concerned with the smooth approximations $\omega_\epsilon(x) = \langle \omega, \theta_\epsilon(x-\cdot)\rangle$, the second one with white noise.

Lemma 71 *i) If $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ is a white noise and f is bounded measurable on $\mathbb{T}^2 \times \mathbb{T}^2$, then for every $p \geq 1$ there is a constant $C_p > 0$ such that, for all $\epsilon > 0$,*

$$\mathbb{E}\left[\left|\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle\right|^p\right] \leq C_p \left\|f\right\|_\infty^p .$$

ii) We have $\mathbb{E}\left[\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle\right] = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \delta^\epsilon_{x-y} f(x,y)\,dxdy$.

iii) If f is symmetric, then

$$\mathbb{E}\left[\left|\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle - \mathbb{E}\left[\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle\right]\right|^2\right] = 2\int_{\left(\mathbb{T}^2\right)^4} \delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2} f(x_1,y_1)\, f(x_2,y_2)\,dx_1dy_1dx_2dy_2.$$

Proof. i) It is sufficient to prove the claim for integer values of p. We have

$$\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_\epsilon(x)\,\omega_\epsilon(y)\, f(x,y)\,dxdy$$

$$\mathbb{E}\left[\left|\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle\right|^p\right] = \int_{\left(\mathbb{T}^2\right)^{2p}} \mathbb{E}\left[\prod_{i=1}^p \left(\omega_\epsilon(x_i)\,\omega_\epsilon(y_i)\right)\right] \prod_{i=1}^p f(x_i,y_i)\,dx_1dy_1\cdots dx_pdy_p.$$

From Isserlis-Wick theorem,

$$\mathbb{E}\left[\prod_{i=1}^p \left(\omega_\epsilon(x_i)\,\omega_\epsilon(y_i)\right)\right] = \sum_\pi \prod_{(a,b)\in\pi} \mathbb{E}\left[\omega_\epsilon(a)\,\omega_\epsilon(b)\right] = \sum_\pi \prod_{(a,b)\in\pi} \delta^\epsilon_{a-b}$$

where the sum is over all partitions π of $(x_1, y_1, ..., x_p, y_p)$ in pairs, generically denoted by (a, b). Therefore

$$\begin{aligned}
\mathbb{E}\left[|\langle\omega_\epsilon \otimes \omega_\epsilon, f\rangle|^p\right] &= \sum_\pi \int_{(\mathbb{T}^2)^{2p}} \prod_{(a,b)\in\pi} \delta^\epsilon_{a-b} \prod_{i=1}^p f(x_i, y_i)\, dx_1 dy_1 \cdots dx_p dy_p \\
&\leq \|f\|^p_\infty \sum_\pi \int_{(\mathbb{T}^2)^{2p}} \prod_{(a,b)\in\pi} \delta^\epsilon_{a-b} dx_1 dy_1 \cdots dx_p dy_p \\
&= \|f\|^p_\infty \sum_\pi \left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \delta^\epsilon_{a-b} dadb\right)^p \\
&= \|f\|^p_\infty \sum_\pi \left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \langle\theta_\epsilon(a-\cdot), \theta_\epsilon(b-\cdot)\rangle\, dadb\right)^p \\
&= \|f\|^p_\infty \sum_\pi \left|\mathbb{T}^2\right|^p =: C_p \|f\|^p_\infty
\end{aligned}$$

(the sum has $(2p)!/(2^p p!)$ terms).

ii) We simply have

$$\mathbb{E}\left[\langle\omega_\epsilon \otimes \omega_\epsilon, f\rangle\right] = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \mathbb{E}\left[\omega_\epsilon(x)\,\omega_\epsilon(y)\right] f(x,y)\, dxdy = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \delta^\epsilon_{x-y} f(x,y)\, dxdy.$$

iii) We just develop more carefully

$$\mathbb{E}\left[\langle\omega_\epsilon \otimes \omega_\epsilon, f\rangle^2\right] = \int_{(\mathbb{T}^2)^4} \mathbb{E}\left[\prod_{i=1}^2 \left(\omega_\epsilon(x_i)\,\omega_\epsilon(y_i)\right)\right] \prod_{i=1}^2 f(x_i, y_i)\, dx_1 dy_1 dx_2 dy_2.$$

We have, again from Isserlis-Wick theorem,

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^2 \left(\omega_\epsilon(x_i)\,\omega_\epsilon(y_i)\right)\right] &= \mathbb{E}\left[\omega_\epsilon(x_1)\,\omega_\epsilon(x_2)\right]\mathbb{E}\left[\omega_\epsilon(y_1)\,\omega_\epsilon(y_2)\right] \\
&\quad + \mathbb{E}\left[\omega_\epsilon(x_1)\,\omega_\epsilon(y_2)\right]\mathbb{E}\left[\omega_\epsilon(y_1)\,\omega_\epsilon(x_2)\right] \\
&\quad + \mathbb{E}\left[\omega_\epsilon(x_1)\,\omega_\epsilon(y_1)\right]\mathbb{E}\left[\omega_\epsilon(x_2)\,\omega_\epsilon(y_2)\right] \\
&= \delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2} + \delta^\epsilon_{x_1-y_2}\delta^\epsilon_{y_1-x_2} + \delta^\epsilon_{x_1-y_1}\delta^\epsilon_{x_2-y_2}.
\end{aligned}$$

Hence, using the symmetry,

$$\begin{aligned}
&\mathbb{E}\left[\langle\omega_\epsilon \otimes \omega_\epsilon, f\rangle^2\right] \\
&= \int_{(\mathbb{T}^2)^4} \left(\delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2} + \delta^\epsilon_{x_1-y_2}\delta^\epsilon_{y_1-x_2} + \delta^\epsilon_{x_1-y_1}\delta^\epsilon_{x_2-y_2}\right) f(x_1,y_1)\, f(x_2,y_2)\, dx_1 dy_1 dx_2 dy_2 \\
&= 2\int_{(\mathbb{T}^2)^4} \delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2} f(x_1,y_1)\, f(x_2,y_2)\, dx_1 dy_1 dx_2 dy_2 + \left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \delta^\epsilon_{x-y} f(x,y)\, dxdy\right)^2.
\end{aligned}$$

We have found

$$\mathbb{E}\left[\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle^2\right] - \mathbb{E}\left[\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle\right]^2 = 2\int_{(\mathbb{T}^2)^4} \delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2} f(x_1,y_1) f(x_2,y_2)\, dx_1 dy_1 dx_2 dy_2.$$

■

Corollary 72 *i) If $\omega : \Xi \to C^\infty(\mathbb{T}^2)'$ is a white noise and $f \in H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$, then for every $p \geq 1$ there is a constant $C_p > 0$ such that*

$$\mathbb{E}\left[|\langle \omega \otimes \omega, f\rangle|^p\right] \leq C_p \|f\|_\infty^p .$$

ii) We have $\mathbb{E}\left[\langle \omega \otimes \omega, f\rangle\right] = \int_{\mathbb{T}^2} f(x,x)\, dx$.
iii) If f is symmetric, then

$$\mathbb{E}\left[|\langle \omega \otimes \omega, f\rangle - \mathbb{E}\left[\langle \omega \otimes \omega, f\rangle\right]|^2\right] = 2\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)^2\, dxdy.$$

Proof. Notice that f is continuous and thus bounded and uniformly continuous, on $\mathbb{T}^2$, by Sobolev embedding theorem. Thus we may apply the previous lemma to $\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle$; and we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \delta^\epsilon_{x-y} f(x,y)\, dxdy = \int_{\mathbb{T}^2} f(x,x)\, dx$$

$$\lim_{\epsilon\to 0}\int_{(\mathbb{T}^2)^4} \delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2} f(x_1,y_1) f(x_2,y_2)\, dx_1 dy_1 dx_2 dy_2 = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x_1,y_1)^2\, dx_1 dy_1.$$

From the identity

$$\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \omega_\epsilon(x)\,\omega_\epsilon(y)\, f(x,y)\, dxdy = \langle \omega \otimes \omega, (\theta_\epsilon \otimes \theta_\epsilon) * f\rangle$$

we see that $\mathbb{P}$-almost surely, for every $f \in H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$ we have

$$\lim_{\epsilon\to 0} \langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle = \langle \omega \otimes \omega, f\rangle .$$

We can pass to the limit in all expectations written in the statement of the corollary, due to uniform integrability of $|\langle \omega_\epsilon \otimes \omega_\epsilon, f\rangle|$ (Vitali theorem), coming from property (i) of the lemma. The corollary then follows from these limit properties and the lemma. ■

Remark 73 *In the non symmetric case we simply have*

$$\mathbb{E}\left[|\langle \omega \otimes \omega, f\rangle - \mathbb{E}\left[\langle \omega \otimes \omega, f\rangle\right]|^2\right] = \int\int f^2(x,y)\, dxdy + \int\int f(x,y)\, f(y,x)\, dxdy.$$

Based on the previous key facts we can give a definition of $\langle \omega \otimes \omega, H_\phi \rangle$ when ω is white noise.

Theorem 74 *Let $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ be a white noise and $\phi \in C^\infty\left(\mathbb{T}^2\right)$ be given. Assume that $H_\phi^n \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ are symmetric and approximate H_ϕ in the following sense:*

$$\lim_{n\to\infty} \int\int \left(H_\phi^n - H_\phi\right)^2 (x,y)\, dxdy = 0$$

$$\lim_{n\to\infty} \int H_\phi^n (x,x)\, dx = 0.$$

Then the sequence of r.v.'s $\left\langle \omega \otimes \omega, H_\phi^n \right\rangle$ is a Cauchy sequence in mean square. We denote by

$$\langle \omega \otimes \omega, H_\phi \rangle$$

its limit. Moreover, the limit is the same if H_ϕ^n is replaced by $\widetilde{H}_\phi^n$ with the same properties and such that $\lim_{n\to\infty} \int\int \left(H_\phi^n - \widetilde{H}_\phi^n\right)^2 (x,y)\, dxdy = 0$.

Proof. Since $\lim_{n\to\infty} \int H_\phi^n (x,x)\, dx = 0$, it is equivalent to show that $\left\langle \omega \otimes \omega, H_\phi^n \right\rangle - \int H_\phi^n (x,x)\, dx$ is a Cauchy sequence in mean square. We have

$$\mathbb{E}\left[\left|\left\langle \omega \otimes \omega, H_\phi^n \right\rangle - \int H_\phi^n (x,x)\, dx - \left\langle \omega \otimes \omega, H_\phi^m \right\rangle + \int H_\phi^m (x,x)\, dx\right|^2\right]$$
$$= \mathbb{E}\left[\left|\left\langle \omega \otimes \omega, \left(H_\phi^n - H_\phi^m\right) \right\rangle - \int \left(H_\phi^n - H_\phi^m\right)(x,x)\, dx\right|^2\right]$$

and now we use properties (ii-iii) of the Corollary

$$= 2\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \left(H_\phi^n - H_\phi^m\right)^2 (x,y)\, dxdy.$$

Due to our assumption, this implies the Cauchy property. Hence $\langle \omega \otimes \omega, H_\phi \rangle$ is well defined. The invariance property is prove in a similar way. ■

Remark 75 *It is easy to construct a sequence $H_\phi^n(x,y)$ with the properties above. Recall that $H_\phi(x,y) := \frac{1}{2}K(x-y)\left(\nabla\phi(x) - \nabla\phi(y)\right)$, where K smooth for $x \neq y$, $K(y-x) = -K(x-y)$,*

$$|K(x-y)| \leq \frac{C}{|x-y|}$$

for small values of $|x-y|$. *We set, for* $\epsilon > 0$,

$$K_\epsilon(x) = \begin{cases} K(x)(1-\theta_\epsilon(x)) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases} \tag{4.6}$$

where $\theta_\epsilon(x) = \theta(\epsilon^{-1}x)$, $0 \le \theta \le 1$, θ *is smooth, with support a small ball* $B(0,r)$, *equal to 1 in* $B(0,r/2)$; *and, given any sequence* $\epsilon_n \to 0$ *we set*

$$H_\phi^n(x,y) = \frac{1}{2} K_{\epsilon_n}(x-y)(\nabla\phi(x) - \nabla\phi(y)).$$

Then H_ϕ^n *is smooth;* $H_\phi^n(x,x) = 0$ *hence* $\int H_\phi^n(x,x)\,dx = 0$; *and*

$$\begin{aligned} \lim_{n\to\infty} \int\int \left(H_\phi^n - H_\phi\right)^2(x,y)\,dxdy &= \lim_{n\to\infty} \int\int H_\phi^2(x,y)\,\theta_{\epsilon_n}^2(x-y)\,dxdy \\ &\le \lim_{n\to\infty} \int\int_{|x-y|\le\epsilon_n r} H_\phi^2(x,y)\,dxdy = 0 \end{aligned}$$

(because $H_\phi^2(x,y)$ *is bounded above,* $\theta_{\epsilon_n}^2 \le 1$, *and* $\theta_{\epsilon_n}^2 \neq 0$ *only in* $B(0,\epsilon_n r)$*).*

In fact, what we need in Definition 83 below is a definition of $\int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle\,ds$ and for such purpose the previous result is not so strong; it would allow for instance to define such integral as a Bochner integral in the Hilbert space $L^2(\Xi)$. We prefer to have a stronger meaning and for this purpose we refine the previous result.

Theorem 76 *Let* $\omega_\cdot : \Xi \times [0,T] \to C^\infty(\mathbb{T}^2)'$ *be a measurable map with trajectories of class* $C([0,T];H^{-1-})$. *Assume that* ω_t *is a white noise at every time* $t \in [0,T]$. *Let* H_ϕ^n *be an approximation of* H_ϕ *as above, of class* $H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$. *Then the well defined sequence of real valued process* $\left\{s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle; s \in [0,T]\right\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $L^2(\Xi; L^2(0,T))$.

Proof. The proof is the same as the one of Theorem 74, but we repeat it, due to the importance of the present result. We have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \left|\left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega_s \otimes \omega_s, H_\phi^m \right\rangle + \int H_\phi^m(x,x)\,dx\right|^2 ds\right] \\ &= \int_0^T \mathbb{E}\left[\int_0^T \left|\left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega_s \otimes \omega_s, H_\phi^m \right\rangle + \int H_\phi^m(x,x)\,dx\right|^2\right] ds \\ &= T \cdot \mathbb{E}\left[\left|\left\langle \omega_0 \otimes \omega_0, \left(H_\phi^n - H_\phi^m\right)\right\rangle - \int \left(H_\phi^n - H_\phi^m\right)(x,x)\,dx\right|^2\right] \end{aligned}$$

and now we use properties (ii-iii) of the Corollary

$$= 2\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\left(H_\phi^n - H_\phi^m\right)^2(x,y)\,dxdy.$$

Due to our assumption, this implies the Cauchy property. ■

Definition 77 *Under the assumptions of the previous theorem, we denote by*

$$\{s \mapsto \langle \omega_s \otimes \omega_s, H_\phi\rangle\,; s \in [0,T]\}$$

or more simply by $\langle \omega. \otimes \omega., H_\phi\rangle$ *the process of class* $L^2\left(\Xi; L^2(0,T)\right)$, *limit of the sequence* $\left\{s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n\right\rangle; s \in [0,T]\right\}_{n\in\mathbb{N}}$.

Remark 78 *By the identification* $L^2\left(\Xi; L^2(0,T)\right) = L^2\left(0,T; L^2(\Xi)\right)$, *we may see* $\langle \omega. \otimes \omega., H_\phi\rangle$ *as an element of the class* $L^2\left(0,T; L^2(\Xi)\right)$; *its value at time s is, for a.e. s, an element of* $L^2(\Xi)$; *one may check that it is the same element of* $L^2(\Xi)$ *given by Theorem 74.*

4.2.4 The nonlinear term for modified white noise vorticity

We may generalize a little bit the previous construction. Assume $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ is a random distribution with the property that

$$\mathbb{E}\left[\Phi(\omega)\right] = \mathbb{E}\left[\rho(\omega_{WN})\,\Phi(\omega_{WN})\right]$$

for every measurable function $\Phi : H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$, where $\omega_{WN} : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ is a white noise and $\rho : H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$ is a measurable function such that

$$k_q := \mathbb{E}\left[\rho^q(\omega_{WN})\right] < \infty$$

for some $q > 1$, and $\int \rho d\mu = 1$. This is equivalent to say that the law of ω is absolutely continuous with respect to μ with density ρ satisfying $\int \rho^q d\mu < \infty$.

Lemma 79 *Under the previous assumptions, if* $f \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$, *then:*

i) for every $r \geq 1$ *there is a constant* $C_r > 0$ *such that*

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, f\rangle\right|^r\right] \leq C_r \left\|f\right\|_\infty^r.$$

ii) If f is symmetric, then there exists a constant $C_q > 0$ *such that*

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|\right] \leq C_q \left\|f\right\|_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}^{1/p}$$

where p is the number such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. i) We deduce the claim from

$$\mathbb{E}\left[\left|\langle \omega\otimes\omega, f\rangle\right|^{r}\right] = \mathbb{E}\left[\rho\left(\omega_{WN}\right)\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle\right|^{r}\right]$$
$$\leq \mathbb{E}\left[\rho^{q}\left(\omega_{WN}\right)\right]^{1/q}\mathbb{E}\left[\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle\right|^{rp}\right]^{1/p}.$$

ii) One has

$$\mathbb{E}\left[\left|\langle \omega\otimes\omega, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|\right] = \mathbb{E}\left[\rho\left(\omega_{WN}\right)\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|\right]$$
$$\leq \mathbb{E}\left[\rho^{q}\left(\omega_{WN}\right)\right]^{1/q}\mathbb{E}\left[\left|\langle \omega_{WN}\otimes\omega_{WN}, \phi\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^{p}\right]^{1/p}.$$

Moreover,

$$\mathbb{E}\left[\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^{p}\right]$$
$$\leq \mathbb{E}\left[\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^{2}\right]^{1/2}\mathbb{E}\left[\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^{2p-2}\right]^{1/2}$$
$$= C_q^0\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)^2\,dxdy\right)^{1/2}$$

where

$$C_q^0 := 2\mathbb{E}\left[\left|\langle \omega_{WN}\otimes\omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^{2p-2}\right]^{1/2}$$

is a finite constant, due to property (i) of a previous corollary. We set $C_q = k_q^{1/q}\left(C_q^0\right)^{1/p}$. ■

The next results are the same as those above in the white noise case except that we have a lower order of integrability, nevertheless sufficient for our aims.

Theorem 80 *Under the previous assumptions, assume that $H_\phi^n \in H^{2+}\left(\mathbb{T}^2\times\mathbb{T}^2\right)$ are symmetric and approximate H_ϕ as in Theorem 74. Then the sequence of r.v.'s $\left\langle \omega\otimes\omega, H_\phi^n\right\rangle$ is a Cauchy sequence in $L^1(\Xi)$. We denote by $\langle \omega\otimes\omega, H_\phi\rangle$ its limit. It is the same if H_ϕ^n is replaced by $\widetilde{H}_\phi^n$ with the properties described in Theorem 74.*

Proof. Since $\lim_{n\to\infty}\int H_\phi^n(x,x)\,dx = 0$, it is equivalent to show that $\left\langle \omega\otimes\omega, H_\phi^n\right\rangle - \int H_\phi^n(x,x)\,dx$ is a Cauchy sequence in $L^1(\Xi)$. We have

$$\mathbb{E}\left[\left|\left\langle \omega\otimes\omega, H_\phi^n\right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega\otimes\omega, H_\phi^m\right\rangle + \int H_\phi^m(x,x)\,dx\right|\right]$$
$$= \mathbb{E}\left[\left|\left\langle \omega\otimes\omega, \left(H_\phi^n - H_\phi^m\right)\right\rangle - \int\left(H_\phi^n - H_\phi^m\right)(x,x)\,dx\right|\right]$$

and now we use property (ii) of the Corollary

$$\leq C_q \left\| H_\phi^n - H_\phi^m \right\|_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}^{1/p}.$$

Due to our assumptions, this implies the Cauchy property. Hence $\langle \omega \otimes \omega, H_\phi \rangle$ is well defined. The invariance property is prove in a similar way. ■

Theorem 81 *Let $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$ be a function such that $\int \rho_t^q d\mu \leq C$ for some constants $C > 0$, $q > 1$, where μ is the law of white noise; and $\int \rho_t d\mu = 1$ for every $t \in [0,T]$. Let $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ be a measurable map with trajectories of class $C\left([0,T]; H^{-1-}\right)$. Assume that the law of ω_t is $\rho_t d\mu$, at every time $t \in [0,T]$. Let H_ϕ^n be an approximation of H_ϕ as above, of class $H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$. Then the well defined sequence of real valued process $\left\{ s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle ; s \in [0,T] \right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1\left(\Xi; L^1(0,T)\right)$.*

Proof. As in previous proofs, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left| \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega_s \otimes \omega_s, H_\phi^m \right\rangle + \int H_\phi^m(x,x)\,dx \right| ds \right] \\
&= \int_0^T \mathbb{E}\left[\left| \left\langle \omega_s \otimes \omega_s, \left(H_\phi^n - H_\phi^m\right) \right\rangle - \int \left(H_\phi^n - H_\phi^m\right)(x,x)\,dx \right| \right] ds \\
&\leq C_q T \left\| H_\phi^n - H_\phi^m \right\|_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}^{1/p}
\end{aligned}$$

■

Definition 82 *Under the assumptions of the previous theorem, we denote by $\langle \omega_\cdot \otimes \omega_\cdot, H_\phi \rangle$ the process of class $L^1\left(\Xi; L^1(0,T)\right)$, limit of the sequence $\left\{ s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle ; s \in [0,T] \right\}_{n\in\mathbb{N}}$.*

4.2.5 Definition of the weak vorticity formulation

Definition 83 *We say that a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ with trajectories of class $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ is a white noise solution of Euler equations if ω_t is a white noise at every time $t \in [0,T]$ and for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, we have the following identity P-a.s., uniformly in time,*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle\, ds.$$

Here $\langle \omega_t, \phi \rangle$ is a.s. a continuous function of time because we assume that trajectories of ω are of class $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$, and $\int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle\, ds$ is the continuous process obtained by integration of the $L^2(0,T)$-process provided by Definition 77.

In the case of the previous definition, in addition, we may require that $\omega_\cdot$ is a time-stationary process. In a sense, the law of white noise is an invariant measure, although we do not have a proper Markov structure allowing us to talk about invariant measures in the classical sense.

Using Definition 82 we may generalize the previous definition to the following case:

Definition 84 *Let $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$ satisfy $\int \rho_t^q d\mu \leq C$ for some constants $C > 0$, $q > 1$, where μ is the law of white noise; and $\int \rho_t d\mu = 1$ for every $t \in [0,T]$. Let $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ be a measurable map with trajectories of class $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$, such that ω_t has law $\rho_t d\mu$, for every $t \in [0,T]$. We say that ω is a $\rho-$white noise solution of Euler equations if for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, $t \mapsto \langle \omega_t, \phi \rangle$ is continuous and we have the following identity P-a.s., uniformly in time,*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds.$$

4.3 Random point vortex dynamics

Let us introduce some notations. In $\left(\mathbb{T}^2\right)^N$, denote by Δ_N the generalized diagonal

$$\Delta_N = \left\{ (x_1, ..., x_N) \in \left(\mathbb{T}^2\right)^N : x_i = x_j \text{ for some } i \neq j,\ i, j = 1, ..., n \right\}.$$

Then introduce the set of unlabelled and labelled finite sequences of different points

$$F_N \mathbb{T}^2 = \left\{ (x_1, ..., x_N) \in \left(\mathbb{T}^2\right)^N : (x_1, ..., x_N) \in \Delta_N^c \right\}$$

$$\mathcal{L} F_N \mathbb{T}^2 = \left\{ \left((\xi_1, x_1), ..., (\xi_N, x_N) \right) \in \left(\mathbb{R} \times \mathbb{T}^2\right)^N : (x_1, ..., x_N) \in \Delta_N^c \right\}$$

and the unlabelled and labelled configuration space

$$C_N \mathbb{T}^2 = F_N \mathbb{T}^2 / \Sigma_N$$

$$\mathcal{L} C_N \mathbb{T}^2 = \mathcal{L} F_N \mathbb{T}^2 / \Sigma_N$$

where Σ_N is the group of permutations of coordinates. This set, $\mathcal{L} C_N \mathbb{T}^2$, is in bijection with the set of discrete signed measures with n-point support:

$$\mathcal{M}_N\left(\mathbb{T}^2\right) = \left\{ \mu \in \mathcal{M}\left(\mathbb{T}^2\right) : \exists X \in C_N \mathbb{T}^2 : |\mu|\left(X^c\right) = 0, \mu(x) \neq 0 \text{ for every } x \in X \right\}.$$

We do not use extensively these notations but they may help to formalize further the topics we are going to describe.

4.3.1 Definition for a.e. initial condition

Inspired by the results proved above in Chapter 3 consider, for every $N \in \mathbb{N}$, the finite dimensional dynamics in $\left(\mathbb{T}^2\right)^N$

$$\frac{dX_t^{i,N}}{dt} = \sum_{j=1}^{N} \frac{1}{\sqrt{N}} \xi_j K\left(X_t^{i,N} - X_t^{j,N}\right) \qquad i = 1, ..., N \tag{4.7}$$

with initial condition $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \left(\mathbb{T}^2\right)^N \backslash \Delta_N$, where as above K is the Biot-Savart kernel on $\mathbb{T}^2$; we set $K(0) = 0$ so that the self-interaction (namely when $j = i$) in the sum does not count. The intensities $\xi_1, ..., \xi_N$ are (random) numbers of any sign. One can consider (4.7) as a dynamics on the configuration space $C_N \mathbb{T}^2$. This system corresponds also to the time-evolution of a vorticity distribution concentrated at positions $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$:

$$\omega_t^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_t^n}.$$

Let $\otimes_N Leb_{\mathbb{T}^2}$ be Lebesgue measure on $\left(\mathbb{T}^2\right)^N$. From Chapter 1 we know:

Theorem 85 *For every $(\xi_1, ..., \xi_N) \in \mathbb{R}^N$ and for $\otimes_N Leb_{\mathbb{T}^2}$- almost every $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \Delta_N^c$, there is a unique solution $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ of system (4.7), with the property that $\left(X_t^{1,N}, ..., X_t^{N,N}\right) \in \Delta_N^c$ for all $t \geq 0$. Moreover, considering the initial condition as a random variable with distribution $\otimes_N Leb_{\mathbb{T}^2}$, the stochastic process $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is stationary, with invariant marginal law $\otimes_N Leb_{\mathbb{T}^2}$.*

When this occurs, the measure-valued process $\omega_t^N = \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \delta_{X_t^n}$ satisfies, for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, the identity

$$\begin{aligned}\frac{d}{dt}\left\langle \omega_t^N, \phi \right\rangle &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \frac{d}{dt} \phi\left(X_t^n\right) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \nabla \phi\left(X_t^n\right) \cdot \sum_{j=1}^{N} \frac{1}{\sqrt{N}} \xi_j K\left(X_t^n - X_t^j\right) \\ &= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \nabla \phi(x) \cdot K(x-y)\, \omega_t^N(dx)\, \omega_t^N(dy)\end{aligned}$$

and therefore

$$\left\langle \omega_t^N, \phi \right\rangle = \left\langle \omega_0^N, \phi \right\rangle + \int_0^t \left\langle \omega_s^N \otimes \omega_s^N, H_\phi \right\rangle ds.$$

4.3.2 Random point vortices, at time $t = 0$, converging to white noise, and their time evolution

On a probability space $(\Xi, \mathcal{F}, \mathbb{P})$, let (ξ_n) be an i.i.d. sequence of $N(0,1)$ r.v.'s and (X_0^n) be an i.i.d. sequence of $\mathbb{T}^2$-valued r.v.'s, independent of (ξ_n) and uniformly distributed. Denote by

$$\lambda_N^0 := \otimes_N \left(N(0,1) \otimes Leb_{\mathbb{T}^2}\right)$$

the law of the random vector

$$\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right).$$

For every $N \in \mathbb{N}$, let us consider also the measure-valued vorticity field

$$\omega_0^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_0^n}.$$

Remark 86 *Since product Lebesgue measure does not charge the generalized diagonal Δ_N, the law λ_N^0 can be seen as a probability measure on the set of labelled ordered different points $\mathcal{L}F_N\mathbb{T}^2$ (see the beginning of Section 4.3). It is an exchangeable measure (namely invariant by permutations) and thus it induces a probability measure on the labelled configuration space $\mathcal{L}C_N\mathbb{T}^2$. It also induces a probability measure on $\mathcal{M}_N\left(\mathbb{T}^2\right)$ or, what we need below, on $H^{-1-}\left(\mathbb{T}^2\right)$. We shall denote this induced measure on discrete measures or on distributions by $\mu_N^0(d\omega)$. Defined the measurable map $\mathcal{T}_N : \left(\mathbb{R} \times \mathbb{T}^2\right)^N \to H^{-1-}\left(\mathbb{T}^2\right)$ as*

$$\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right) \overset{\mathcal{T}_N}{\mapsto} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_0^n}$$

we have (with the push-forward notation)

$$\mu_N^0 = (\mathcal{T}_N)_* \lambda_N^0.$$

The random distribution ω_0^N is centered, becuase

$$\mathbb{E}\left[\xi_n \left\langle \delta_{X_0^n}, \varphi \right\rangle\right] = 0$$

(true since ξ_n and $\left\langle \delta_{X_0^n}, \varphi \right\rangle$ are independent and ξ_n is centered). Let us denote by Q_N the covariance operator of ω_0^N, defined as

$$\langle Q_N \varphi, \psi \rangle = \mathbb{E}\left[\left\langle \omega_0^N, \varphi \right\rangle \left\langle \omega_0^N, \psi \right\rangle\right]$$

for all $\varphi, \psi \in C^{\infty}\left(\mathbb{T}^2\right)$. We have

$$\begin{aligned}
\langle Q_N \varphi, \psi \rangle &= \frac{1}{N} \sum_{n,m=1}^{N} \mathbb{E}\left[\xi_n \xi_m \left\langle \delta_{X_0^n}, \varphi \right\rangle \left\langle \delta_{X_0^m}, \psi \right\rangle\right] \\
&= \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[\xi_n^2\right] \mathbb{E}\left[\left\langle \delta_{X_0^n}, \varphi \right\rangle \left\langle \delta_{X_0^n}, \psi \right\rangle\right] \\
&= \mathbb{E}\left[\xi_1^2\right] \mathbb{E}\left[\varphi\left(X_0^1\right) \psi\left(X_0^1\right)\right] \\
&= \int_{\mathbb{T}^2} \varphi(x) \psi(x)\, dx
\end{aligned}$$

hence ω_0^N has the same covariance as white noise, but obviously it is not Gaussian. However, we have proved it converges to the White Noise measure, see Theorem 61, that we repeat here for convenience, due to little change of notations (here we have also the dynamics):

Proposition 87 *If ω_{WN} denotes white noise, then*

$$\omega_0^N \overset{Law}{\rightharpoonup} \omega_{WN}$$

where convergence takes place in $H^{-1-\delta}$ for every $\delta > 0$. Moreover, convergence takes place in H^{-1-}.

The proof was given in Chapter 3; but we recall here a basic (and simple) estimate used in the proof, since it will play a role below:

$$\mathbb{E}\left[\left\|\xi_n \delta_{X_0^n}\right\|_{H^{-1-\delta}}^2\right] < \infty. \tag{4.8}$$

As a consequence of Theorem 85 we have:

Proposition 88 *Consider the vortex dynamics with random intensities $(\xi_1, ..., \xi_N)$ and random initial positions $\left(X_0^1, ..., X_0^N\right)$ distributed as λ_N^0. For a.e. value of $\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right)$ the dynamics $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is well defined in Δ_N^c for all $t \geq 0$, and the associated measure-valued vorticity ω_t^N satisfies the weak vorticity formulation. The stochastic process ω_t^N is stationary in time and space-homogeneous; in particular the law of $\left(\left(\xi_1, X_t^1\right), ..., \left(\xi_N, X_t^N\right)\right)$ is λ_N^0 at any time $t \geq 0$.*

Proof. The first claims are obvious consequences of Theorem 85. Given $(\xi_1, ..., \xi_N)$, the process $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is stationary. Hence, denoted $(\xi_1, ..., \xi_N)$ by ξ and $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ by X_t, for every $0 \leq t_1 \leq ... \leq t_n$ and bounded measurable F, the random variable (conditional expectation given the σ-field generated by ξ)

$$\mathbb{E}\left[F\left(\left(\xi, X_{t_1+h}\right), ..., \left(\xi, X_{t_n+h}\right)\right) |\xi\right]$$

is independent of h (in the equivalence class of conditional expectation). Therefore its expectation, namely $\mathbb{E}\left[F\left(\left(\xi, X_{t_1+h}\right), ..., \left(\xi, X_{t_n+h}\right)\right)\right]$, is independent of h, which implies that (ξ, X_t) (and therefore ω_t^N) is a stationary process. Space homogeneity is not used below and thus we do not prove it, but it is not difficult due to the symmetries of the system. ■

4.3.3 Integrability properties of the random point vortices

Let ω_t^N be given by Proposition 88. It satisfies estimates similar to those of white noise.

Lemma 89 *Assume $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ is symmetric, bounded and measurable. Then, for every $p \geq 1$ and $\delta > 0$ there are constants $C_p, C_{p,\delta} > 0$ such that*

$$\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^p\right] \leq C_p \left\|f\right\|_\infty^p$$

$$\mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta}}^p\right] \leq C_{p,\delta}$$

and moreover

$$\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^2\right] = \frac{3}{N} \int f^2\left(x,x\right) dx + \left(\int f\left(x,x\right) dx\right)^2 + 2\int\int f^2\left(x,y\right) dxdy.$$

Proof. Step 1. It is sufficient to consider integer values of p. One has

$$\begin{aligned}
\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^p\right] &= \mathbb{E}\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f\left(x,y\right) \omega_t^N\left(dx\right) \omega_t^N\left(dy\right)\right)^p \\
&= \int_{\left(\mathbb{T}^2\right)^{2p}} \mathbb{E}\left[\prod_{i=1}^p f\left(x_i, y_i\right) \prod_{i=1}^p \left(\omega_t^N\left(dx_i\right) \omega_t^N\left(dy_i\right)\right)\right] \\
&= \frac{1}{N^p} \sum_{k_1,h_1,...,k_p,h_p=1}^N \mathbb{E}\left[\prod_{i=1}^p \xi_{k_i}\xi_{h_i}\right] \mathbb{E}\left[\prod_{i=1}^p f\left(X_t^{k_i}, X_t^{h_i}\right)\right].
\end{aligned}$$

We replace here Isserlis-Wick theorem by a combinatorial argument based on the independence of the r.v.'s ξ_i. Denote by $\mathcal{P}_p$ the family of all $(2p)$-ples $(k_1, h_1, ..., k_p, h_p)$ that are "paired", namely such that we may split $(k_1, h_1, ..., k_p, h_p)$ in p pairs such that in each pair the two elements have the same value; an example is when $h_1 = k_1, \ldots, h_p = k_p$. Notice that we do not require that the values in different pairs are different. One has $\mathbb{E}\left[\prod_{i=1}^p \xi_{k_i}\xi_{h_i}\right] = 0$ if $(k_1, h_1, ..., k_p, h_p) \notin \mathcal{P}_p$, hence

$$\begin{aligned}
\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^p\right] &= \frac{1}{N^p} \sum_{(k_1,h_1,...,k_p,h_p)\in\mathcal{P}_p} \mathbb{E}\left[\prod_{i=1}^p \xi_{k_i}\xi_{h_i}\right] \mathbb{E}\left[\prod_{i=1}^p f\left(X_t^{k_i}, X_t^{h_i}\right)\right] \\
&\leq \left\|f\right\|_\infty^p \frac{C_p'}{N^p} Card\left(\mathcal{P}_p\right)
\end{aligned}$$

where C'_p is a constant that bounds from above $\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]$ independently of the index. The cardinality of $\mathcal{P}_p$ is bounded above by $C''_p N^p$ for another constant $C''_p > 0$ (the idea is that given any one of the N values of k_1, either h_1 or k_2 or one of the next indexes is equal to k_1, and this constraints the variability of that index to one value; then repeat p times this argument). Therefore $\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f\right\rangle^p\right] \leq \|f\|_\infty^p C'_p C''_p$. This proves the first claim of the lemma, with $C_p = C'_p C''_p$.

Step 2. Similarly,

$$\begin{aligned}
\mathbb{E}\left[\left\|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\xi_n\delta_{X_t^n}\right\|_{H^{-1-\delta/2}}^{2p}\right] &= \mathbb{E}\left[\left(\left\langle \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\xi_n\delta_{X_t^n}, \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\xi_n\delta_{X_t^n}\right\rangle_{H^{-1-\delta/2}}\right)^p\right] \\
&= \frac{1}{N^p}\mathbb{E}\left[\left(\sum_{n,m=1}^{N}\xi_n\xi_M\left\langle\delta_{X_t^n},\delta_{X_t^m}\right\rangle_{H^{-1-\delta/2}}\right)^p\right] \\
&= \frac{1}{N^p}\sum_{k_1,h_1,\dots,k_p,h_p=1}^{N}\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]\mathbb{E}\left[\prod_{i=1}^{p}\left\langle\delta_{X_t^{k_i}},\delta_{X_t^{h_i}}\right\rangle_{H^{-1-\delta/2}}\right] \\
&= \frac{1}{N^p}\sum_{(k_1,h_1,\dots,k_p,h_p)\in\mathcal{P}_p}\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]\mathbb{E}\left[\prod_{i=1}^{p}\left\langle\delta_{X_0^{k_i}},\delta_{X_0^{h_i}}\right\rangle_{H^{-1-\delta/2}}\right] \\
&\leq C_{p,\delta}
\end{aligned}$$

because we use the same bounds above for $\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]$ and $Card\left(\mathcal{P}_p\right)$ and a trivial uniform bound on $\mathbb{E}\left[\prod_{i=1}^{p}\left\langle\delta_{X_0^{k_i}},\delta_{X_0^{h_i}}\right\rangle_{H^{-1-\delta/2}}\right]$ due to the property $\left\|\delta_{X_0^i}\right\|_{H^{-1-\delta/2}} \leq C$ showed in the proof of Proposition 87.

Step 3.

$$\begin{aligned}
\mathbb{E}\left[\left\langle\omega_t^N\otimes\omega_t^N, f\right\rangle^2\right] &= \mathbb{E}\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)\,\omega_t^N(dx)\,\omega_t^N(dy)\right)^2 \\
&= \mathbb{E}\int_{(\mathbb{T}^2)^4} f(x,y)\, f\left(x',y'\right)\omega_t^N(dx)\,\omega_t^N(dy)\,\omega_t^N\left(dx'\right)\omega_t^N\left(dy'\right) \\
&= \frac{1}{N^2}\sum_{ijkh=1}^{N}\mathbb{E}\left[f\left(X_t^i,X_t^j\right)f\left(X_t^k,X_t^h\right)\right]\mathbb{E}\left[\xi_i\xi_j\xi_k\xi_h\right].
\end{aligned}$$

In this sum there are various terms. The term with $i=j=k=h$ is

$$\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left[f\left(X_t^i,X_t^i\right)f\left(X_t^i,X_t^i\right)\right]\mathbb{E}\left[\xi_i^4\right] = \frac{\mathbb{E}\left[\xi^4\right]}{N}\int f^2(x,x)\,dx.$$

Then there are terms with $j = i$, $h = k$:

$$\begin{aligned}&\frac{1}{N^2}\sum_{i\neq k=1}^{N} E\left[\xi_i^2\right] E\left[\xi_k^2\right] \mathbb{E}\left[f\left(X_t^i, X_t^i\right) f\left(X_t^k, X_t^k\right)\right] \\ &= \frac{E\left[\xi^2\right]^2}{N^2}\sum_{i\neq k=1}^{N} \mathbb{E}\left[f\left(X_t^i, X_t^i\right)\right] \mathbb{E}\left[f\left(X_t^k, X_t^k\right)\right] \\ &\leq E\left[\xi^2\right]^2 \left(\int f(x,x)\, dx\right)^2 .\end{aligned}$$

Then there are terms with $k = i$, $h = j$:

$$\frac{E\left[\xi^2\right]^2}{N^2}\sum_{i\neq j=1}^{N} \mathbb{E}\left[f\left(X_t^i, X_t^j\right) f\left(X_t^i, X_t^j\right)\right] \leq E\left[\xi^2\right]^2 \int\int f^2(x,y)\, dxdy.$$

Finally, then there are terms with $k = j$, $h = i$: (here we use symmetry)

$$\frac{E\left[\xi^2\right]^2}{N^2}\sum_{i\neq j=1}^{N} \mathbb{E}\left[f\left(X_t^i, X_t^j\right) f\left(X_t^j, X_t^i\right)\right] \leq E\left[\xi^2\right]^2 \int\int f^2(x,y)\, dxdy.$$

■

4.4 Main results

Denote by μ the law of White Noise. We first formulate our version of Albeverio-Cruzeiro result [2].

Theorem 90 *There exists a probability space $(\Xi, \mathcal{F}, P)$ with the following properties.*

i) There exists a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ such that $\omega_\cdot$ is a time-stationary white noise solution of Euler equations, in the sense of Definition 83.

ii) On $(\Xi, \mathcal{F}, P)$ one can define a subsequence of the random point vortex system described in Section 4.3.2 which converges P-a.s. to the solution of point (i) in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$.

iii) On $(\Xi, \mathcal{F}, P)$ one can define a sequence of functions $\omega^{(n)}(\theta, t, x)$, $(\theta, t, x) \in \Xi \times [0,T] \times \mathbb{T}^2$, such that for $\mathbb{P}$-a.e. $\theta \in \Xi$ the functions $(t,x) \mapsto \omega^{(n)}(\theta, t, x)$ are L^∞-solutions of 2D Euler equations, and converge to $\omega_\cdot(\theta)$ in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$.

We prove also a generalization to $\rho-$white noise solutions; the assumption on ρ_0 is presumably too restrictive but further investigation is needed for more generality.

Theorem 91 *Given $\rho_0 \in C_b\left(H^{-1-}\left(\mathbb{T}^2\right)\right)$ such that $\rho_0 \geq 0$ and $\int \rho_0 d\mu = 1$, there exist a probability space $(\Xi, \mathcal{F}, P)$, a bounded measurable function $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0, \|\rho_0\|_\infty]$ and a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ such that $\omega_\cdot$ is a $\rho-$white noise solution of Euler equations, in the sense of Definition 84. It is also the limit P-a.s. in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ of a suitable sequence of random point vortices and of L^∞-solutions.*

4.4.1 Remarks on disintegration, uniqueness an Gaussianity

In this section we discuss several limits of the previous results and open problems arising from them.

Consider the law Q, on path space $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$, of a solutions provided by Theorem 90 (similarly for Theorem 91). If we disintegrate Q with respect to the marginal law at time $t = 0$ (namely the white noise law μ for Theorem 90 or law $\rho_0 d\mu$ for Theorem 91), we find a probability kernel $Q(\cdot, \omega_0)$, indexed by $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$, such that for μ-a.e. $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$ the probability measure $Q(\cdot, \omega_0)$ is concentrated on solutions of Euler equations (in the sense described above). But $Q(\cdot, \omega_0)$ is not of the form $\delta_{\omega_\cdot^{\omega_0}}$, namely it is not concentrated on a single solution $\omega_t^{\omega_0}$ with initial condition ω_0; or at least we do not know this information. In the language of [7], we have a superposition solution that we do not know to be a graph. For μ-a.e. $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$, we have at least one solution ω of Euler equations, but we could have many; also in the sense of the Lagrangian flows described in [7], see below.

In the case of Theorem 91 on $\rho-$white noise solutions, we are certainly far away from any uniqueness claim, even in law. Presumably one should try first to investigate uniqueness of ρ_t, maybe with tools related to those of [8], [9], [22], [31], which already looks a formidable task.

In the case however of Theorem 90, due to fact that the law at any time t is uniquely determined, it could seem that a statement of uniqueness in law is not far (notice that uniqueness in law would also imply that the full sequence of point vortices converges to it, in law). And perhaps a statement of uniqueness of Lagrangian flows. These are however open problems, potentially of very difficult solution. Let us mention where two approaches, both based on uniqueness of the 1-dimensional marginals, meet essential difficulties.

One approach is by the criteria of uniqueness for martingale solutions of stochastic equations (applicable in principle to deterministic equations with random solutions). Take as an example Theorem 6.2.3 of [57]. It does not apply here, at the present stage of our understanding, since we do not have any information of uniqueness of 1-dimensional marginals starting from generic deterministic initial conditions. As remarked above, by disintegration we may construct solutions $Q(\cdot, \omega_0)$ (in the sense of the martingale problem; we do not develop the details) for μ-a.e. $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$, but we do not know the uniqueness of their 1-point marginals.

A second approach is described in [7], see Theorem 16. It requires the validity of comparison principle, a variant of 1-point marginal uniqueness, for the associated continuity equation. The comparison principle should hold in a convex class of solutions (denoted by $\mathcal{L}_b$ in [7]); if only this, one could take the class defined by the rule that it is white noise at every time. However, the class $\mathcal{L}_b$ in [7] has to satisfy also a monotonicity property (see (14) in [7], used in essential way in Theorem 18), which is not satisfied by the trivial class defined by being white noise at every time. If we enlarge the class to have the monotonicity property, we are faced with a very difficult question of uniqueness - or comparison principle - for weak solutions of the continuity equation associated to Euler equations, which is an open problem.

The k-dimensional time marginals are not easily identified by the Euler equations or by the random point vortex dynamics. The question is, given $0 \leq t_1 < \cdots < t_k \leq T$, to understand the limit as $N \to \infty$ of the marginal $\left(\omega_{t_1}^N, ..., \omega_{t_k}^N\right)$, given by

$$\left(\omega_{t_1}^N, ..., \omega_{t_k}^N\right) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \left(\delta_{X_{t_1}^n}, ..., \delta_{X_{t_k}^n}\right).$$

This is an open problem.

For Burgers equations with white noise initial conditions, thanks to special representation formulae, it was possible to compute the two-point distribution, see [32]. Here we do not see yet a method. But, also due to the comparison with [32], one should be aware that there is no reason why k-dimensional time marginals are Gaussian! Nonlinearity, still preserving a Gaussian initial condition, should distroy Gaussianity at the level of the process.

Another example of nonlinear equation with stationary solutions having Gaussian 1-dimensional marginals is KPZ equation or the stochastic Burgers equations, see [35], [33], [34].

4.4.2 Proof of Theorem 90

Consider the Polish space $\mathcal{X} = C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ with the metric $d_{\mathcal{X}}(\omega_\cdot, \omega'_\cdot)$ defined in Section **??**. J. Simon [55], in Corollary 8, gives a useful class of compact sets in this space, generalizing the more classical Aubin-Lions compactness lemma (and Ascoli-Arzelà criterion). Let us explain the result of Simon in our context. Take $\delta \in (0,1)$, $\gamma > 3$ (this special choice of γ is due to the estimates below) and consider the spaces

$$X = H^{-1-\delta/2}\left(\mathbb{T}^2\right), \qquad B = H^{-1-\delta}\left(\mathbb{T}^2\right), \qquad Y = H^{-\gamma}\left(\mathbb{T}^2\right).$$

We have

$$X \subset B \subset Y$$

with compact dense embeddings and we also have, for a suitable constant $C > 0$ and for

$$\theta = \frac{\delta/2}{\gamma - 1 - \delta/2}$$

the interpolation inequality

$$\|\omega\|_B \leq C \|\omega\|_X^{1-\theta} \|\omega\|_Y^{\theta}$$

for all $\omega \in X$. These are preliminary assumptions of Corollary 8 of [55]. Then such Corollary, in the second part, in the particular case $r_1 = 2$, states that a bounded family F in

$$L^{p_0}(0,T;X) \cap W^{1,2}(0,T;Y)$$

is relatively compact in

$$C([0,T];B)$$

if

$$\frac{\theta}{2} > \frac{1-\theta}{p_0}.$$

Here p_0 is any number in $[1,\infty]$. We apply this result to our spaces X, B, Y, taking p_0 large enough to have the previous inequality. More precisely, we use the following statement (notice that $\frac{1-\theta}{\theta} = \frac{\gamma-1-\delta}{\delta/2}$):

Lemma 92 *Let $\delta > 0$, $\gamma > 3$ be given. If*

$$p_0 > \frac{\gamma - 1 - \delta}{\delta/2}$$

then

$$L^{p_0}\left(0,T;H^{-1-\delta/2}\left(\mathbb{T}^2\right)\right) \cap W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$$

is compactly embedded into

$$C\left([0,T];H^{-1-\delta}\left(\mathbb{T}^2\right)\right).$$

In fact we need compactness in $\mathcal{X}$. Denote by $L^{\infty-}\left(0,T;H^{-1-}\left(\mathbb{T}^2\right)\right)$ the space of all functions of class $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$ for any $p_0 > 0$ and $\delta > 0$, endowed with the metric

$$d_{L_t^{\infty-}(H^{-1-})}\left(\omega_\cdot,\omega'_\cdot\right) = \sum_{n=1}^{\infty} 2^{-n}\left(\left(\int_0^T \left\|\omega_t - \omega'_t\right\|_{H^{-1-\frac{1}{n}}}^n\right)^{1/n} \wedge 1\right).$$

It is a simple exercise to check that:

Corollary 93 *Let $\gamma > 3$ be given. Then*

$$\mathcal{Y} := L^{\infty-}\left(0,T;H^{-1-}\left(\mathbb{T}^2\right)\right) \cap W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$$

is compactly embedded into $\mathcal{X}$.

Let Q^N be the law of ω^N on Borel subsets of $\mathcal{X}$. We want to prove that the family $\{Q^N\}_{N\in\mathbb{N}}$ is tight in this space. In order to prove this, it is sufficient to prove that the family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in the space $\mathcal{Y}$ given by the previous corollary. For this purpose, it is sufficient to prove that $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$ and in each $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$, for any $p_0>0$ and $\delta>0$. Let us prove these conditions.

The family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$ (by Chebyshev inequality) because

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left[\int_0^T\left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0}dt\right]<\infty.$$

This inequality (that we could conceptually summarize as the "compactness in space") comes from stationarity of ω_t^N:

$$\mathbb{E}\left[\int_0^T\left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0}dt\right]=\int_0^T\mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0}\right]dt\le C_{p_0,\delta}T$$

by Lemma 89.

To prove "compactness in time", namely the property that the family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$, we use the equation, in its weak vorticity formulation. We have, for all $\phi\in C^\infty\left(\mathbb{T}^2\right)$,

$$\left\langle\omega_t^N,\phi\right\rangle=\left\langle\omega_0^N,\phi\right\rangle+\int_0^t\left\langle\omega_s^N\otimes\omega_s^N,H_\phi\right\rangle ds$$

where P-a.s. the function $s\mapsto\left\langle\omega_s^N\otimes\omega_s^N,H_\phi\right\rangle$ is continuous (the trajectories of point vortices are continuous and never touch the diagonal), hence, P-a.s., the function $t\mapsto\left\langle\omega_t^N,\phi\right\rangle$ is continuously differentiable and $\partial_t\left\langle\omega_t^N,\phi\right\rangle=\left\langle\omega_t^N\otimes\omega_t^N,H_\phi\right\rangle$. Thus

$$\begin{aligned}\mathbb{E}\left[\left|\partial_t\left\langle\omega_t^N,\phi\right\rangle\right|^2\right]&=\mathbb{E}\left[\left|\left\langle\omega_t^N\otimes\omega_t^N,H_\phi\right\rangle\right|^2\right]\\&\le C\left\|H_\phi\right\|_\infty^2\le C\left\|D^2\phi\right\|_\infty^2\end{aligned}$$

by Lemma 89. Then we apply this inequality to $\phi=e_k$ and get

$$\mathbb{E}\left[\left|\partial_t\left\langle\omega_t^N,e_k\right\rangle\right|^2\right]\le C\left|k\right|^4.$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\int_0^T\left\|\partial_t\omega_t^N\right\|_{H^{-\gamma}}^2dt\right]&=\mathbb{E}\left[\int_0^T\sum_k\left(1+|k|^2\right)^{-\gamma}\left|\left\langle\partial_t\omega_t^N,e_k\right\rangle\right|^2dt\right]\\&\le C\mathbb{E}\left[\int_0^T\sum_k\left(1+|k|^2\right)^{-\gamma}|k|^4\,dt\right]<\infty\end{aligned}$$

for $2\gamma - 4 > 2$, hence $\gamma > 3$. The estimate for $\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-\gamma}}^2 dt\right]$ is similar to the one for "compactness in space" above. By Chebyshev inequality, $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$.

We have proved that the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $\mathcal{Y}$ and thus it is tight in $\mathcal{X}$. From Prohorov theorem, it is relatively compact in $\mathcal{X}$. Let $\left\{Q^{N_k}\right\}_{k\in\mathbb{N}}$ be a subsequence which converges weakly, in $\mathcal{X}$, to a Borel probability measure Q. First, convergence in $\mathcal{X}$ implies that Q is invariant by time-shift (because Q^N is; by shift we mean shift of finite dimensional distributions such that all involved time points are in $[0,T]$) and the marginal at any time is the law of white noise, by Proposition 87 (recall that ω_t^N is stationary, hence this proposition applies at every time).

By Skorokhod representation theorem, there exist a new probability space $\left(\widehat{\Xi},\widehat{\mathcal{F}},\widehat{P}\right)$ and r.v.'s $\widehat{\omega}^{N_k}$, $\widehat{\omega}$ with values in $\mathcal{X}$, such that the laws of $\widehat{\omega}^{N_k}$ and $\widehat{\omega}$ are Q^{N_k} and Q respectively, and $\widehat{\omega}^{N_k}$ converges P-a.s. to $\widehat{\omega}$ in the topology of $\mathcal{X}$; since $\mathcal{X}$ is made of functions of time, we may see $\widehat{\omega}^{N_k}$ and $\widehat{\omega}$ as stochastic processes, $\widehat{\omega}_t^{N_k}$ and $\widehat{\omega}_t$ being the result of application of the projection at time t. We are going to check that $\widehat{\omega}$, or more precisely another process closely defined, is the solution claimed by the theorem. We already know it has trajectories of class $C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$, it is time stationary and with marginal being a white noise. We have to show that it satisfies the equation, in the sense specified by the definitions.

We have to enlarge the probability space $\left(\widehat{\Xi},\widehat{\mathcal{F}},\widehat{P}\right)$ to be sure it contains certain independent r.v.'s we need in the construction. Denote by $\left(\widetilde{\Xi},\widetilde{\mathcal{F}},\widetilde{P}\right)$ a probability space where, for every N, it is defined a random permutation $\widetilde{s}_N : \widetilde{\Xi} \to \Sigma_N$, uniformly distributed. Define the new probability space

$$(\Xi,\mathcal{F},P) := \left(\widehat{\Xi}\times\widetilde{\Xi},\widehat{\mathcal{F}}\otimes\widetilde{\mathcal{F}},\widehat{P}\otimes\widetilde{P}\right)$$

and the new processes

$$\omega^{N_k} = \widehat{\omega}^{N_k}\circ\pi_1, \qquad \omega = \widehat{\omega}\circ\pi_1, \qquad s_N = \widetilde{s}_N\circ\pi_2$$

where π_1 and π_2 are the projections on $\widehat{\Xi}\times\widetilde{\Xi}$. We adopt a little abuse of notation here, because we indicate the final spaces and processes like the original ones, but we shall try to clarify everywhere which ones we are investigating. Notice that the properties of convergence and of the laws of the processes ω^{N_k} and ω are the same as those of $\widehat{\omega}^{N_k}$ and $\widehat{\omega}$.

Lemma 94 *The process $\omega_t^{N_k}$ (the one on the new probability space) can be represented in the form $\frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\xi_i\delta_{X_t^{i,N_k}}$, where*

$$\left(\left(\xi_1,X_0^{1,N_k}\right),...,\left(\xi_{N_k},X_0^{N_k,N_k}\right)\right) \tag{4.9}$$

is a random vector with law λ_N^0 and $\left(X_t^{1,N_k},...,X_t^{N_k,N_k}\right)$ solves system (4.7) with initial condition $\left(X_0^{1,N_k},...,X_0^{N_k,N_k}\right)$.

Proof. Step 1. Let us list a few preliminary facts; we omit some detail in the proofs; we extensively use the notations at the beginning of Section 4.3.

Identify for a second $\mathbb{T}^2$ with $[0,1)^2$. On $[0,1)^2$, consider the lexicographic order: $x = (a,b)$ is smaller than $y = (c,d)$ either if $a < c$ or if $a = c$ but $b < d$. It is a total order. We write $<_L$ for the strict lexicographic order just defined. Let us denote by $\mathcal{L}\Lambda_N^1 \subset \mathcal{L}F_N\mathbb{T}^2$ the set of strings $((\xi_1,x_1),...,(\xi_N,x_N))$ such that $x_1 <_L ... <_L x_N$, with x_i seen as elements of $[0,1)^2$. The set $\mathcal{L}F_N\mathbb{T}^2$ is partitioned in $N!$ subsets $\mathcal{L}\Lambda_N^1,...,\mathcal{L}\Lambda_N^{N!}$ obtained applying to $\mathcal{L}\Lambda_N^1$ each one of the $N!$ permutations of indexes.

Given $\omega \in \mathcal{M}_N\left(\mathbb{T}^2\right)$, there is a unique element $\{(\xi_i,x_i), i = 1,..,N\} \in \mathcal{L}C_N\mathbb{T}^2 = \mathcal{L}F_N\mathbb{T}^2/\Sigma_N$ such that $\omega = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i\delta_{x_i}$. Notice that the indexing $i = 1,..,N$ here, a priori, is not canonical. However, we may use the lexicographic order, and the fact that point are disjoint, to attribute the indexes $i = 1,..,N$ to the elements of the set $\{(\xi_i,x_i), i = 1,..,N\}$, in such a way that $((\xi_1,x_1),...,(\xi_N,x_N)) \in \mathcal{L}\Lambda_N^1$. This way, we have uniquely defined maps $\omega \overset{h_1}{\mapsto} (\xi_1,x_1)$, ..., $\omega \overset{h_N}{\mapsto} (\xi_N,x_N)$, from $\mathcal{M}_N\left(\mathbb{T}^2\right)$ to $\mathbb{R}\times\mathbb{T}^2$.

On $\mathcal{M}_N\left(\mathbb{T}^2\right) \subset H^{-1-}\left(\mathbb{T}^2\right)$ let us put the topology induced by $d_{H^{-1-}}$ and consider the functions of class $C\left([0,T];\mathcal{M}_N\left(\mathbb{T}^2\right)\right)$. The set $\mathcal{M}_N\left(\mathbb{T}^2\right)$ is measurable in $H^{-1-}\left(\mathbb{T}^2\right)$, and the set $C\left([0,T];\mathcal{M}_N\left(\mathbb{T}^2\right)\right)$ is measurable in $C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$ (the proof is not difficult arguing on suitable close subfamilies of $\mathcal{M}_N\left(\mathbb{T}^2\right)$, constrained by the minimal distance between elements in the support).

If $\omega_t^N = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i\delta_{X_t^{i,N}}$ comes from the vortex point dynamics with an initial condition such that coalescence does not occur, then $\omega_\cdot^N \in C\left([0,T];\mathcal{M}_N\left(\mathbb{T}^2\right)\right)$: to prove this, one has to use the embedding of $H^{-1-}\left(\mathbb{T}^2\right)$ into Hölder continuous functions, in evaluating

$$\sup_{\|\phi\|_{H^{-1-\delta}}\leq 1}\left|\sum_{i=1}^N \xi_i\left(\phi\left(X_t^{i,N}\right)-\phi\left(X_s^{i,N}\right)\right)\right|.$$

Conversely, if $\omega_\cdot^N \in C\left([0,T];\mathcal{M}_N\left(\mathbb{T}^2\right)\right)$, then there exist functions $x_\cdot^{i,N} \in C\left([0,T];\mathbb{T}^2\right)$ and numbers ξ_i, $i = 1,...,N$, such that $\omega_t^N = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i\delta_{x_t^{i,N}}$; the lengthy proof requires identification of these functions locally in time by means of very concentrated test functions. The indexing $i = 1,...,N$ of this functions however cannot correspond to lexicographic order: to have lexicographic order at every time we should accept jumps in time (these jumps occur every time the first coordinates of two points exchange their order, also due to the difference between $\mathbb{T}^2$ and $[0,1)^2$). Let us impose lexicographic order only at time $t = 0$ (in doing so there is no problem to identify $\mathbb{T}^2$ with $[0,1)^2$) and then accept that particles exchange lexicographic order later in time, with the advantage that $x_\cdot^{i,N} \in C\left([0,T];\mathbb{T}^2\right)$. Thus we have uniquely defined the maps $\omega_\cdot^N \overset{\widetilde{h}_1}{\mapsto} \left(\xi_1, x_\cdot^{1,N}\right)$, ..., $\omega_\cdot^N \overset{\widetilde{h}_N}{\mapsto} \left(\xi_N, x_\cdot^{N,N}\right)$ from

$C\left([0,T];\mathcal{M}_N\left(\mathbb{T}^2\right)\right)$ to $\mathbb{R}\times C\left([0,T];\mathbb{T}^2\right)$: at time zero we impose $x_0^{1,N} <_L ... <_L x_0^{N,N}$ (at later times this may be not true anymore). These maps are measurable.

Finally let us discuss the last preliminary fact we need below. Given a probability measure ρ on $\mathcal{L}F_N\mathbb{T}^2$, assume it is exchangeable, namely its law is invariant by permutation of the indexes; it is thus uniquely determined by its restriction to $\mathcal{L}\Lambda_N^1$. Consider ρ restricted to $\mathcal{L}\Lambda_N^1$, remormalized by $N!$ so to be a probability measure; call $\widehat{\rho}$ such measure. We have a one-to-one correspondence between ρ and $\widehat{\rho}$, measures on $\mathcal{L}F_N\mathbb{T}^2$ and $\mathcal{L}\Lambda_N^1$ respectively.

In particular, given a measure $\widehat{\rho}$ on $\mathcal{L}\Lambda_N^1$, we may reconstruct an exchangeable measure on $\mathcal{L}F_N\mathbb{T}^2$, the unique one that restricted to $\mathcal{L}\Lambda_N^1$ gives values proportional to $\widehat{\rho}$ up to $N!$. Assume more, namely that $\widehat{\rho}$ on $\mathcal{L}\Lambda_N^1$ is the law of a vector $\left(\left(\widehat{\xi}_1,\widehat{X}_1\right),...,\left(\widehat{\xi}_N,\widehat{X}_N\right)\right)$, defined on a probability space $\left(\widehat{\Xi},\widehat{\mathcal{F}},\widehat{P}\right)$. Enlarge the probability space as described before the lemma, incorporating independent permutations $\widetilde{s}_N:\widetilde{\Xi}\to\Sigma_N$. On the product space $(\Xi,\mathcal{F},P)$, with the notations above plus $(\xi_i,X_i)=\left(\widehat{\xi}_i,\widehat{X}_i\right)\circ\pi_1$, consider the new vector

$$((\xi_1^*,X_1^*),...,(\xi_N^*,X_N^*)) := \left(\left(\xi_{\widetilde{s}_N(1)},X_{\widetilde{s}_N(1)}\right),...,\left(\xi_{\widetilde{s}_N(N)},X_{\widetilde{s}_N(N)}\right)\right).$$

This vector takes values in $\mathcal{L}F_N\mathbb{T}^2$, not in $\mathcal{L}\Lambda_N^1$ as the previous one $\left(\left(\widehat{\xi}_1,\widehat{X}_1\right),...,\left(\widehat{\xi}_N,\widehat{X}_N\right)\right)$. We claim its law is ρ, in the correspondence $\rho\leftrightarrow\widehat{\rho}$ described above. Indeed, $((\xi_1^*,X_1^*),...,(\xi_N^*,X_N^*))$ is exchangeable, because given a single deterministic permutation s, $\widetilde{s}_N\circ s$ is uniformly distributed. And conditioning to have $X_{\widetilde{s}_N(1)} <_L ... <_L X_{\widetilde{s}_N(N)}$ is like conditioning to have $\widetilde{s}_N=id$, which gives $\widehat{\rho}$. Let us call *shuffling* the procedure illustrated here of composition with independent permutations, to get the exchangeable distribution from a distribution on $\mathcal{L}\Lambda_N^1$.

Step 2. Now let us prove the lemma. The law of $\widehat{\omega}_\cdot^{N_k}$, being the same as the law of the original process, is concentrated on $C\left([0,T];\mathcal{M}_{N_k}\left(\mathbb{T}^2\right)\right)$. Hence, by the measurable maps $\widetilde{h}_i$ described above, it defines random elements $\left(\widehat{\xi}_1,\widehat{X}_\cdot^{1,N_k}\right)$, ..., $\left(\widehat{\xi}_{N_k},\widehat{X}_\cdot^{N_k,N_k}\right)$ in $\mathbb{R}\times C\left([0,T];\mathbb{T}^2\right)$. One has $\widehat{\omega}_t^{N_k}=\frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\widehat{\xi}_i\delta_{\widehat{X}_t^{i,N_k}}$; therefore we have proved a first claim of the lemma (in fact we shall redefine the random vector but the redefinition will not change this statement). We still have to prove that λ_N^0 is the law of (4.9) (in fact we still have to define properly (4.9)) and $\left(\widehat{X}_t^{1,N_k},...,\widehat{X}_t^{1,N_k}\right)$ solves system (4.7).

Since the original process $\omega_\cdot^{N_k}$ had the property that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\left\langle\omega_t^{N_k},\phi\right\rangle-\left\langle\omega_0^{N_k},\phi\right\rangle-\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\nabla\phi(x)\cdot K(x-y)\,\omega_s^{N_k}(dx)\,\omega_s^{N_k}(dy)\,ds\right|\wedge 1\right]=0$$

for every $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$, the same property holds for the new process $\widehat{\omega}_t^{N_k}$ (because they have the same law), hence $\widehat{P}$-a.s. it holds

$$\sup_{t\in[0,T]}\left|\left\langle \widehat{\omega}_t^{N_k},\phi\right\rangle - \left\langle \widehat{\omega}_0^{N_k},\phi\right\rangle - \int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\nabla\phi\left(x\right)\cdot K\left(x-y\right)\widehat{\omega}_s^{N_k}\left(dx\right)\widehat{\omega}_s^{N_k}\left(dy\right)ds\right| = 0$$

on a dense countable set of $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$, which implies (using the structure $\widehat{\omega}_t^{N_k} = \frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\widehat{\xi}_i\delta_{\widehat{X}_t^{i,N_k}}$) that $\left(\widehat{X}_t^{1,N_k},...,\widehat{X}_t^{N_k,N_k}\right)$ satisfies (4.7). Below we shall redefine this process but the redefinition will not change this property.

It remains to understand the law of (4.9). We have constructed the random vector $\left(\left(\widehat{\xi}_1,\widehat{X}_0^{1,N_k}\right),...,\left(\widehat{\xi}_{N_k},\widehat{X}_0^{N_k,N_k}\right)\right)$, with $\widehat{X}_0^{1,N} <_L ... <_L \widehat{X}_0^{N,N}$. We apply the shuffling procedure described at the end of Step 1, hence redefining all r.v.'s and processes by composition with random permutations. The result is an initial random vector of the form (4.9) and the associated process $\left(X_t^{1,N_k},...,X_t^{N_k,N_k}\right)$. The modifications introduced by shuffling do not change the representation $\omega_t^{N_k} = \frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\xi_i\delta_{X_t^{i,N_k}}$ (now $\omega_t^{N_k}$ is the process defined before the lemma) and the fact that $\left(X_t^{1,N_k},...,X_t^{N_k,N_k}\right)$ solves system (4.7). We claim that the new initial random vector (4.9) has law λ_N^0. By construction the vector (4.9) is exchangeable and its law is the unique exchangeable law on $\mathcal{L}F_N\mathbb{T}^2$ corresponding to a certain probability measure $\widehat{\rho}$ on $\mathcal{L}\Lambda_N^1$ that we now describe. Since λ_N^0 has this property, we deduce that λ_N^0 is the law of (4.9). Let us describe $\widehat{\rho}$. It is the law of $\left(\left(\widehat{\xi}_1,\widehat{X}_0^{1,N_k}\right),...,\left(\widehat{\xi}_{N_k},\widehat{X}_0^{N_k,N_k}\right)\right)$, random vector constructed through the unique maps h_i, hence $\widehat{\rho}$ is the push forward under $(h_1,...,h_N)$ of the law of $\omega_0^{N_k}$; call it $\pi_{t=0}Q^{N_k}$. These correspondences are bijections and, as already said, if we start by λ_N^0 and push it forward (in opposite direction) to a law on $\omega_0^{N_k}$ we find $\pi_{t=0}Q^{N_k}$. Thus we have the identification. ■

Given $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$ and $t \in [0,T]$, we are going to prove that

$$E\left[\left|\langle\omega_t,\phi\rangle - \langle\omega_0,\phi\rangle - \int_0^t\langle H_\phi,\omega_s\otimes\omega_s\rangle\,ds\right|\wedge 1\right] = 0.$$

This implies that $\langle\omega_t,\phi\rangle = \langle\omega_0,\phi\rangle - \int_0^t\langle H_\phi,\omega_s\otimes\omega_s\rangle\,ds$ with P-probability one, at time t. Since the processes involved are continuous, this implies that the identity holds uniformly in time, with P-probability one.

Based on the identity

$$\left\langle\omega_t^{N_k},\phi\right\rangle - \left\langle\omega_0^{N_k},\phi\right\rangle - \int_0^t\left\langle H_\phi,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds = 0$$

and the general fact that $|x+y|\wedge 1 \le (|x|\wedge 1)+(|y|\wedge 1)$, one has the inequality

$$
\begin{aligned}
&E\left[\left|\langle\omega_t,\phi\rangle-\langle\omega_0,\phi\rangle-\int_0^t\langle H_\phi,\omega_s\otimes\omega_s\rangle\,ds\right|\wedge 1\right]\\
&\le E\left[\left(\left|\langle\omega_t,\phi\rangle-\left\langle\omega_t^{N_k},\phi\right\rangle\right|\right)\wedge 1\right]+E\left[\left(\left|\langle\omega_0,\phi\rangle-\left\langle\omega_0^{N_k},\phi\right\rangle\right|\right)\wedge 1\right]\\
&+E\left[\left(\left|\int_0^t\left\langle H_\phi,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds-\int_0^t\langle H_\phi,\omega_s\otimes\omega_s\rangle\,ds\right|\right)\wedge 1\right].
\end{aligned}
$$

We have, for $\phi\in C^\infty\left(\mathbb{T}^2\right)$ and $t\in[0,T]$,

$$
\lim_{k\to\infty}E\left[\left(\left|\langle\omega_t,\phi\rangle-\left\langle\omega_t^{N_k},\phi\right\rangle\right|\right)\wedge 1\right]=0
$$

simply because we have a.s. convergence in $C\left([0,T];H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$. Hence it remains to prove

$$
\lim_{k\to\infty}E\left[\left(\left|\int_0^t\left\langle H_\phi,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds-\int_0^t\langle H_\phi,\omega_s\otimes\omega_s\rangle\,ds\right|\right)\wedge 1\right]=0
$$

which is the most demanding part of the passage to the limit. Let us consider a smooth (of class H^{2+} is sufficient) approximation H_ϕ^δ of H_ϕ, $\delta>0$, with the property $H_\phi^\delta(x,x)=0$ (see Remark 75). We have

$$
\lim_{n\to\infty}E\left[\left(\left|\int_0^t\left\langle H_\phi^\delta,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds-\int_0^t\left\langle H_\phi^\delta,\omega_s\otimes\omega_s\right\rangle ds\right|\right)\wedge 1\right]=0
$$

again because of a.s. convergence of ω^{N_k} to ω in $C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$ and thus of $\omega^{N_k}\otimes\omega^{N_k}$ to $\omega\otimes\omega$ in $C\left([0,T];H^{-2-}\left(\mathbb{T}^2\times\mathbb{T}^2\right)\right)$. Therefore

$$
\begin{aligned}
&\limsup_{k\to\infty}E\left[\left(\left|\int_0^t\left\langle H_\phi,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds-\int_0^t\langle H_\phi,\omega_s\otimes\omega_s\rangle\,ds\right|\right)\wedge 1\right]\\
&\le E\left[\left(\left|\int_0^t\left\langle H_\phi-H_\phi^\delta,\omega_s\otimes\omega_s\right\rangle ds\right|\right)\wedge 1\right]+\sup_{k\in\mathbb{N}}E\left[\left(\left|\int_0^t\left\langle H_\phi-H_\phi^\delta,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds\right|\right)\wedge 1\right].
\end{aligned}
$$

We know that

$$
\begin{aligned}
&E\left[\left(\left|\int_0^t\left\langle H_\phi-H_\phi^\delta,\omega_s\otimes\omega_s\right\rangle ds\right|\right)\wedge 1\right]\le\int_0^t E\left[\left|\left\langle H_\phi-H_\phi^\delta,\omega_s\otimes\omega_s\right\rangle\right|\right]ds\\
&\le C\int_0^t E\left[\left|\left\langle H_\phi-H_\phi^\delta,\omega_s\otimes\omega_s\right\rangle\right|^2\right]^{1/2}ds
\end{aligned}
$$

and the last term is arbitrarily small with δ, due to Corollary 72 (a little argument is needed because $H_\phi-H_\phi^\delta$ is not smooth but the computation is similar to the Cauchy property of Theorem 74). It remain to show that

$$
E\left[\left(\left|\int_0^t\left\langle H_\phi-H_\phi^\delta,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds\right|\right)\wedge 1\right]
$$

is small for small δ, uniformly in k. But this case is similar to the previous one, using now Lemma 89. We have proved parts (i) and (ii) of Theorem 90.

Finally, let us prove part (iii) of Theorem 90. From part (ii) we know that on $(\Xi, \mathcal{F}, P)$ we can define a subsequence $\frac{1}{\sqrt{N_k}} \sum_{n=1}^{N_k} \xi_n \delta_{X_t^n}$ of the random point vortex system which converges P-a.s. to the solution $\omega.$ of point (i) in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$. For each N_k, by Corollary 117, part (ii), we can construct a sequence of functions $\left\{\omega_k^{(n)}(\theta,t,x)\right\}_{n\in\mathbb{N}}$, $(\theta,t,x) \in \Xi \times [0,T] \times \mathbb{T}^2$, such that for $\mathbb{P}$-a.e. $\theta \in \Xi$ the functions $(t,x) \mapsto \omega_k^{(n)}(\theta,t,x)$ are L^∞-solutions of 2D Euler equations, and converge, as $n \to \infty$ to $\frac{1}{\sqrt{N_k}} \sum_{n=1}^{N_k} \xi_n \delta_{X_t^n}$ in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ (the convergence claimed in Corollary 117, part (ii), implies convergence in the weak sense of measures uniformly in time, which implies convergence in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$). The functions $\omega_k^{(n)}$ are measurable from $(\Xi, \mathcal{F}, P)$ to $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ with Borel σ-algebra: the full proof requires several steps but the idea is to reduce the question (by means of general arguments of functional analysis) to the measurability in θ, for every t and $\phi \in C^\infty\left(\mathbb{T}^2\right)$ of the real-valued random variable $\theta \mapsto \int_{\mathbb{T}^2} \phi(x)\, \omega_k^{(n)}(\theta,t,x)\, dx$; and prove this by inspection into the proof of existence of L^∞-solutions (in this step one has to use the analogous measurability of the initial conditions used in Corollary 117, part (ii), and uniqueness).

After these preliminaries, replace a.s. convergence with convergence in probability, on $(\Xi, \mathcal{F}, P)$ with values in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$, which is metrizable. Denote a metric by d. By a diagonal procedure, we may now construct a sequence $\omega_k^{(n_k)}(\theta,t,x)$ which converges to $\omega.$ in probability on $(\Xi, \mathcal{F}, P)$, in the topology of $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$. Then there exists a subsequence which converges P-a.s., in the topology of $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$. This completes the proof.

4.5 Proof of Theorem 91

Recall the definitions of $\lambda_N^0(d\theta)$, $\mathcal{T}_N$, $\mu_N^0(d\omega)$ from Remark 86.

Lemma 95 *Given a measurable function $\rho : H^{-1-\delta}\left(\mathbb{T}^2\right) \to [0,\infty)$ such that $\int_{H^{-1-\delta}(\mathbb{T}^2)} \rho(\omega)\, \mu_N^0(d\omega) < \infty$, the measure $\lambda_N^\rho(d\theta) := \rho(\mathcal{T}_N(\theta))\, \lambda_N^0(d\theta)$ on $\left(\mathbb{R} \times \mathbb{T}^2\right)^N$ has the property that its image measure $\mu_N^\rho(d\omega)$ on $H^{-1-\delta}\left(\mathbb{T}^2\right)$ under the map $\mathcal{T}_N$ is $\rho(\omega)\, \mu_N^0(d\omega)$.*

Proof. By definition of $\mu_N^\rho(d\omega)$ and $\lambda_N^\rho(d\theta)$, for every non-negative measurable function F we have

$$\begin{aligned}\int_{H^{-1-\delta}(\mathbb{T}^2)} F(\omega)\,\mu_N^\rho(d\omega) &= \int_{\mathbb{R}^N\times\mathbb{R}^{2N}} F(\mathcal{T}_N(\theta))\,\lambda_N^\rho(d\theta)\\ &= \int_{\mathbb{R}^N\times\mathbb{R}^{2N}} F(\mathcal{T}_N(\theta))\,\rho(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta)\\ &= \int_{H^{-1-\delta}(\mathbb{T}^2)} F(\omega)\,\rho(\omega)\,\mu_N^0(d\omega).\end{aligned}$$

■

We may now prove Theorem 91. Given $\rho_0 \in C_b\left(H^{-1-}\left(\mathbb{T}^2\right)\right)$, $\rho_0 \geq 0$, $\int \rho_0 d\mu = 1$ (μ here is the white noise Gaussian law on $H^{-1-}\left(\mathbb{T}^2\right)$), there is a constant $C_N > 0$ such that $C_N \int_{H^{-1-\delta}(\mathbb{T}^2)} \rho_0(\omega)\,\mu_N^0(d\omega) = 1$, for any $\delta > 0$. Since μ_N^0 converges weakly to μ on $H^{-1-}\left(\mathbb{T}^2\right)$ and ρ_0 is continuous and bounded on $H^{-1-}\left(\mathbb{T}^2\right)$, we deduce $\lim_{N\to\infty} C_N = 1$. Let us consider, on Borel sets of $\left(\mathbb{R}\times\mathbb{T}^2\right)^N$, the finite positive measure $C_N\rho_0(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta)$. By the lemma, its image measure on $H^{-1-}\left(\mathbb{T}^2\right)$ under the map $\mathcal{T}_N$ is $C_N\rho_0(\omega)\,\mu_N^0(d\omega)$ (we apply the lemma to $\rho(\omega) := C_N\rho_0(\omega)$). The point vortex dynamics is well defined for a.e. $\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right) \in \left(\mathbb{R}\times\mathbb{T}^2\right)^N$ with respect to $C_N\rho_0(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta)$, because this fact holds for $\lambda_N^0(d\theta)$. Denote by ω_t^N the vorticity of this point vortex dynamics; the law of ω_0^N is $C_N\rho_0(\omega)\,\mu_N^0(d\omega)$.

Denote by Φ_t^N the map in $H^{-1-}\left(\mathbb{T}^2\right)$, defined a.s. with respect to μ_N^0, which gives $\omega_t^N = \Phi_t^N\omega_0^N$. The law of ω_t^N has the form

$$C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\mu_N^0(d\omega)$$

where $\left(\Phi_t^N\right)^{-1}$ is the inverse map of Φ_t^N and it is defined for μ_N^0-a.e. $\omega \in H^{-1-}\left(\mathbb{T}^2\right)$. Indeed, for every non-negative measurable function F we have

$$\begin{aligned}\mathbb{E}\left[F\left(\omega_t^N\right)\right] = \mathbb{E}\left[F\left(\Phi_t^N\omega_0^N\right)\right] &= \int_{H^{-1-\delta}(\mathbb{T}^2)} F\left(\Phi_t^N\omega\right) C_N\rho_0(\omega)\,\mu_N^0(d\omega)\\ &= \int_{H^{-1-\delta}(\mathbb{T}^2)} F(\omega)\,C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\left(\Phi_t^N\right)_*\mu_N^0(d\omega)\end{aligned}$$

but $\left(\Phi_t^N\right)_*\mu_N^0 = \mu_N^0$, see Proposition 88.

Therefore, for every non-negative measurable function F on $H^{-1-}\left(\mathbb{T}^2\right)$, one has

$$\mathbb{E}\left[F\left(\omega_t^N\right)\right] = \mathbb{E}\left[C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right)F\left(\omega_{WN}^N\right)\right]$$

where ω_{WN}^N denotes the random point vortices initial condition with law μ_N^0.

Let Q^N be the law of ω^N on Borel subsets of the space $\mathcal{X}$, as in the previous section. We want to prove that the family $\{Q^N\}_{N\in\mathbb{N}}$ is tight in $\mathcal{X}$, by proving that it is bounded in probability in $\mathcal{Y}$ (see previous section). The family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$, because

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0} dt\right] &= \int_0^T \mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0}\right] dt \\
&= \int_0^T \mathbb{E}\left[C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right)\left\|\omega_{WN}^N\right\|_{H^{-1-\delta}}^{p_0}\right] dt \\
&\leq C_N\left\|\rho_0\right\|_\infty T\mathbb{E}\left[\left\|\omega_{WN}^N\right\|_{H^{-1-\delta}}^{p_0}\right] \leq C_{p_0,\delta}C_N\left\|\rho_0\right\|_\infty T
\end{aligned}$$

(see the estimate of the previous section). It is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$, by the same arguments given in the previous section, because

$$\begin{aligned}
&\mathbb{E}\left[\left|\left\langle\omega_t^N\otimes\omega_t^N,H_\phi\right\rangle\right|^2\right] \\
&= \mathbb{E}\left[C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right)\left|\left\langle\omega_{WN}^N\otimes\omega_{WN}^N,H_\phi\right\rangle\right|^2\right] \\
&\leq C_N\left\|\rho_0\right\|_\infty \mathbb{E}\left[\left|\left\langle\omega_{WN}^N\otimes\omega_{WN}^N,H_\phi\right\rangle\right|^2\right] \\
&\leq C_N\left\|\rho_0\right\|_\infty C\left\|H_\phi\right\|_\infty^2 \leq C_N\left\|\rho_0\right\|_\infty C\left\|D^2\phi\right\|_\infty^2
\end{aligned}$$

(all the other steps of the proof are the same). This proves tightness in $\mathcal{X}$.

Repeating the arguments of the previous section (we use Prohorov and Skorokhod theorems) we extract a subsequence N_k, construct a new probability space, denoted by $(\Xi,\mathcal{F},P)$ and processes $\omega_t^{N_k}$, ω_t with trajectories in $\mathcal{X}$, such that the laws of ω^{N_k} and ω are Q^{N_k} and Q respectively, and ω^{N_k} converges to ω in the topology of $\mathcal{X}$, P-a.s.; and the structure of ω^{N_k} as sum of delta Dirac is identified, namely Lemma 94 is still true in the case treated here (the proof does not require modifications). The only difference is that here the law of (4.9) is $C_N\rho_0(\omega)\,\mu_N^0(d\omega)$. Let us first prove that the law of ω_t on $H^{-1-}\left(\mathbb{T}^2\right)$, called herewith μ_t, is absolutely continuous with respect to μ (the law of white noise) with bounded density. For every $F\in C_b\left(H^{-1-}\left(\mathbb{T}^2\right)\right)$, we have

$$\begin{aligned}
\int F(\omega)\,\mu_t(d\omega) &= \lim_{N\to\infty}\mathbb{E}\left[F\left(\omega_t^N\right)\right] = \lim_{N\to\infty}\mathbb{E}\left[C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right)F\left(\omega_{WN}^N\right)\right] \\
&\leq \left\|\rho_0\right\|_\infty \lim_{N\to\infty}\mathbb{E}\left[F\left(\omega_{WN}^N\right)\right] = \left\|\rho_0\right\|_\infty\int F(\omega)\,\mu(d\omega).
\end{aligned}$$

This implies $\mu_t << \mu$ with bounded density, denoted in the sequel by ρ_t.

We can pass to the limit as in the previous section. Inspection in that proof reveals that we have only to explain why $E\left[\left(\left|\int_0^t\left\langle H_\phi-H_\phi^\delta,\omega_s\otimes\omega_s\right\rangle ds\right|\right)\wedge 1\right]$ and

$$E\left[\left(\left|\int_0^t\left\langle H_\phi-H_\phi^\delta,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds\right|\right)\wedge 1\right] \tag{4.10}$$

are small for small δ, uniformly in k for the second term. We have

$$E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s \right\rangle ds\right|\right) \wedge 1\right] \leq C \int_0^t E\left[\left|\left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s \right\rangle\right|^2\right]^{1/2} ds$$

$$= C \int_0^t E\left[\rho_s\left(\omega_{WN}\right)\left|\left\langle H_\phi - H_\phi^\delta, \omega_{WN} \otimes \omega_{WN} \right\rangle\right|^2\right]^{1/2} ds$$

$$\leq C \int_0^t E\left[\left|\left\langle H_\phi - H_\phi^\delta, \omega_{WN} \otimes \omega_{WN} \right\rangle\right|^2\right]^{1/2} ds$$

that is arbitrarily small with δ, due to Corollary 72. The proof for (4.10) is similar.

4.6 Proof of Theorem 67

We have proved, see Theorem 90 part (i) and (iii), that there exist a probability space $(\Xi, \mathcal{F}, P)$ and a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ such that $\omega_\cdot$ is a time-stationary white noise solution of Euler equations, in the sense of Definition 83, and that there exists a sequence of functions $\omega^{(n)}\left(\theta, t, x\right)$, $(\theta, t, x) \in \Xi \times [0,T] \times \mathbb{T}^2$, such that for $\mathbb{P}$-a.e. $\theta \in \Xi$ the functions $(t,x) \mapsto \omega^{(n)}\left(\theta, t, x\right)$ are L^∞-solutions of 2D Euler equations, and converge to $\omega_\cdot(\theta)$ in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$. Apparently this result readily implies Theorem 67, just looking at single elements $\theta \in \Xi$. Indeed this is true concerning part (ii) of Theorem 67, the convergence statement of L^∞-solutions to the distributional ones. What could be not clear a priori is in which sense a single path of $\omega_\cdot$ is a solution of Euler equations in vorticity form, because we have defined the nonlinear term of such form using a probabilistic procedure. In a sense, the process $\omega_\cdot$ is like the solution of an Itô equation (becuase the nonlinear term is define a limit in probability) and now we want to interpret the equation for single realizations.

We clarify this issue here, giving rise to the particular formulation adopted in part (i) of Theorem 67. We know that:

- ω_0 is distributed as a white noise, hence it takes values in $H^{-1-}\left(\mathbb{T}^2\right) \setminus \left(H^{-1}\left(\mathbb{T}^2\right) \cup \mathcal{M}\left(\mathbb{T}^2\right)\right)$ and it is a full μ-measure set, where μ is the enstrophy Gaussian measure;
- there exists a set $\Xi_1 \in \mathcal{F}$ with $P\left(\Xi_1\right) = 1$ such that for all $\theta \in \Xi_1$ one has $\omega_\cdot(\theta) \in C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$.

Moreover, for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, the following two claims hold true:

- for P-a.e. $\theta \in \Xi$, $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle (\theta)$ is well defined as $L^2(0,T)$-limit of a subsequence of $s \mapsto \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_\phi^n \right\rangle$ (Definition 77 identifies $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle$ by an $L^2(\Xi)$-limit, from which we can extract a subsequence which converges P-almost surely)

- for P-a.e. $\theta \in \Xi$, we have the identity uniformly in time:

$$\langle \omega_t(\theta), \phi \rangle = \langle \omega_0(\theta), \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle (\theta)\, ds.$$

Therefore, if $\mathcal{D}$ is a countable set in $C^\infty(\mathbb{T}^2)$, applying a diagonal procedure to extract a single subsequence with P-a.s. convergence of $\left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle$, we can find a set $\Xi_2 \in \mathcal{F}$ with $P(\Xi_2) = 1$ such that for all $\theta \in \Xi_2$:

- for every $\phi \in \mathcal{D}$, $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle (\theta)$ is well defined as $L^2(0,T)$-limit of a subsequence of $s \mapsto \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_\phi^n \right\rangle$
- for every $\phi \in \mathcal{D}$, we have the identity above uniformly in time.

Putting together $\Xi_{1,2} := \Xi_1 \cap \Xi_2$, for all $\theta \in \Xi_{1,2}$ the function $\omega_\cdot(\theta)$ satisfies the conditions of Theorem 67, part (i), for all $\phi \in \mathcal{D}$. We have thus proved such claim, limited to $\phi \in \mathcal{D}$.

Assume $\mathcal{D}$ is also dense in $C^\infty(\mathbb{T}^2)$; precisely we shall use density in $H^{-\gamma}(\mathbb{T}^2)$ for some $\gamma > 3$. Given $\phi \in H^{-\gamma}(\mathbb{T}^2)$, take $\phi_k \to \phi$ in $H^{-\gamma}(\mathbb{T}^2)$, $\phi_k \in \mathcal{D}$. We have

$$\begin{aligned}
&\int_0^T \left| \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_\phi^n - H_\phi^m \right\rangle \right|^2 ds \\
&\leq 2 \int_0^T \left| \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_{\phi_k}^n - H_{\phi_k}^m \right\rangle \right|^2 ds + 2 \int_0^T \left| \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_{\phi_k - \phi}^n - H_{\phi_k - \phi}^m \right\rangle \right|^2 ds
\end{aligned}$$

hence, to get that $s \mapsto \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_\phi^n \right\rangle$ is Cauchy in $L^2(0,T)$ it is sufficient to prove that

$$\int_0^T \left| \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_{\phi_k - \phi}^n \right\rangle \right|^2 ds$$

is small uniformly in n, if k is large enough. Let us prove that this property is true in a set $\Xi_3 \in \mathcal{F}$ with $P(\Xi_3) = 1$. Then the proof of Theorem 67, part (i), will be complete, considering $\theta \in \Xi_{1,2,3} := \Xi_1 \cap \Xi_2 \cap \Xi_3$.

Consider the distribution $g_s^n(\theta)$ defined as

$$\langle g_s^n(\theta), \phi \rangle := \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_\phi^n \right\rangle.$$

We have

$$\begin{aligned}
\|g_s^n(\theta)\|_{H^{-\gamma}}^2 &= \sum_k \left(1 + |k|^2\right)^{-\gamma} \left| \langle g_s^n(\theta), e_k \rangle \right|^2 \\
&= \sum_k \left(1 + |k|^2\right)^{-\gamma} \left| \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_{e_k}^n \right\rangle \right|^2
\end{aligned}$$

$$\mathbb{E}\left[\int_0^T \|g_s^n\|_{H^{-\gamma}}^2 \, ds\right] = \sum_k \left(1+|k|^2\right)^{-\gamma} \mathbb{E}\left[\int_0^T \left|\left\langle \omega_s \otimes \omega_s, H_{e_k}^n \right\rangle\right|^2 ds\right]$$
$$\leq CT \sum_k \left(1+|k|^2\right)^{-\gamma} \|e_k\|_{C^2}^2 \leq CT \sum_k \left(1+|k|^2\right)^{-\gamma} |k|^4$$

and this is finite when $\gamma > 3$. Hence there is a set $\Xi_3 \in \mathcal{F}$ with $P(\Xi_3) = 1$, such that $\int_0^T \|g_s^n(\theta)\|_{H^{-\gamma}}^2 \, ds < \infty$ for all $\theta \in \Xi_3$. For such θ we have

$$\int_0^T \left|\left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_{\phi_k - \phi}^n \right\rangle\right|^2 ds = \int_0^T |\langle g_s^n(\theta), \phi_k - \phi\rangle|^2 \, ds \leq C(\theta) \|\phi_k - \phi\|_{H^\gamma}^2$$

where $C(\theta) := \int_0^T \|g_s^n(\theta)\|_{H^{-\gamma}}^2 \, ds < \infty$. Hence we have the required property. The proof is complete.

4.7 Remarks on ρ-white noise solutions

4.7.1 The continuity equation

Let μ be the law of white noise. Following [23], [22] and related literature, let us denote by $\mathcal{FC}_{b,T}^1$ the set of all functionals $F : [0,T] \times C^\infty(\mathbb{T}^2)' \to \mathbb{R}$ of the form $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle \omega, \phi_1\rangle, ..., \langle \omega, \phi_n\rangle) g_i(t)$, with $\phi_1, ..., \phi_n \in C^\infty(\mathbb{T}^2)$, $\widetilde{f}_i \in C_b^1(\mathbb{R}^n)$, $g_i \in C^1([0,T])$ with $g_i(T) = 0$. Given $F \in \mathcal{FC}_{b,T}^1$, denote by $D_\omega F(t,\omega)$ the function

$$\sum_{i=1}^m \sum_{j=1}^n \partial_j \widetilde{f}_i(\langle \omega, \phi_1\rangle, ..., \langle \omega, \phi_n\rangle) g_i(t) \phi_j.$$

Definition 96 *Given $F \in \mathcal{FC}_{b,T}^1$, we set*

$$\langle D_\omega F(t,\omega), b(\omega)\rangle := \sum_{i=1}^m \sum_{j=1}^n \partial_j \widetilde{f}_i(\langle \omega, \phi_1\rangle, ..., \langle \omega, \phi_n\rangle) g_i(t) \left\langle \omega \otimes \omega, H_{\phi_j} \right\rangle$$

where $\left\langle \omega \otimes \omega, H_{\phi_j} \right\rangle$, $j = 1, ..., n$, are the elements of $L^2(\Xi)$ given by Theorem 74. Hence $\langle D_\omega F(t,\omega), b(\omega)\rangle$ is an element of $C\left([0,T]; L^2(\Xi)\right)$.

Definition 97 *We say that a bounded measurable function $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$ is a bounded weak solution of the continuity equation*

$$\partial_t \rho_t + \mathrm{div}_\mu(\rho_t b) = 0 \tag{4.11}$$

with initial condition ρ_0, if

$$\int_0^T \int_{H^{-1-\delta/2}} (\partial_t F(t,\omega) + \langle D_\omega F(t,\omega), b(\omega)\rangle) \rho_t(\omega) \mu(d\omega) \, dt = -\int_{H^{-1-\delta/2}} F(0,\omega) \rho_0(\omega) \mu(d\omega)$$

for all $F \in \mathcal{FC}_{b,T}^1$.

Proposition 98 *Any function ρ given by Theorem 91 is a bounded weak solution of the continuity equation (4.11).*

Proof. Let ω be a solution of Euler equations given by Theorem 91, with the associated density function ρ. Given $F \in \mathcal{FC}^1_{b,T}$ of the form $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i(t)$, we know that

$$\langle \omega_t, \phi_j \rangle = \langle \omega_0, \phi_j \rangle + \int_0^t \left\langle \omega_s \otimes \omega_s, H_{\phi_j} \right\rangle ds$$

for every $j = 1, ..., n$. Here P-a.s. the function $s \mapsto \left\langle \omega_s \otimes \omega_s, H_{\phi_j} \right\rangle$ is of class $L^2(0,T)$. Hence $\langle \omega_t, \phi_j \rangle$ is differentiable a.s. in time. We have, P-a.s., a.s. in time,

$$\begin{aligned}
&\partial_t (F(t,\omega_t)) \\
&= \sum_{i=1}^m \sum_{j=1}^n \partial_j \widetilde{f}_i (\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i(t) \partial_t \langle \omega_t, \phi_j \rangle + \sum_{i=1}^m \widetilde{f}_i (\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i'(t) \\
&= \langle D_\omega F(t,\omega_t), b(\omega_t) \rangle + \partial_t F(t,\omega)|_{\omega=\omega_t}
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_0^T \int_{H^{-1-\delta/2}} (\partial_t F(t,\omega) + \langle D_\omega F(t,\omega), b(\omega) \rangle) \rho_t(\omega) \mu(d\omega) dt \\
&= \int_0^T \mathbb{E}\left[\partial_t F(t,\omega)|_{\omega=\omega_t} + \langle D_\omega F(t,\omega_t), b(\omega_t) \rangle\right] dt \\
&= \int_0^T \mathbb{E}\left[\partial_t (F(t,\omega_t))\right] dt = \int_0^T \partial_t \mathbb{E}\left[F(t,\omega_t)\right] dt \\
&= \mathbb{E}\left[F(T,\omega_T)\right] - \mathbb{E}\left[F(0,\omega_0)\right] \\
&= -\int_{H^{-1-\delta/2}} F(0,\omega) \rho_0(\omega) \mu(d\omega)
\end{aligned}$$

where the exchange of time-derivative and expectation is possible due to the boundedness of terms in F; and we have used $g_i(T) = 0$. ■

The analysis of this continuity equation deserves more attention; we have just mentioned here as a starting point of future investigations.

4.7.2 An open problem

We have treated above the problem of approximating Albeverio-Cruzeiro solution by smoother solutions of the Euler equations. Let us mention a sort of dual problem, that can be formulated thanks to Theorem 91.

Given $\overline{\omega}_0 \in L^\infty\left(\mathbb{T}^2\right)$, there exists a unique solution $\overline{\omega}_t$ in $L^\infty\left(\mathbb{T}^2\right)$ of the Euler equations (point 1 of the Introduction). For every $\epsilon > 0$, consider the density

$$\rho_0^{(\epsilon)}(\omega) = \frac{1}{Z_R} \exp\left(-\frac{d_{H^{-1-}}\left(\omega, \overline{\omega}_0\right)^2}{2\epsilon}\right)$$

defined on $H^{-1-}\left(\mathbb{T}^2\right)$, where

$$Z_R = \int_{H^{-1-\delta}(\mathbb{T}^2)} \exp\left(-\frac{d_{H^{-1-}}\left(\omega, \overline{\omega}_0\right)^2}{2\epsilon}\right) \mu\left(d\omega\right).$$

Let $\omega_t^{(\epsilon)}$ be a ρ-white noise solution, provided by Theorem 91, corresponding to this initial density $\rho_0^{(\epsilon)}$. Can we prove that $\omega_t^{(\epsilon)}$ converges, in a suitable sense, to $\overline{\omega}_t$?

We do not know the solution of this problem. Let us only remark that it looks similar to the question of vortex point approximation of solutions of Euler equations, solved in a smoothed Biot-Savart kernel scheme by [**?**] and in great generality by [53]. Also, very roughly, reminds large deviation approximations of smooth paths by diffusion processes.

Theorems 90 and 91 give some intuition into Albeverio-Cruzeiro solution and its variants, as a limit of random point vortices. A positive solution of the previous problem would add more.

Chapter 5

Other models with white noise initial condition

5.1 HJB equation for interface growth

There are several equations in the literature describing growth (of interfaces, for instance). A paradigm for surface growth is the following equation. Assume we have a $(d-1)$-dimensional oriented surface in $\mathbb{R}^d$ that moves ("growths") by displacing points in the direction of the (conventionally chosen) outer normal to the surface; with the infinitesimal amount of displacement equal at every point, in an infinitesimal time. In other words, the surface "shifts" with speed one in the direction of the outer normal. Assume we may describe the surface by a time-dependent graph

$$y = h(t,x), \qquad x \in \mathbb{R}^{d-1},\, y \in \mathbb{R}.$$

The outer normal to the graph is (chosing the upward orientation)

$$\frac{(\nabla h(t,x), 1)}{\sqrt{|\nabla h(t,x)|^2 + 1}}.$$

Thus in a very short time Δt, very close to a point x_0, the parametrized surface

$$x \mapsto (x, h(t,x))$$

moves approximatively into the parametrized surface

$$x \mapsto \left(x + \Delta t \frac{\nabla h(t,x)}{\sqrt{|\nabla h(t,x)|^2 + 1}}, h(t,x) + \Delta t \frac{1}{\sqrt{|\nabla h(t,x)|^2 + 1}} \right).$$

Assume $|\nabla h(t,x)|$ small, so that approximatively this is equal to

$$x \mapsto \left(x, h(t,x) + \Delta t \left(1 - \frac{1}{2}|\nabla h(t,x)|^2\right)\right)$$

(recall that for $f(t) = \frac{1}{\sqrt{t+1}}$ we have $f'(0) = -\frac{1}{2}$). It means that

$$\partial_t h(t,x) \sim 1 - \frac{1}{2}|\nabla h(t,x)|^2 .$$

We have found the Hamilton-Jacobi-Bellman (HJB) type equation

$$\partial_t h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 = 1.$$

This equation is derived under the assumption of small $|\nabla h(t,x)|$ but it looks a plausible model in general: at positions x where $|\nabla h(t,x)|^2$ is large (hence quite steep graph) the vertical displacement of the graph is bigger (drawing a picture, it is exactly what we expect).

5.2 Random surface growth

Assume the growth of the surface is random. Let us think to an example: a colony of bacteria. Assume to observe a layer of bacteria in a box, the bacteria lying on the bottom of the box. Assume that mostly the bacteria on the free surface duplicate, due to larger availability of oxygen and other nutrients. The colony increases, namely the free surface becomes higher and higher. But the process is random: each bacterium duplicates at random. In the average we observe an increase of the surface, but at microscopic level there are random fluctuations.

Another example: take a piece of paper and put an angle in contact with fire: an interface will develop between the burned area and the intact one, but the profile of such interface is often not regular.

There are two natural viewpoints to include such randomness. One is to assume that the interface growth described above by a deterministic equation is affected by noise. It is a very natural viewpoint, but we do not consider it now. Another viewpoint is that the profile we observe at a certain time is irregular, so we have irregular initial conditions, well described by a random process (in the space variable), and we observe the deterministic evolution of this profile.

In dimension one, if we decide that, due to universality considerations (usually called Donsker invariance principle) the typical shape of a random profile should be Brownian (or Gaussian Free Field under periodic conditions or maybe in higher dimensions), what we are looking for is a solution of the above HJB equation which is a Brownian motion at time zero.

It is not clear a priori that the two requirements do not contradict each other. For instance, we derived the HJB equation above under the condition that the space derivative of h was small; this is obviosly not true for Brownian motion! Therefore is it not clear a priori that this program is meaningful. And indeed, its solution is still an open problem. However, in the literature, some discrete models start to appear whose continuum limit looks like a Brownian solution to HJB equation (see [44]).

Example 99 *In this example we illustrate a few elments of a potential discrete model that could converge to the above problem; we do not claim that it converges; we describe it only to give the idea of what we mean by an approximating discrete model. Consider the following discrete random dynamics. The configuration at time t is a piecewise linear continuous path $x \mapsto h_n(t,x)$, linear on each interval of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ with $k \in \mathbb{Z}$. The interpretation of $x \mapsto h_n(t,x)$ is of profile of an interface, or hight function. Assume the difference*

$$\left| h_n\left(t, \frac{k+1}{n}\right) - h_n\left(t, \frac{k}{n}\right) \right|$$

is always equal to a value ϵ_n to be determined. Assume the interface $h_n(t,x)$ growths with the following rule: at each point $\frac{k}{n}$, if $h_n\left(t,\frac{k+1}{n}\right) > h_n\left(t,\frac{k}{n}\right) > h_n\left(t,\frac{k-1}{n}\right)$ or the opposite inequalities, the value $h_n\left(t,\frac{k}{n}\right)$ cannot change; if

$$h_n\left(t, \frac{k+1}{n}\right) > h_n\left(t, \frac{k}{n}\right) < h_n\left(t, \frac{k-1}{n}\right)$$

then $h_n\left(t,\frac{k}{n}\right)$, in the unit of time Δt, will become equal to $h_n\left(t,\frac{k}{n}\right) + 2\epsilon_n$ with probability $p_n \Delta t$; if

$$h_n\left(t, \frac{k+1}{n}\right) < h_n\left(t, \frac{k}{n}\right) > h_n\left(t, \frac{k-1}{n}\right)$$

then $h_n\left(t,\frac{k}{n}\right)$, in the unit of time Δt, will become equal to $h_n\left(t,\frac{k}{n}\right) - 2\epsilon_n$ with probability $\widetilde{p}_n \Delta t$; and $\widetilde{p}_n << p_n$. We assume changes at different positions are independent, in the positions that can change.

If we could assume a "rule of simplicity", we could conjecture that the path $h_n(t,x)$ has roughly the statistics of a random walk; of course proving this claim is intricate, since the increments of a classical random walk are independent and here the increments are created by the dynamics of the interface, which is spatially structured (but we assumed independent changes). If $h_n(t,x)$ has roughly the statistics of a random walk, with the classical parabolic choice $\epsilon_n^2 = \frac{1}{n}$ in the limit $n \to \infty$ it weakly converges to a Brownian motion.

5.3 Burgers equation with random initial condition

Consider now the 1-dimensional case, $x \in \mathbb{R}$ or $x \in \mathbb{T}$. If h solves the HJB equation

$$\partial_t h(t,x) + \frac{1}{2} \left| \partial_x h(t,x) \right|^2 = 1$$

then

$$u(t,x) := \partial_x h(t,x)$$

satisfies Burger equation

$$\partial_t u(t,x) + \frac{1}{2}\partial_x \left(u^2(t,x)\right) = 0.$$

If $x \mapsto h(t,x)$ is Brownian motion (or GFF in the periodic case) then $x \mapsto u(t,x)$ is White Noise. Thus we are faced with a problem very similar in spirit with the one we have solved for the 2D Euler equations: construct solutions of Burger equation (which is quadratic like Euler equations) which is White Noise in space (and one can show that it is natural to expect White Noise at every time). Unfortunately the problem is much more difficult and completely open. Let us see the difficulty.

The problem is the definition of

$$\frac{1}{2}\int \partial_x u^2 \phi dx = -\frac{1}{2}\int u^2 \partial_x \phi dx$$

when ϕ is smooth and u is White Noise. Formally it is

$$\int u^2 \partial_x \phi dx = \int\int \delta(x-y)\, u(x)\, u(y)\, \partial_x \phi(x)\, dx$$

which is similar but more singular than the 2D Euler case. Let us approximate it by

$$\langle u \otimes u, f^\epsilon \rangle$$

where $f^\epsilon(x,y) = \delta^\epsilon(x-y)\,\partial_x\phi(x)$ and δ^ϵ is a smooth approximation of δ, say

$$\delta^\epsilon(x) = \epsilon^{-1}\theta\left(\epsilon^{-1}x\right)$$

with θ satisfying usual assumptions, including symmetry for simplicity. Recall the following lemma (proved in the 2D periodic case, but the proof is the same in 1d and other cases), that we state in the case of non-symmetric f^ϵ:

Lemma 100 *If f is smooth and u is white noise, then:*
i) $\mathbb{E}[\langle u \otimes u, f\rangle] = \int_{\mathbb{T}^2} f(x,x)\,dx$.

Corollary 101 *ii)* $\mathbb{E}\left[\left|\langle u \otimes u, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^2\right] = \int\int f(x,y)^2\,dxdy + \int\int f(x,y)\,f(y,x)\,dxdy.$

Corollary 102 *For $f^\epsilon(x,y) = \delta^\epsilon(x-y)\,\partial_x\phi(x)$, one has*

$$\begin{aligned} \mathbb{E}[\langle u \otimes u, f^\epsilon\rangle] &= 0 \\ \mathbb{E}\left[|\langle u \otimes u, f^\epsilon\rangle|^2\right] &= \epsilon^{-1}C_\epsilon \end{aligned}$$

where

$$C_\epsilon = \int\int \epsilon^{-1}\theta^2\left(\epsilon^{-1}(x-y)\right)\frac{1}{2}\left|\partial_x\phi(x) + \partial_x\phi(y)\right|^2 dxdy \overset{\epsilon\to 0}{\to} 2\,\|\theta\|_{L^2}^2 \int |\partial_x\phi(x)|^2\,dx.$$

Proof. From the definition of f^ϵ,

$$\int f^\epsilon(x,x)\,dx = \int \delta^\epsilon(0)\,\partial_x\phi(x)\,dx = \epsilon^{-1}\theta(0)\int \partial_x\phi(x)\,dx = 0$$

$$\begin{aligned} & \int\int f^\epsilon(x,y)^2\,dxdy + \int\int f^\epsilon(x,y)\,f^\epsilon(y,x)\,dxdy \\ = \;& \int\int \epsilon^{-2}\theta^2\left(\epsilon^{-1}(x-y)\right)\left(|\partial_x\phi(x)|^2 + \partial_x\phi(x)\,\partial_x\phi(y)\right)dxdy \\ = \;& \epsilon^{-1}\int\int \epsilon^{-1}\theta^2\left(\epsilon^{-1}(x-y)\right)\frac{1}{2}\,|\partial_x\phi(x) + \partial_x\phi(y)|^2\,dxdy. \end{aligned}$$

■

Recall the analysis of the kinetic energy for the distributional solutions of 2D Euler equestions; in that case the average energy was infinite; subtracting this infinite contribution, a finite expression remained that we called renormalized energy. Here it is different. Due to a special cancellation the average is even equal to zero, but the variance is infinite.

5.4 Weak versus strong KPZ universality

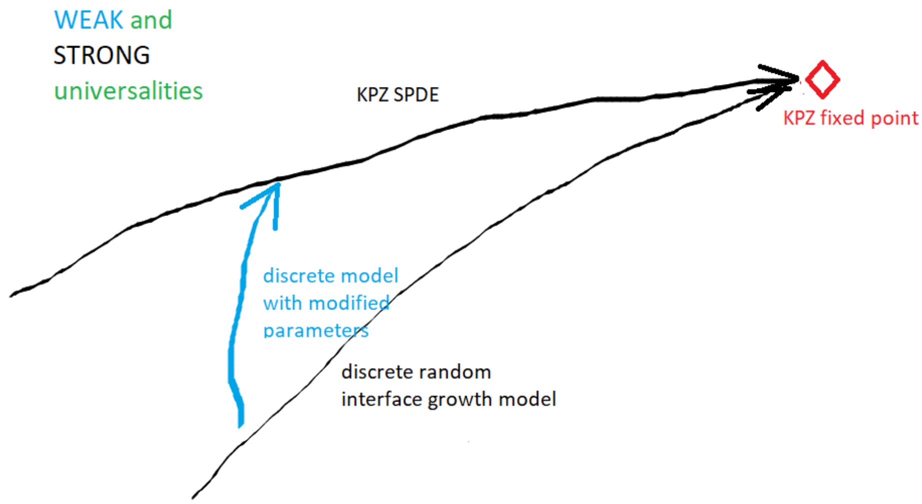
Following a seminal paper of Kardar, Parisi and Zhang [36], it is believed that the HJB equation with Brownian space solution is the universal model of random growth phenomena, in the sense that any such model, suitably rescaled, should converge to such universal model (similarly to the fact that a sum of independent random variables, $S_n = X_1 + ... + X_n$, suitably rescaled, should converge to a standard Gaussian law, under standard conditions).

This is called the *strong KPZ universality.* Since it proved to be incredibly difficult to confirm by rigorous theorems, a weaker concept was detected, called *weak KPZ universality.* It is a little more technical to be described. Preliminary, let us describe the so called "KPZ SPDE".

Consider a modified model with respect to the one described above: a stochastic partial differential equation (SPDE) of HJB type

$$\partial_t h(t,x) + \frac{1}{2}\,|\partial_x h(t,x)|^2 = \nu\partial_{xx}^2 h(t,x) + \sigma\xi(t,x)$$

where $\xi(t,x)$ is the so called space-time white noise and a dissipation has been included. In a sense, this corresponds to the idea already mentioned above of adding noise to the deterministic HJB equation in order to take into account random fluctuations; and dissipation is introduced to dissipate the energy injected by noise, in order to have a stationary regime in the statistical sense. Thus this is a sort of alternative model to describe growth and random fluctuations. However, it introduces two new constants, ν and σ, which in practice are expected to be very small. The deterministic HJB equation is a sort of ideal limit when



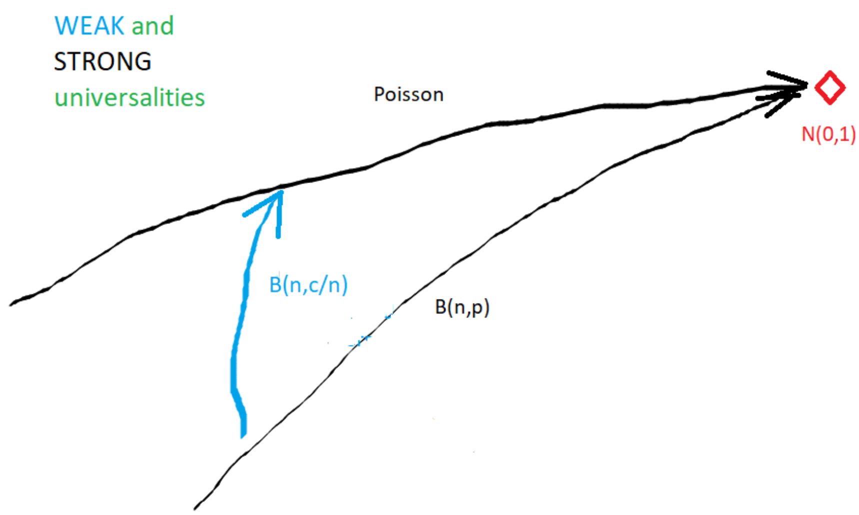
ν and σ go to zero. One can prove that the SPDE above is solvable (see [35]) and has a stationary solution that is GFF at every time, with constants of the field depending on ν and σ; when ν and σ go to zero with a suitable relation, the same GFF is invariant for all non zero values of ν and σ and thus we expect to see a limit solution that is GFF and solves the deterministic HJB equation.

After this introduction, let us explain the weak universality. Since the SPDE above is solvable, if we have a discrete (or any other) approximate model of random growth, instead of trying to prove it converges to the Brownian solution of the deterministic HJB equation one could try to prove that it is close to the existing solution of the KPZ SPDE; in a sense, when the parameter (ϵ, n, ...) of the approximating model tends to its limit, we should change the values of ν and σ to observe closedness to the SPDE solutions. In Mathematics however closedness is a vague concept; we should try to reformulate the problem as a limit. In order to do so, one can parametrize also certan probabilities (by ϵ, n, ...) of the approximate model in order to prove that, under this extra parametrization, the model converges to a solution of the SPDE with *fixed* values of ν and σ.

We recognize that this description is poor and vague; for this reason, let us recall an analogous limit problem in classical probability, with the hope it increases the intuition about the difference between weak and strong universality.

Central limit and rare event theorems

1. Let $\{X_n\}$ be a sequence of independent Bernoulli random variables of parameter p (we write $X_n \sim B(1,p)$). We know that $S_n = X_1 + ... + X_n$ is a Binomial r.v. with parameters



n and p ($S_n \sim B(n,p)$). Moreover, we know that

$$\frac{S_n - np}{\sqrt{np(1-p)}} \overset{\text{in law}}{\to} N(0,1).$$

2. Let $N_n \sim \mathcal{P}(np)$, Poisson of parameter np; $\mathbb{E}[N_n] = np$, $Var[N_n] = np$. The standardized of N_n has a Gaussian limit:

$$\frac{N_n - np}{\sqrt{np}} \overset{\text{in law}}{\to} N(0,1).$$

Indeed, $\mathbb{E}[\exp(itN_n)] = \exp\left(np\left(e^{it}-1\right)\right)$,

$$\begin{aligned}\log \mathbb{E}\left[\exp\left(it\frac{N_n - np}{\sqrt{np}}\right)\right] &= np\left(\exp\left(\frac{it}{\sqrt{np}}\right) - 1 - \frac{it}{\sqrt{np}}\right) \\ &\sim np\frac{1}{2}\left(\frac{it}{\sqrt{np}}\right)^2 = -\frac{t^2}{2}\end{aligned}$$

and we can apply Lévy theorem.

3. If we rescale p with n like $p_n = \frac{\lambda}{n}$, for some $\lambda > 0$, so that $S_n \sim B\left(n, \frac{\lambda}{n}\right)$, we have

$$S_n \overset{\text{in law}}{\to} \mathcal{P}(\lambda).$$

Compared with the case of KPZ, we could say that the CLT is a strong universality result, the rare event theorem is a weak universality one.

Chapter 6

Appendix

6.1 Invariant measures

Given a measurable space $(E, \mathcal{E})$ and a measurable map $T : E \to E$, the push-forward of a measure ν on $(E, \mathcal{E})$ under T, denoted by $T_\sharp \nu$, is the measure on $(E, \mathcal{E})$ defined by

$$(T_\sharp \nu)(A) = \nu\left(T^{-1}A\right)$$

for all $A \in \mathcal{E}$ (the definition is meaningful also between different measurable spaces). We say that μ is invariant for T if

$$T_\sharp \mu = \mu.$$

It follows that

$$(T^n)_\sharp \mu = \mu$$

for every positive integer n, where T^n is the composition $T \circ ... \circ T$ made n times. If φ_t, $t \geq 0$, is a family of measurable maps in E, $\varphi_t : E \to E$, we say that μ is invariant for φ_t if

$$(\varphi_t)_\sharp \mu = \mu$$

for every $t \geq 0$; the typical example we have in mind is the flow associated to a well-posed differential equation

$$\frac{dx(t)}{dt} = f(x(t)). \tag{6.1}$$

Trivial examples of invariant measures are the delta Dirac at fixed points; others are measures distributed along periodic trajectories; for Hamiltonian systems, Lebesgue measure is invariant (Liouville theorem).

Uniqueness of invariant measures, in the deterministic context, is an extremely difficult and maybe useless problem; very interesting situations present even an infinity of invariant measures, amongh which only one has special physical properties (we have in mind the theory of RSB measures for axiom A dynamical systems); every fixed point or periodic orbit carries an invariant measure but often these are not the interest ones, for instance when these fixed points or periodic orbits are unstable. Noise has the typical effect to give uniqueness (along with suitable ergodic properties) of invariant measures.

Existence is, on the contrary, subject to a relatively general theory. The simplest case is when T is a continuous map in a compact metric space (X, d):

Theorem 103 (Krylov-Bogoliubov) *If T is a continuous map in a compact metric space (X, d), then there exists at least one invariant probability measure.*

Proof. Given a probability measure ν_0, define

$$\begin{aligned} \nu_n &= (T^n)_\sharp \nu_0 \\ \mu_n &= \frac{1}{n} \sum_{i=1}^{n} \nu_n. \end{aligned}$$

Since (X, d) is compact, the family (μ_n) is tight, hence by Prohorov theorem there exits a subsequence (μ_{n_k}) weakly convergent to a probability measure μ. Since T is continuous, one can easily see that $T_\sharp \mu_{n_k}$ weakly converges to $T_\sharp \mu$. But

$$\begin{aligned} (T)_\sharp \mu_{n_k} &= \frac{1}{n_k} \sum_{i=1}^{n_k} \left(T^{n+1}\right)_\sharp \nu_0 \\ &= \frac{1}{n_k} \sum_{i=1}^{n_k} (T^n)_\sharp \nu_0 + \frac{1}{n_k} \left(T_\sharp \nu_0 - \left(T^{n+1}\right)_\sharp \nu_0\right). \end{aligned}$$

It is easy to see that the measures $\frac{1}{n_k}\left(T_\sharp \nu_0 - \left(T^{n+1}\right)_\sharp \nu_0\right)$ weakly converge to zero, hence we deduce, in the limit as $k \to \infty$, $T_\sharp \mu = \mu$. ■

Remark 104 *Also the family (ν_n) is tight but if (ν_{n_k}) is a convergent subsequence, we cannot prove that $T_\sharp \nu_{n_k}$ weakly converges.*

When X is not compact, in order to implement the previous strategy we need tightness of (μ_n). Recall that a family F of probability measures on a metric space (X, d) is tight if for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset X$ such that $\mu(K_\epsilon) > 1 - \epsilon$ for all $\mu \in F$. And recall that, by Prohorov theorem, if F is tight then there exists a sequence $(\mu_n) \subset F$ and a probability measure μ such that μ_n weakly converges to μ, where weak convergence means $\int \phi d\mu_n \to \int \phi d\mu$ for every bounded continuous function ϕ. In order to repeat the previous proof we need that (μ_n) (the time averages) is a tight family. A checkable sufficient condition is that (ν_n) are tight.

Proposition 105 *Let T be a continuous map in a (not necessarily compact) metric space (X, d). Given a probability measure ν_0, define $\nu_n = (T^n)_\sharp \nu_0$. If the family (ν_n) is tight, then there exists at least one invariant probability measure.*

For the proof, it is sufficient to notice that tightness of (ν_n) impies tightness of (μ_n); then repeat the proof of Krylov-Bogoliubov theorem.

Example 106 *In $\mathbb{R}^d$, consider the differential equation (6.1) with locally Lipschitz continuous f. Assume there are constants $\lambda, C > 0$ such that*

$$\langle f(x), x\rangle \leq -\lambda |x|^2 + C.$$

Take $\nu_0 = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^d$ and let ν_t be the push forward of δ_{x_0} by the solution map (it is simply $\nu_t = \delta_{x(t|x_0)}$). Then the family (ν_t) is tight (prove it) and there exists at least one invariant measure (not necessarily of delta Dirac type).

Example 107 *If $f(x) = Ax + B(x) + b$ with $A = A^T < 0$, B a locally Lipschitz continuous map such that*

$$\langle B(x), x\rangle = 0$$

(the nonlinearities related to fluid dynamics have usually this property) and b is a given vector, then the assumptions of the previous example are fulfilled.

We have discussed existence of invariant measures to enlarge a bit the framework of these lectures but for our purposes, usually, just existence is not sufficient. We need to know explicit quantitative informations on the invariant measures, in order to use it to prove well posedness results for a.e. initial condition. A question arising in this direction is: how to prove that a given measure is invariant for a given system; another one is: when we still do not have the dynamics - since our purpose is to prove that the dynamics is well defined a.s. - but we have a measure that presumably will be invariant a posteriori, how to use this information to construct the dynamics.

6.2 Existence of a sequence of independent Gaussian variables

We have extensively used the fact that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a sequence $\{G_n\}_{n\in\mathbb{N}}$ of independent Gaussian random variables, say $N(0,1)$ (or with any other specified laws). The shortest explanation is the existence of the countable product measure

$$\mathbb{P} := \otimes_{\mathbb{N}} N(0,1)$$

on the product space

$$(\Omega, \mathcal{F}) := \left(\mathbb{R}^{\mathbb{N}}, \otimes_{\mathbb{N}} \mathcal{B}(\mathbb{R})\right).$$

The proof of this existence is not trivial. Whence we have it, it is sufficient to define G_n equal to the n-th component of $\omega \in \mathbb{R}^{\mathbb{N}}$.

Another very different construction is possible. Let

$$(\Omega, \mathcal{F}, \mathbb{P}) := ([0,1], \mathcal{B}([0,1]), Leb).$$

Call X_n the n-th digit of $\omega \in [0,1]$ written in binary form. One can prove that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent Bernoulli $B\left(1, \frac{1}{2}\right)$ random variables. Using a bijection between $\mathbb{N}$ and $\mathbb{Q}$, we may rewrite $\{X_n\}_{n\in\mathbb{N}}$ as a double index sequence

$$\left\{X_n^{(k)}\right\}_{n,k\in\mathbb{N}}.$$

Then we set

$$U^{(k)} = \sum_{n=1}^{\infty} 2^{-n} X_n^{(k)}$$

and we can prove that $\left\{U^{(k)}\right\}_{k\in\mathbb{N}}$ is a sequence of independent uniform $\mathcal{U}(0,1)$ random variables. Finally, if Φ denotes the distribution function of the standard Gaussian, the random variables

$$G_n = \Phi^{-1}\left(U^{(n)}\right)$$

satisfy the required conditions.

6.3 Convergence and Gaussian r.v.

We recall the following fact: if $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of Gaussian random variables that converges in law to a random variable X, then X is Gaussian and mean and variances of X_n converge to mean and variances of X. The result is a fortiori true when $\{X_n\}_{n\in\mathbb{N}}$ converges in probability or in mean square, that we have used above.

Recall also the classical Central limit theorem. It has the following generalization to Hilbert spaces, see [39].

Theorem 108 (CLT) *Let H be a separable Hilbert space, X be a random vector in H with $\mathbb{E}\left[\|X\|^2\right] < \infty$ and, for notational simplicity, $\mathbb{E}[X] = 0$, and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent random vectors in H distributed as X. Then*

$$\frac{X_1 + ... + X_n}{\sqrt{n}}$$

converges in law to a centered Gaussian μ measure on H, with covariance Q given by

$$\langle Qh, k\rangle = \mathbb{E}[\langle X, h\rangle \langle X, k\rangle]$$

for all $h, k \in H$ (in other words, the covariance of μ is equal to the covariance of X).

We do not give the full proof but only explain one aspect. Assume we have proved that $\frac{X_1+\ldots+X_n}{\sqrt{n}}$ is tight. Let us prove that every weakly convergent subsequence has a limit the same Gaussian measure μ, with the above stated covariance, hence the full sequence $\frac{X_1+\ldots+X_n}{\sqrt{n}}$ converges weakly to such Gaussian measure. Let us prove, thus, that the limit of a convergent subsequence $\frac{X_1+\ldots+X_{n_k}}{\sqrt{n_k}}$ is a Gaussian measure μ with the same covariance as X (hence the Gaussian measure is unique). From the assumption $\mathbb{E}\left[\|X\|^2\right]<\infty$, given $h\in H$ the r.v.'s $\langle X_n,h\rangle$ are i.i.d. and have finite second moment, and zero mean, hence

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\langle X_n,h\rangle \stackrel{\mathcal{L}}{\to} N\left(0,\mathbb{E}\left[\langle X,h\rangle^2\right]\right).$$

Recall that convergence in law is stable by composition with continuous functions. Then from the above assumed property that $\frac{X_1+\ldots+X_{n_k}}{\sqrt{n_k}}$ converges in law to μ it follows that $\left\langle \frac{1}{\sqrt{n_k}}\sum_{i=1}^{n}X_{n_k},h\right\rangle$ converges in law to $\pi_h\mu$. By uniqueness of the limit in law, $\pi_h\mu$ is $N\left(0,\mathbb{E}\left[\langle X,h\rangle^2\right]\right)$. We have proved that μ is Gaussian. Repeating the argument for the map $x\mapsto \pi_{h,k}x:=(\langle x,h\rangle,\langle x,k\rangle)$, we get that $\pi_{h,k}\mu$ is a Gaussian vector with covariance matrix

$$\begin{pmatrix} \mathbb{E}\left[\langle X,h\rangle^2\right] & \mathbb{E}\left[\langle X,h\rangle\langle X,k\rangle\right] \\ \mathbb{E}\left[\langle X,h\rangle\langle X,k\rangle\right] & \mathbb{E}\left[\langle X,k\rangle^2\right] \end{pmatrix}.$$

In particular

$$\mathbb{E}\left[\langle X,h\rangle\langle X,k\rangle\right]=\int_{\mathbb{R}^2} st\left(\pi_{h,k}\mu\right)(dsdt)=\int_H \langle x,h\rangle\langle x,k\rangle\,\mu(dx)=\langle Qh,k\rangle.$$

6.4 Functional analysis

We use several facts of Functional analysis in these lectures. One used above is that any separable Hilbert space H has a complete orthonormal system $\{e_n\}_{n\in\mathbb{N}}$; and given a compact selfadjoint operator Q in H, there is one such $\{e_n\}_{n\in\mathbb{N}}$ made of eigenvectors of Q.

We have used moreover that $L^2(\Omega;H)$ is complete.

6.5 Point vortices and vortex patches

In this appendix we revisit the theorem of convergence of vortex patches to point vortices, proved first by [**?**] and elaborated in successive references, including [17] to which we are also inspired. Those proofs are in full space; here we work on the torus, with the necessary corrections. The corollary at the end of the section is tuned for application to the random case treated in this paper.

There is some similarity with Proposition 1. However, the result proved here is much more advanced.

Given $N \in \mathbb{N}$, $T > 0$, $\alpha_1, ..., \alpha_N \in \mathbb{R}$, $x_1(0), ..., x_N(0) \in \mathbb{T}^2$, consider on the time interval $[0,T]$ the motion of the point vortices $x_1(t), ..., x_N(t)$ on $\mathbb{T}^2$

$$\frac{dx_i(t)}{dt} = \sum_{j \neq i} \alpha_j K(x_i(t) - x_j(t)), \qquad i = 1, ..., N$$

with initial conditions $x_1(0), ..., x_N(0)$. Assume, for this initial condition, global existence and uniqueness outside the generalized diagonal Δ_N; set

$$r_0 = \min\{d(x_i(t), x_j(t)); t \in [0,T], i \neq j\} \tag{6.2}$$

where d is the distance on the torus; the minimal distance r_0 is a positive real number. Set also

$$\alpha := \max\{|\alpha_i|; i = 1, ..., N\} \tag{6.3}$$

Given $\epsilon \in \left(0, \frac{r_0}{4}\right)$ (in particular $\epsilon < 1$), we consider the bounded measurable function $\omega_0 : \mathbb{T}^2 \to \mathbb{R}$ given by

$$\omega_0(x) = \sum_{i=1}^N \alpha_i \omega_i(0,x)$$

with

$$\omega_i(0,x) = \frac{1}{\pi\epsilon^2} 1_{B(x_i(0),\epsilon)}(x) \qquad i = 1, ..., N$$

where $B(x_i(0), \epsilon)$ denotes the ball in $\mathbb{T}^2$ of center $x_i(0)$ and radius ϵ. The results below remain true just assuming that $\omega_i(0)$ is a bounded probability density with support in $B(x_i(0), \epsilon)$, for every $i = 1, ..., N$. Denote by $\omega(t,x)$ the unique L^∞-solution (by this we mean solutions of class $L^\infty([0,T] \times \mathbb{T}^2) \cap C([0,T]; L^p(\mathbb{T}^2))$ for every $p \in [1, \infty)$ satisfying (4.2) and thus (4.3)) of 2D Euler equations on $\mathbb{T}^2$ with initial condition ω_0; see for instance [60], [61], [18], [40], [41], [?] for this classical result. For any $g \in L^\infty(\mathbb{T}^2)$, denote by $BS[g]$ the Biot-Savart law, namely the velocity field given by

$$BS[g](x) = \int_{\mathbb{T}^2} K(x-y) g(y) dy$$

where K is the Biot-Savart kernel on the torus; locally around $x = 0$, say for $|x|_\infty \leq 1/2$, we have

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} + R(x)$$

where $R(x)$ is smooth, $R(-x) = -R(x)$, hence $R(0) = 0$. Then we may decompose $\omega(t,x)$ in the form

$$\omega(t,x) = \sum_{i=1}^N \alpha_i \omega_i(t,x)$$

where, for each $i = 1, ..., N$, $\omega_i(t, x)$ is the unique L^∞-solution of the *linear* transport equation (we invoke [28] result for this existence and uniqueness result)

$$\partial_t \omega_i(t, x) + BS[\omega(t, \cdot)](x) \cdot \nabla \omega_i(t, x) = 0$$

with initial condition $\omega_i(0, x)$. Indeed, $\sum_{i=1}^N \xi_i \omega_i$ is also an L^∞-solution of the 2D Euler equations on $\mathbb{T}^2$ with initial condition ω_0, hence it is equal to $\omega(t, x)$ by uniqueness.

Recall that $\omega(t, x)$ satisfies the weak vorticity formulation

$$\frac{d}{dt} \int_{\mathbb{T}^2} \phi(x) \omega(t, x) dx = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} H_\phi(x, y) \omega(t, x) \omega(t, y) dxdy$$

for every $\phi \in C^\infty(\mathbb{T}^2)$. Its components $\omega_i(t, x)$ satisfy

$$\frac{d}{dt} \int_{\mathbb{T}^2} \phi(x) \omega_i(t, x) dx = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} H_\phi(x, y) \omega_i(t, x) \omega_i(t, y) dxdy + \int_{\mathbb{T}^2} \omega_i(t, x) u_{\widehat{i}}(t, x) \cdot \nabla \phi(x) dx$$

where

$$u_{\widehat{i}} = BS[\omega_{\widehat{i}}], \qquad \omega_{\widehat{i}} = \omega - \alpha_i \omega_i = \sum_{j \neq i} \alpha_j \omega_j.$$

Theorem 109 *There is a constant $C > 0$, depending only on (N, T, r_0, α), with the following properties. For every $\epsilon \in (0, 1)$ such that $C\epsilon^{1/5} \leq \frac{r_0}{8}$, the functions $\omega_i(t)$ defined above (which depend on ϵ) have support contained in $B(x_i(t), C\epsilon^{1/5})$ for every $i = 1, ..., N$ and $t \in [0, T]$.*

The proof will occupy the rest of the section. It is based on [17], [**?**]. When we write the index i it is always meant that it varies in $\{1, ..., N\}$; we also omit to write all the times that the values of parameter ϵ is at least included in $\left(0, \frac{r_0}{4}\right)$. We denote by $C > 0$ a generic constant that depends only on (N, T, r_0, α). We shall often use the fact that $\omega_i(t)$ are probability measures.

Consider the fluid-particle-motion associated to ω: there exists a continuous function $(t, x_0) \mapsto x(t|x_0)$ from $[0, T] \times \mathbb{T}^2 \to \mathbb{T}^2$ such that $x(0|x_0) = x_0$ and

$$\frac{d}{dt} x(t|x_0) = BS[\omega(t, \cdot)](x(t|x_0)).$$

It has additional regularity properties, including some Hölder continuity in space, see [**?**], or [14] on the torus. Due to the representation

$$\omega(t, x(t|x_0)) = \omega(0, x_0)$$

we may write $\omega_i(t,x)$ in the form

$$\omega_i(t,x) = \frac{1}{\pi\epsilon^2} 1_{S_i(t,\epsilon)}(x)$$

where $S_i(t,\epsilon)$, the support of $\omega_i(t,\cdot)$, is the image of $B(x_i(0),\epsilon)$ under the flow $x(t|x_0)$. We have $Leb(S_i(t,\epsilon)) = \pi\epsilon^2$.

From the previous facts, there exists some $\tau > 0$ such that the supports of $\omega_i(t)$ are contained in $B\left(x_i(t), \frac{r_0}{4}\right)$, for $t \in [0,\tau]$. We work on any such interval $[0,\tau]$. At the end we shall show that we can take $\tau = T$, if we choose ϵ small enough. Let us remark that our constants will depend on (N, T, r_0, α) but not on τ.

Lemma 110 *On $[0,\tau]$, for $x, x' \in B\left(x_i(t), \frac{r_0}{4}\right)$ we have*

$$|BS[\omega_j(t,\cdot)](x)| \le C$$
$$\left|BS[\omega_j(t,\cdot)](x) - BS[\omega_j(t,\cdot)](x')\right| \le C\left|x - x'\right|$$

for all $j \neq i$, and thus the same inequalities hold for $u_{\widehat{i}}(t,\cdot)$ in place of $BS[\omega_j(t,\cdot)]$.

Proof. If $x \in B\left(x_i(t), \frac{r_0}{4}\right)$ and $y \in B\left(x_j(t), \frac{r_0}{4}\right)$ with $j \neq i$ we have $|K(x-y)| \le C$, constant depending on r_0 (this because $|x-y| > \frac{r_0}{2}$). Therefore

$$|BS[\omega_j(t,\cdot)](x)| \le \int_{\mathbb{T}^2} |K(x-y)|\,\omega_j(t,y)\,dy \le C\int_{\mathbb{T}^2} \omega_j(t,y)\,dy = C$$

where C depends on N, α, r_0. The proof of the second inequality is similar. ■

Let us introduce mean, variance and tail-mass of the distribution $\omega_i(t,x)$:

$$M_i(t) = \int_{\mathbb{T}^2} x\omega_i(t,x)\,dx, \qquad V_i(t) = \int_{\mathbb{T}^2} |x - M_i(t)|^2\,\omega_i(t,x)\,dx$$
$$m_i(t,h) = \int_{|x-M_i(t)|>h} \omega_i(t,x)\,dx = \frac{1}{\pi\epsilon^2} Leb\left(S_i(t,\epsilon) \cap B(M_i(t),h)^c\right), \qquad \text{for } h \in (0,1).$$

From the weak formulation for ω_i above we have

$$\frac{dM_i(t)}{dt} = \int_{\mathbb{T}^2} \omega_i(t,x)\,u_{\widehat{i}}(t,x)\,dx$$

because $H_x(x,y) = 0$. We may prove that $V_i(t)$ and $m_i(t,h)$ are small when ϵ is small.

Lemma 111 *On $[0,\tau]$, we have*

$$V_i(t) \le C\epsilon^2$$

$$m_i(t,h) \le C\frac{\epsilon^2}{h^2}, \qquad Leb\left(S_i(t,\epsilon) \cap B(M_i(t),h)^c\right) \le C\frac{\epsilon^4}{h^2}.$$

Proof. From $V_i(t) = \int_{\mathbb{T}^2} |x|^2 \omega_i(t,x)\,dx - M_i(t)^2$ and the weak formulation for ω_i above, we have

$$\begin{aligned}\frac{dV_i(t)}{dt} &= \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_{|x|^2}(x,y)\,\omega_i(t,x)\,\omega_i(t,y)\,dxdy \\ &+ 2\int_{\mathbb{T}^2} \omega_i(t,x)\,u_{\widehat{i}}(t,x)\cdot(x - M_i(t))\,dx\end{aligned}$$

where, for $|x-y|_\infty \le 1/2$ (and due to the term $\omega_i(t,x)\,\omega_i(t,y)$ the integration is restricted to very close x, y)

$$\begin{aligned}H_{|x|^2}(x,y) &= K(x-y)\cdot(x-y) = R(x-y)\cdot(x-y) \\ &\le (|x - M_i(t)| + |y - M_i(t)|)^2 \le 2|x - M_i(t)|^2 + 2|y - M_i(t)|^2 .\end{aligned}$$

Moreover $\int_{\mathbb{T}^2} \omega_i(t,x)(x - M_i(t))\,dx = 0$ hence

$$\int_{\mathbb{T}^2} \omega_i(t,x)\,u_{\widehat{i}}(t,x)\cdot(x - M_i(t))\,dx = \int_{\mathbb{T}^2} \omega_i(t,x)\left(u_{\widehat{i}}(t,x) - u_{\widehat{i}}(t, M_i(t))\right)\cdot(x - M_i(t))\,dx.$$

Using Lemma 110,

$$\frac{dV_i(t)}{dt} \le CV_i(t)$$

that implies the inequality for $V_i(t)$, by Gronwall lemma. The estimate for $m_i(t,h)$ follows simply from Chebyshev inequality. ■

We may now prove that $x_i(t)$ and $M_i(t)$ are close when ϵ is small.

Lemma 112 *On $[0,\tau]$, we have*

$$|x_i(t) - M_i(t)| \le C\epsilon.$$

Proof. From the following reformulations of the equations for $x_i(t)$ and $M_i(t)$

$$\begin{aligned}\frac{dx_i(t)}{dt} &= \sum_{j\neq i} \alpha_j BS[\omega_j(t,\cdot)](x_i(t)) \\ &+ \sum_{j\neq i} \alpha_j \left(K(x_i(t) - x_j(t)) - BS[\omega_j(t,\cdot)](x_i(t))\right)\end{aligned}$$

$$\begin{aligned}\frac{dM_i(t)}{dt} &= \sum_{j\neq i} \alpha_j BS[\omega_j(t,\cdot)](M_i(t)) \\ &+ \sum_{j\neq i} \alpha_j \int_{\mathbb{T}^2} \omega_i(t,x)\left(BS[\omega_j(t,\cdot)](x) - BS[\omega_j(t,\cdot)](M_i(t))\right)dx\end{aligned}$$

and from Lemma 110 we deduce

$$\frac{d}{dt}|x_i(t)-M_i(t)| \le C|x_i(t)-M_i(t)| + \int_{\mathbb{T}^2} |K(x_i(t)-x_j(t)) - K(x_i(t)-y)|\,\omega_j(t,y)\,dy + C\int_{\mathbb{T}^2} \omega_i(t,x)\,|x-M_i(t)|\,dx.$$

The second term is bounded by

$$C\int_{\mathbb{T}^2} |x_j(t)-y|\,\omega_j(t,y)\,dy \le C|x_i(t)-M_i(t)| + C\epsilon$$

(we have used Lemma 111 in the last term); similarly the third term by $C\epsilon$; hence

$$\frac{d}{dt}|x_i(t)-M_i(t)| \le C|x_i(t)-M_i(t)| + C\epsilon$$

which implies the result. ■

At time $t=0$, the decay in h of $m_i(t,h)$ is obviously much faster than $C\frac{\epsilon^2}{h^2}$:

$$m_i(0,h) = 0 \text{ for } h > \epsilon.$$

Therefore we expect a much better decay also at later times. The following lemma works in this direction.

Lemma 113 *On* $[0,\tau]$, *we have*

$$m_i(t,2h) \le m_i(0,h) + C\int_0^t \left(\frac{\epsilon^2}{h^8}+1\right) m_i(s,h)\,ds.$$

Proof. Let $\chi_h(x)$ be a smooth approximation of $1_{|x|>h}$, defined as follows: $\chi_h(x) = 1_{|x|>h}$ for $|x|<h$ and $|x|>2h$, $\chi_h(x) \in [0,1]$ for all x, smooth radially symmetric (hence $\nabla\chi_h(x)\cdot x^\perp = 0$), with $|\nabla\chi_h(x)| \le \frac{C}{h}$, $\left|D^2\chi_h(x)\right| \le \frac{C}{h^2}$. Set

$$\mu_i(t,h) = \int_{\mathbb{T}^2} \chi_h(x-M_i(t))\,\omega_i(t,x)\,dx.$$

It is an approximation of $m_i(t,h)$, in the sense that (from $\chi_h(x) \le 1_{|x|>h} \le \chi_{h/2}(x)$)

$$\mu_i(t,h) \le m_i(t,h) \le \mu_i(t,h/2).$$

From the weak vorticity formulation (easily extended to time-dependent test functions) one has

$$\frac{d}{dt}\mu_i(t,h) = I_1 + I_2$$

$$I_1 = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_{\chi_h(\cdot - M_i(t))}(x,y)\,\omega_i(t,x)\,\omega_i(t,y)\,dxdy$$
$$I_2 = \int_{\mathbb{T}^2} \omega_i(t,x)\left(u_{\hat{i}}(t,x) - \frac{dM_i(t)}{dt}\right)\cdot\nabla\chi_h(x - M_i(t))\,dx$$

where

$$H_{\chi_h(\cdot - M_i(t))}(x,y) = \frac{1}{2}K(x-y)\cdot\left(\nabla\chi_h(x - M_i(t)) - \nabla\chi_h(y - M_i(t))\right).$$

For the easy term I_2 we have (from Lemma 110 and the first claim of the present lemma)

$$\begin{aligned}
I_2 &= \int_{\mathbb{T}^2}\omega_i(t,x)\left(\int_{\mathbb{T}^2}\left(u_{\hat{i}}(t,x) - u_{\hat{i}}(t,y)\right)\omega_i(t,y)\,dy\right)\cdot\nabla\chi_h(x - M_i(t))\,dx \\
&\le \frac{C}{h}\int_{|x-M_i(t)|>h}\int_{|y-M_i(t)|>h}\omega_i(t,x)\,\omega_i(t,y)\,dydx \\
&+ \frac{C}{h}\int_{h<|x-M_i(t)|<2h}\int_{|y-M_i(t)|\le h}|x-y|\,\omega_i(t,x)\,\omega_i(t,y)\,dydx \\
&\le \frac{C}{h}m_i(t,h)^2 + Cm_i(t,h) \\
&\le C\left(\frac{\epsilon^2}{h^3}+1\right)m_i(t,h).
\end{aligned}$$

Let us treat now the difficult term I_1, which we rewrite (t, i are given) with the change of variables $x' = x - M_i(t)$, $y' = y - M_i(t)$:

$$I_1 = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\frac{1}{2}K(x'-y')\cdot\left(\nabla\chi_h(x') - \nabla\chi_h(y')\right)\omega_i\left(t,x'+M_i(t)\right)\omega_i\left(t,y'+M_i(t)\right)dx'dy'.$$

Since $(\nabla\chi_h(x') - \nabla\chi_h(y')) = 0$ for $|x\prime| < h, |y\prime| < h$ and $K(x'-y')\cdot(\nabla\chi_h(x') - \nabla\chi_h(y'))$ is symmetric, we may bound

$$I_1 \le \int\int_{|x\prime|>h, y'\in\mathbb{R}^2}\left|K(x'-y')\cdot\left(\nabla\chi_h(x') - \nabla\chi_h(y')\right)\right|\omega_i\left(t,x'+M_i(t)\right)\omega_i\left(t,y'+M_i(t)\right)dx'dy'$$

and now split the integral as $\int\int_{|x\prime|>h,y'\in\mathbb{R}^2} = I_{11}+I_{12}$ where I_{11} is the integral $\int\int_{|x\prime|>h,|y\prime|>h^3}$ and I_{12} the integral $\int\int_{|x\prime|>h,|y\prime|\le h^3}$. For I_{11}, namely for $|x\prime| > h, |y\prime| > h^3$, we simply use the fact that

$$\left|K(x'-y')\cdot\left(\nabla\chi_h(x') - \nabla\chi_h(y')\right)\right| \le C\left\|D^2\chi_h\right\|_\infty \le \frac{C}{h^2}$$

to get (also from the first bound)

$$I_{11} \le \frac{C}{h^2}m_i(t,h)\,m_i\left(t,h^3\right) \le C\frac{\epsilon^2}{h^8}m_i(t,h).$$

For I_{12}, namely for $|x\prime| > h, |y\prime| \leq h^3$, we have

$$\left|K\left(x'-y'\right)\cdot\left(\nabla\chi_h\left(x'\right)-\nabla\chi_h\left(y'\right)\right)\right| = \left|K\left(x'-y'\right)\cdot\nabla\chi_h\left(x'\right)\right|$$

$$\leq \left|\frac{1}{2\pi}\left(\frac{\left(x'-y'\right)^{\perp}}{\left|x'-y'\right|^2}-\frac{\left(x\prime\right)^{\perp}}{\left|x\prime\right|^2}\right)\cdot\nabla\chi_h\left(x\prime\right)\right| + \left|R\left(x'-y'\right)\cdot\nabla\chi_h\left(x\prime\right)\right|$$

because $(x\prime)^{\perp}\cdot\nabla\chi_h(x\prime) = 0$. Since the norm of the differential of $y' \mapsto \frac{(x'-y')^{\perp}}{|x'-y'|^2}$ around $y' = 0$ is bounded above by $\frac{C}{|x'-y'|^2}$, the first term in the above sum is bounded by $\frac{C}{h}\frac{|y'|}{(h-h^3)^2} \leq C$. Since $\nabla\chi_h(x\prime) = 0$ for $|x\prime| > 2h$, to bound the term $|R(x'-y')\cdot\nabla\chi_h(x\prime)|$ we just need to consider $h < |x\prime| \leq 2h, |y\prime| > h^3$; in particular $|x'-y'| \leq 3h$; since R is smooth and null at zero, for such values of x', y' we deduce $|R(x'-y')| \leq C|x'-y'| \leq Ch$ and thus $|R(x'-y')\cdot\nabla\chi_h(x\prime)| \leq C$. Therefore $I_{12} \leq Cm_i(t,h)$. Summarizing,

$$\frac{d}{dt}\mu_i(t,h) \leq C\left(\frac{\epsilon^2}{h^8}+\frac{\epsilon^2}{h^3}+C\right)m_i(t,h).$$

The result easily follows. ■

Corollary 114 *For every $K > 0$ and $\delta \in \left(0,\frac{1}{4}\right)$, we have*

$$\lim_{\epsilon\to 0}\epsilon^{-K}\sup_{t\in[0,\tau]} m_i\left(t,\epsilon^{\frac{1}{4}-\delta}\right) = 0.$$

Proof. For every h such that $h > \epsilon$ (so that $m_i(0,h) = 0$) and $\frac{\epsilon^2}{h^8} \leq C$ we have

$$m_i(t,2h) \leq C\int_0^t m_i(s,h)\,ds.$$

By iteration it follows

$$m_i(t,2^n h) \leq C^n\frac{t^n}{n!}\sup_{s\in[0,t]} m_i(s,h)$$

and using Stirling formula and the trivial bound (we already know more) $\sup_{s\in[0,\tau]} m_i(s,h) \leq C$ we get

$$m_i(t,2^n h) \leq \left(\frac{CTe}{n}\right)^n.$$

Given ϵ, choose $h = \epsilon^{1/4}$ and n such that $2^n \sim \epsilon^{-\delta}$ with $\delta \in \left(0,\frac{1}{4}\right)$. The result of the corollary can be checked by a simple computation. ■

Lemma 115 *Let $x_0 \in B(x_i(0), \epsilon)$ and $h \geq \epsilon^{1/5}$ be given. For every $t \in [0, \tau]$, either*

$$|x(t|x_0) - M_i(t)| \leq h$$

or $|x(t|x_0) - M_i(t)| > h$ and

$$\begin{aligned}\frac{d}{dt}|x(t|x_0) - M_i(t)| &\leq C|x(t|x_0) - M_i(t)| + C\epsilon \\ &\quad + \frac{1}{|x(t|x_0) - M_i(t)|}\frac{1}{|x(t|x_0) - M_i(t)| - h}C\epsilon\end{aligned}$$

where $\zeta > 0$ is arbitrary and C_ζ depends also on ζ.

Proof. We have to prove the above inequality for $\frac{d}{dt}|x(t|x_0) - M_i(t)|$ for those (x_0, t) such that $|x(t|x_0) - M_i(t)| > h$. Thus, when needed, we restrict to such condition. From

$$\begin{aligned}\frac{dM_i(t)}{dt} &= u_{\widehat{i}}(t, M_i(t)) \\ &\quad + \sum_{j \neq i} \alpha_j \int_{\mathbb{T}^2} \omega_i(t, x)\left(BS[\omega_j(t, \cdot)](x) - BS[\omega_j(t, \cdot)](M_i(t))\right)dx\end{aligned}$$

$$\frac{d}{dt}x(t|x_0) = u_{\widehat{i}}(t, x(t|x_0)) + \alpha_i BS[\omega_i(t, \cdot)](x(t|x_0))$$

and the inequality

$$\begin{aligned}&\int_{\mathbb{T}^2} \omega_i(t, x)\left|BS[\omega_j(t, \cdot)](x) - BS[\omega_j(t, \cdot)](M_i(t))\right|dx \\ &\leq C\int_{\mathbb{T}^2} \omega_i(t, x)|x - M_i(t)|\,dx \leq C\epsilon\end{aligned}$$

(we have used Lemma 110 and Lemma 111) we deduce

$$\begin{aligned}\frac{d}{dt}|x(t|x_0) - M_i(t)| &\leq C|x(t|x_0) - M_i(t)| + C\epsilon \\ &\quad + \alpha_i BS[\omega_i(t, \cdot)](x(t|x_0))\frac{x(t|x_0) - M_i(t)}{|x(t|x_0) - M_i(t)|}.\end{aligned}$$

Thus we have to control the last term, namely the effect of the i-blob on fluid particles in the region of the blob itself. The intuition is the following one: if the fluid particle is outside the blob, it should rotate around it as a consequence of its effect; if it is inside, the motion can be arbitrary, but bounded by the size of the blob. We try to implement this idea. The last term above is equal to $\frac{\alpha_i}{2\pi}(I_1+I_2+I_3)$ where

$$I_1 = \int_{|y-M_i(t)|\leq h} \frac{(x(t|x_0)-y)^\perp}{|x(t|x_0)-y|^2} \cdot \frac{y-M_i(t)}{|x(t|x_0)-M_i(t)|} \omega_i(t,y)\,dy$$
$$I_2 = \int_{|y-M_i(t)|> h} \frac{(x(t|x_0)-y)^\perp}{|x(t|x_0)-y|^2} \cdot \frac{x(t|x_0)-M_i(t)}{|x(t|x_0)-M_i(t)|} \omega_i(t,y)\,dy$$
$$I_3 = 2\pi \int_{\mathbb{T}^2} R(x(t|x_0)-y) \cdot \frac{x(t|x_0)-M_i(t)}{|x(t|x_0)-M_i(t)|} \omega_i(t,y)\,dy.$$

The correction term (between full space and torus) is bounded as above (R is smooth and $R(0)=0$):

$$I_3 \leq C \int_{\mathbb{T}^2} |x(t|x_0)-y|\,\omega_i(t,y)\,dy \leq C\,|x(t|x_0)-M_i(t)| + C\epsilon.$$

For I_1 we can write

$$I_1 \leq \frac{1}{|x(t|x_0)-M_i(t)|} \int_{|y-M_i(t)|\leq h} \frac{1}{|x(t|x_0)-y|}\,|y-M_i(t)|\,\omega_i(t,y)\,dy.$$

Here we use the condition $x(t|x_0) \notin B(M_i(t),h)$: for $|y-M_i(t)|\leq h$,

$$\frac{1}{|x(t|x_0)-y|} \leq \frac{1}{|x(t|x_0)-M_i(t)|-h}$$

hence (by Lemma 111)

$$I_1 \leq \frac{1}{|x(t|x_0)-M_i(t)|} \frac{1}{|x(t|x_0)-M_i(t)|-h} C\epsilon.$$

Finally, let us discuss the most difficult term I_2. It is equal to

$$I_2 = \frac{1}{\pi\epsilon^2} \frac{x(t|x_0)-M_i(t)}{|x(t|x_0)-M_i(t)|} \cdot \int_{S_i(t,\epsilon)\cap B(M_i(t),h)^c} \frac{(x(t|x_0)-y)^\perp}{|x(t|x_0)-y|^2}\,dy.$$

Since we assume $x_0 \in B(x_i(0),\epsilon)$, we have $x(t|x_0) \in S_i(t,\epsilon)$, and also $x(t|x_0) \in B(M_i(t),h)^c$ by the assumption at the beginning of the proof, hence the singularity in the Biot-Savart kernel is in the domain of integration. It is interesting to remark that, using only the bound of Lemma 111, it is not possible to prove that this term is small. It is necessary to use Corollary 114. We have indeed, for every $\zeta > 0$ and $K > 0$, by Hölder inequality and Corollary 114,

$$I_2 \leq C_\zeta \left(\int_{|y-M_i(t)|>h} \omega_i(t,y)^{2+\zeta}\,dy \right)^{1/(2+\zeta)} \leq C_\zeta \epsilon^{-\frac{2+2\zeta}{2+\zeta}} m_i(t,h)^{1/(2+\zeta)}$$
$$\leq C_{\zeta,K} \epsilon^{-\frac{2+2\zeta}{2+\zeta}} \left(\epsilon^K\right)^{1/(2+\zeta)} = C_{\zeta,K} \epsilon^{\frac{K-2-2\zeta}{2+\zeta}}$$

where C_ζ and $C_{\zeta,K}$ are positive constants depending also on ζ and K (we have used an upper bound for $\omega_i(t,y)$ by $(\pi\epsilon)^{-2}$). The result of the lemma follows by a proper choice of ζ and K. ■

Corollary 116 *On* $[0,\tau]$, *for every* $x_0 \in B(x_i(0),\epsilon)$ *we have*

$$|x(t|x_0) - x_i(t)| \le C\epsilon^{1/5}. \tag{6.4}$$

Proof. In view of Lemma 112, it is sufficient to prove that

$$|x(t|x_0) - M_i(t)| \le C\epsilon^{1/5}. \tag{6.5}$$

Choose $h = \epsilon^{1/5}$ in the statement of the previous lemma; h is much larger than ϵ. Initial points $x_0 \in B(x_i(0),\epsilon)$ have the property $|x(0|x_0) - M_i(0)| < 2h$, because $M_i(0) = x_i(0)$. Let t_0 be any time in $[0,\tau]$ when $|x(t_0|x_0) - M_i(t_0)| = 2h$ and $|x(t|x_0) - M_i(t)| > 2h$ in some interval of the form $(t_0,t_1) \subset [0,\tau]$; take it maximal, in the sense that either $t_1 = \tau$ or $|x(t_1|x_0) - M_i(t_1)| = 2h$. If there are no such times, it means that $|x(t|x_0) - M_i(t)| \le 2\epsilon^{1/5}$ on $[0,\tau]$ and thus (6.5) holds true. If there are such times, in the complementary part of $[0,\tau]$ inequality (6.5) holds true, and thus we have to prove it in the maximal intervals (t_0,t_1) just defined. Let (t_0,t_1) be one of such intervals. We have

$$\frac{1}{|x(t|x_0) - M_i(t)|}\frac{1}{|x(t|x_0) - M_i(t)| - h} \le \frac{1}{2h^2}$$

and thus, recalling also that $h = \epsilon^{1/5}$, we deduce

$$\begin{aligned}\frac{d}{dt}|x(t|x_0) - M_i(t)| &\le C|x(t|x_0) - M_i(t)| + C\epsilon + C\frac{\epsilon}{\epsilon^{2/5}} \\ &\le C|x(t|x_0) - M_i(t)| + C\epsilon^{3/5}.\end{aligned}$$

Hence

$$|x(t|x_0) - M_i(t)| \le |x(t_0|x_0) - M_i(t_0)|\, e^{CT} + Te^{CT}C\epsilon^{3/5} \le C\epsilon^{1/5}$$

by Gronwall lemma and the property $|x(t_0|x_0) - M_i(t_0)| = h = \epsilon^{1/5}$. Thus inequality (6.5) holds true in (t_0,t_1). ■

We may now complete the proof of the theorem. Let C be the constant in (6.4); choose any ϵ such that $C\epsilon^{1/5} \le \frac{r_0}{8}$. From (6.4), the support of $\omega_i(t,\cdot)$ is contained in $B\left(x_i(t), \frac{r_0}{8}\right)$ for all $t \in [0,\tau]$ and all $i = 1,...,N$. By an easy argument of maximality, we can take $\tau = T$. Thus (6.4) holds on $[0,T]$, which means that the support of $\omega_i(t,\cdot)$ is contained in $B\left(x_i(t), C\epsilon^{1/5}\right)$ for all $t \in [0,T]$ and all $i = 1,...,N$.

We conclude the section with an obvious corollary.

Corollary 117 *i) Given $N \in \mathbb{N}$, $T > 0$, $\alpha_1, ..., \alpha_N \in \mathbb{R}$, $x_1(0), ..., x_N(0) \in \mathbb{T}^2$, such that the motion of the point vortices $x_1(t), ..., x_N(t)$ on $\mathbb{T}^2$ with this initial condition and intensities exists uniquely outside Δ_N on $[0,T]$, there exists a sequence of L^∞ solutions $\omega^{(n)}(t,x)$ of 2D Euler equations such that*

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} \phi(x)\, \omega^{(n)}(t,x)\, dx = \sum_{i=1}^N \alpha_i \phi(x_i(t))$$

for all $t \in [0,T]$ and $\phi \in C(\mathbb{T}^2)$.

ii) Given $N \in \mathbb{N}$, $T > 0$, let $\alpha_1, ..., \alpha_N$ be real valued random variables and $x_1(0), ..., x_N(0)$ be random variables taking values in $\mathbb{T}^2$, defined on a probability space $(\Xi, \mathcal{F}, \mathbb{P})$, such that the point vortex motion with this initial condition and intensities exists uniquely outside Δ_N on $[0,T]$, with $\mathbb{P}$-probability 1. Choose any sequence $\epsilon_n \to 0$ and the initial conditions

$$\omega^{(n)}(\theta, 0, x) = \sum_{i=1}^N \frac{\alpha_i(\theta)}{\pi {\epsilon_n}^2} 1_{B(x_i(\theta,0),\epsilon_n)}(x)$$

parametrized by $\theta \in \Xi$; let $\omega^{(n)}(\theta, t, x)$, defined on $\Xi \times [0,T] \times \mathbb{T}^2$, be the corresponding L^∞ solutions of 2D Euler equations, parametrized by $\theta \in \Xi$. Then, for $\mathbb{P}$-a.e. $\theta \in \Xi$, the following hold:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{T}^2} \phi(x)\, \omega^{(n)}(\theta,t,x)\, dx - \sum_{i=1}^N \alpha_i \phi(x_i(\theta,t)) \right| = 0$$

for all $\phi \in C(\mathbb{T}^2)$.

Proof. i) Choose $\epsilon_n \to 0$; eventually it satisfies the condition of the theorem. Denote by $\omega^{(n)}(0,x)$ the corresponding initial condition as defined above for the theorem. The corresponding solution $\omega^{(n)}(t,x)$ has the form

$$\omega^{(n)}(t,x) = \sum_{i=1}^N \alpha_i \omega_i^{(n)}(t,x)$$

and, by the theorem, eventually the components $\omega_i^{(n)}(t,\cdot)$ have support in $B\left(x_i(t), C\epsilon^{1/5}\right)$. When this happens

$$\left| \int_{\mathbb{T}^2} \phi(x)\, \omega_i^{(n)}(t,x)\, dx - \phi(x_i(t)) \right| \leq \int_{\mathbb{T}^2} |\phi(x) - \phi(x_i(t))|\, \omega_i^{(n)}(t,x)\, dx \leq k_\phi\left(C\epsilon^{1/5}\right)$$

where $k_\phi(r)$ is the modulus of continuity of the uniformly continuous function ϕ. This implies che claim.

ii) The argument of point (i) applies for $\mathbb{P}$-a.e. $\theta \in \Xi$. ■

Bibliography

[1] M. Aizenman, A sufficient condition for the avoidance of sets by measure preserving flows in $\mathbb{R}^n$, *Duke Math. J.* **45** (1978), 809-813.

[2] S. Albeverio and A. B. Cruzeiro, Global flows with invariant (Gibbs) measures for Euler and Navier–Stokes two-dimensional fluids, *Comm. Math. Phys.* **129** (1990), 431-444.

[3] S. Albeverio and B. Ferrario, 2D vortex motion of an incompressible ideal fluid: the Koopman-von Neumann approach, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **6** (2003), 155-165.

[4] S. Albeverio and B. Ferrario, *Some Methods of Infinite Dimensional Analysis in Hydrodynamics: An Introduction*, In SPDE in Hydrodynamic: Recent Progress and Prospects, G. Da Prato and M. Röckner Eds., CIME Lectures, Springer-Verlag, Berlin 2008.

[5] S. Albeverio and R. Høegh-Krohn, Stochastic flows with stationary distribution for two-dimensional inviscid fluids, *Stochastic Process. Appl.* **31** (1989), 1-31.

[6] S. Albeverio, M. Ribeiro de Faria, and R. Høegh-Krohn, Stationary measures for the periodic Euler flow in two dimensions, *J. Statist. Phys.* **20** (1979), 585-595.

[7] L. Ambrosio, *Transport equation and Cauchy problem for non-smooth vector fields*, CIME Lectures, Springer-Verlag, Berlin 2006.

[8] L. Ambrosio, A. Figalli, On flows associated to Sobolev vector fields in Wiener spaces: an approach à la DiPerna-Lions, *J. Funct. Anal.* **256** (2009), 179-214.

[9] L. Ambrosio, D. Trevisan, Well posedness of Lagrangian flows and continuity equations in metric measure spaces, *Anal. PDE* **7** (2014), 1179-1234.

[10] N. Berestycki, Introduction to the Gaussian Free Field and Liouville Quantum Gravity, 2015.

[11] V. I. Bogachev, *Gaussian Measures*, American Mathematical Society 1998.

[12] V. Bogachev, E. M. Wolf, Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions, *J. Funct. Anal.* **167** (1999), 1-68.

[13] J. Bourgain, Invariant measures for the 2d-defocusing nonlinear Schrodinger equation, *Comm. Math. Phys.* **176** (1996), 421-445.

[14] Z. Brzezniak, F. Flandoli, M. Maurelli, Existence and Uniqueness for Stochastic 2D Euler Flows with Bounded Vorticity, *Arch. Rational Mech. Anal.* **221** (2016), 107-142.

[15] N. Burq, N. Tzvetkov, Random data Cauchy theory for supercritical wave equations I. Local theory, *Invent. Math.* **173** (2008), 449-475.

[16] N. Burq, N. Tzvetkov, Random data Cauchy theory for supercritical wave equations II. A global existence result, *Invent. Math.* **173** (2008), 477-496.

[17] P. Buttà, C. Marchioro, Long time evolution of concentrated Euler flows with planar symmetry, to appear on *SIAM J. Math. Anal.*

[18] J.-Y. Chemin, *Perfect incompressible fluids*, volume 14 of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Oxford University Press, New York, 1998.

[19] F. Cipriano, The two-dimensional Euler equation: a statistical study, *Comm. Math. Phys.* **201** (1999), 139-154.

[20] A. B. Cruzeiro, Unicité de solutions d'équations di érentielles sur l'espace de Wiener, *J. Funct. Anal.* **58** (1984), 335-347.

[21] G. Da Prato, *An Introduction to Infinite Dimensional Analysis*, Springer, oppure appunti SNS.

[22] G. Da Prato, F. Flandoli and M. Rockner, Uniqueness for continuity equations in Hilbert spaces with weakly differentiable drift, *Stoch. PDE: Anal. Comp.* **2** (2014), 121-145.

[23] G. Da Prato, F. Flandoli, M. Rockner, Existence of absolutely continuous solutions for continuity equations in Hilbert spaces, preprint.

[24] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

[25] F. Delarue, F. Flandoli, D. Vincenzi, Noise prevents collapse of Vlasov–Poisson point charges, *Comm. Pure Appl. Math.* **67** (2014), n. 10, 1700-1736.

[26] C. De Lellis, L. Székelyhidi, The Euler equations as a differential inclusion, *Annals of Math.* **170** (2009), n. 3, 1417-1436.

[27] J.-M. Delort, Existence of vortex sheets in dimension two. (Existence de nappes de tourbillon en dimension deux.), *J. Am. Math. Soc.* **4** (1991), 553-586.

[28] R. J. DiPerna, P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98** (1989), 511–547.

[29] R. DiPerna, A. Majda, Concentrations in regularizations for two-dimensional incompressible flow, *Comm. Pure Appl. Math.* **40** (1987), 301-345.

[30] D. Dürr, M. Pulvirenti, On the vortex flow in bounded domains, *Comm. Math. Phys.* **85**, 256-273 (1982).

[31] Shizan Fang, Dejun Luo, Transport equations and quasi-invariant flows on the Wiener space, *Bulletin des Sciences Mathématiques* **134** (2010), 295-328.

[32] L. Frachebourg, P. A. Martin, Exact statistical properties of the Burgers equation, *J. Fluid Mech.* **417** (2000), 323-349.

[33] M. Gubinelli, M. Jara, Regularization by noise and stochastic Burgers equations, *Stoch. Partial Diff. Eq.: Anal. Comp.* **1** (2013), 325-350.

[34] M. Gubinelli, N. Perkowski, KPZ Reloaded, *Comm. Math. Phys.* **349** (2017), 165-269.

[35] M. Hairer, Solving the KPZ equation, *Annals of Mathematics* **178** (2013), 559-664.

[36] M. Kardar, G. Parisi, Y.-C. Zhang, Dynamic scaling of growing interfaces, Physical Review Letters 56 (1986), n. 9, 889-892.

[37] C. Kipnis, C. Landim, *Scaling Limits of Interacting Particle Systems*, Springer-Verlag, New York 1999.

[38] H. H. Kuo, *Gaussian Measures in Banach Spaces*, Springer-Verlag, Berlin 1975.

[39] M. Ledoux, M. Talagrand, *Probability in Banach Spaces*, Springer, New York 2011.

[40] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, volume 1, Incompressible Models, Science Publ., Oxford, 1996.

[41] A. J. Majda, A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Univ. Press, 2002.

[42] C. Marchioro, M. Pulvirenti, Vortices and localization in Euler flows, *Comm. Math. Phys.* **154** (1993), 49-61.

[43] C. Marchioro, M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, volume 96 of Applied Mathematical Sciences, Springer-Verlag, New York, 1994.

[44] K. Matetski, J. Quastel, D. Remenik, The KPZ fixed point, arXiv:1701.00018.v1.

[45] E. Miot, Étude du système dynamique de N tourbillons ponctuels, Journées XUPS, Polytechnique 2015.

[46] T. Oh, Invariance of the white noise for KdV, *Comm. Math. Phys.* **292** (2009) 217-236.

[47] B. X. Hu, J. Miller, Y. Peres, Thick points of the Gaussian Free Field, The Annals of Probab. 38 (2010), No. 2, 896-926.

[48] F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equation, *Meth. Appl. Anal.* **9** (2002), 533-562.

[49] J. Quastel, B. Valkó, KdV preserves white noise, *Comm. Math. Phys.* **277** (2008), 707-714.

[50] G. Richards, Invariance of the Gibbs measure for the periodic quartic gKdV, *Ann. Instit. H. Poincaré (C) Anal. non lin.* **33** (2016), 699-766.

[51] S. Sheffield, Gaussian free fields for mathematicians, 2006.

[52] A. Shnirelman, On the nonuniqueness of weak solution of the Euler equation, *Comm. Pure Appl. Math.* **50**(12):1261-1286, 1997.

[53] S. Schochet, The point-vortex method for periodic weak solutions of the 2-D Euler equations, *Comm. Pure Appl. Math.* **91**, 1-965 (1996).

[54] S. Schochet, The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation, *Comm. Partial Diff. Eq.* **20**, 1995, pp. 1077-1104.

[55] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.* **146** (1987) 65-96.

[56] W. Stannat, L^1-uniqueness of regularized 2D-Euler and stochastic Navier–Stokes equations, *J. Funct. Anal.* **200**(1):101-117, 2003.

[57] D. W. Stroock, S. R. S. Varadhan, *Multidimensional Diffusion Processes*, New York 1979.

[58] A. Symeonide, Invariant measures for the two-dimensional averaged-Euler equations, arXiv 1605.06974.v1.

[59] Nicolai Tzvetkov, Random data wave equations, CIME Lectures 2016, to appear.

[60] W. Wolibner, Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogene, incompressible, pendant un temps infiniment long, *Math. Z.* 37 (1933), 698-726.

[61] V. I. Yudovich, Non-stationary flows of an ideal incompressible liquid, *USSR Comput. Math. and Math. Phys.* **3** (1963), 1407-1456; *Z. Vyčisl. Mat. i Mat. Fiz.* **3** (1963), 1032-1066.