Zero-noise selection for Peano phenomena

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1 General theory of weak solutions to SDEs indexed by noise intensity

Let $(E,\mathcal{E},\mathbb{Q}),(\mathcal{E}_t)_{t\geq 0}$ a standard-filtered probability space equipped with a standard Brownian motion B. Consider on this space the family indexed by $\varepsilon\in(0,1)$ of SDEs in dimension one:

$$X^{(\varepsilon)}_t=\int_0^t b(X^{(\varepsilon)}_s)\,ds+\varepsilon\int_0^t dB_s, \tag{1}$$

where $b\in C_b(\mathbb{R})$. Classical results about SDEs (see e.g. [4], Chapter 9) guarantee strong existence and uniqueness for such SDEs, ε beeing fixed and positive, so that it does make sense to talk about the law of $X^{(\varepsilon)}$. We would like to study the laws of the solutions of (1): in particular, we are interested in the zero-noise limit of the previous equations and how this is related to the deterministic ODE obtained by imposing $\varepsilon=0$ in (1). In order to do this we will introduce a new setting that turns out to be more convenient for our purpose. Let $\Omega:=C(\mathbb{R}^+,\mathbb{R})$ be the space of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$, endowed with the distance:

$$D^{loc}_\infty(\omega,\omega'):=\sum_{k\in\mathbb{N}}2^{-k}\frac{\sup_{t\in[0,k]}|\omega_t-\omega'_t|}{1+\sup_{t\in[0,k]}|\omega_t-\omega'_t|},$$

where ω_t denotes the value of the function $\omega\in\Omega$ at time t. Then (Ω,D^{loc}_∞) is a Polish space and the convergence induced by D^{loc}_∞ is the uniform convergence on compact subsets of $\mathbb{R}^+$. A well known result (see e.g. [5], section 1.3) is that the Borel σ-field $\mathcal{F}$ associated with the distance D^{loc}_∞ coincides with the σ-field generated by the projection maps $\pi_t(\omega)=\omega_t$ and $\mathcal{F}_t:=\sigma\{\pi_s,s\leq t\}$ is a filtration on $(\Omega,\mathcal{F})$. It turns out that $X^{(\varepsilon)}:E\to\Omega$ is measurable, being $\pi_t\circ X^{(\varepsilon)}:E\to\mathbb{R}$ measurable for every $t\geq 0$, hence we can consider $\mathbb{P}^\varepsilon:=X^{(\varepsilon)}_\#(\mathbb{Q})$, the law of $X^{(\varepsilon)}$ on Ω. Here we have some general results about the family of probabilities $\{\mathbb{P}^\varepsilon\}_{\varepsilon\in(0,1)}$ which somehow justify our interest in the zero-noise limit of (1) and indicate a possible way to study this problem.

Proposition 1. The family $\{\mathbb{P}^\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact with respect to the weak convergence of measures.

Proof. The main tool that we are going to use is Prokhorov Theorem.
Let define on Ω the process $t \mapsto M_t(\omega) := \omega_t - \int_0^t b(\omega_s)\,ds$. By the very definition of $\mathbb{P}^\varepsilon$ we have that $\forall \varepsilon \in (0,1)$ the process M is a rescaled Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^\varepsilon)$ endowed with the filtration $(\mathcal{F}_t)$ and its quadratic variation under $\mathbb{P}^\varepsilon$ is $\langle M\rangle_t^\varepsilon = \varepsilon^2 t$.
Let consider now the set, depending on $N \in \mathbb{N}$:

$$L_N := \Big\{ \omega \in \Omega : \forall m \in \mathbb{N}, m \geq N, \forall p \in \mathbb{N}, \forall \,\text{interval}\, I \subseteq [0,m] \\ \text{of lenght} \leq 2^{-p} : \operatorname{osc}(M(\omega), I) \leq 4\sqrt{2\log 2mp2^{-p}} \Big\},$$

where $\operatorname{osc}(M(\omega), I) := \sup_{t\in I}(M_t(\omega)) - \inf_{t\in I}(M_t(\omega))$ denotes the oscillation function. We want to show that $\forall \alpha \in (0,1)$ there is a $N(\alpha)$ such that $\mathbb{P}^\varepsilon(L_{N(\alpha)}) \geq 1-\alpha$ for every $\varepsilon \in (0,1)$. Let divide the interval $[0,m]$ into $m \cdot 2^p$ subintervals I_j of the form $[(j-1)\cdot 2^{-p}, j \cdot 2^{-p}], j = 1, ..., m\cdot 2^p$. We have the following inclusions:

$$\begin{aligned} L_N^c &\subseteq \bigcup_{m\geq N} \bigcup_{p\in\mathbb{N}} \bigcup_{j=1}^{m\cdot 2^p} \Big\{ \operatorname{osc}(M(\omega), I_j) > 2\sqrt{2\log 2mp2^{-p}} \Big\} \\ &\subseteq \bigcup_{m\geq N} \bigcup_{p\in\mathbb{N}} \bigcup_{j=1}^{m\cdot 2^p} \Big\{ \sup_{r\in I_j} \big|M_r(\omega) - M_{(j-1)\cdot 2^{-p}}(\omega)\big| > \sqrt{2\log 2mp2^{-p}} \Big\}. \end{aligned}$$

By Markov Property of Brownian motion we know that for every $c > 0$:

$$\mathbb{P}^\varepsilon\Big\{ \sup_{r\in I_j} \big|M_r(\omega) - M_{(j-1)\cdot 2^{-p}}(\omega)\big| > c \Big\} = \mathbb{P}\Big\{ \sup_{r\leq 2^{-p}} |\varepsilon B_r(\omega)| > c \Big\},$$

where B is a standard Brownian motion under $\mathbb{P}$. By applying Doob's maximal inequality to the stochastic exponential of B we obtain for any $\lambda > 0$:

$$\begin{aligned} \mathbb{P}\Big\{ \sup_{r\leq 2^{-p}} \varepsilon B_r(\omega) > c \Big\} &= \mathbb{P}\Big\{ \sup_{r\leq 2^{-p}} e^{\lambda\varepsilon B_r(\omega)} > e^{\lambda c} \Big\} \\ &\leq \mathbb{P}\Big\{ \sup_{r\leq 2^{-p}} e^{\lambda\varepsilon B_r(\omega) - \frac{\lambda^2\varepsilon^2}{2} r} > e^{\lambda c - \frac{\lambda^2}{2}2^{-p}} \Big\} \\ &\leq \mathbb{E}\left[e^{\lambda\varepsilon B_{2^{-p}}(\omega) - \frac{\lambda^2\varepsilon^2}{2}2^{-p}} \right] \cdot e^{\frac{\lambda^2}{2}2^{-p} - \lambda c} = 1 \cdot e^{\frac{\lambda^2}{2}2^{-p} - \lambda c}. \end{aligned}$$

By taking $\lambda = c \cdot 2^p$ and using simmetry we get:

$$\mathbb{P}^\varepsilon\Big\{ \sup_{r\in I_j} \big|M_r(\omega) - M_{(j-1)\cdot 2^{-p}}(\omega)\big| > c \Big\} \leq 2 \cdot e^{-\frac{c^2}{2}2^p},$$

which after an opportune choice of the parameter c becomes:

$$\mathbb{P}^\varepsilon \Big\{ \sup_{r\in I_j} \big| M_r(\omega) - M_{(j-1)\cdot 2^{-p}}(\omega) \big| > \sqrt{2\log 2mp2^{-p}} \Big\} \leq 2^{1-mp},$$

thus we obtain the following estimate for the probability of L_N^c:

$$\begin{aligned} \mathbb{P}^\varepsilon(L_N^c) &\leq \sum_{m\geq N} \sum_{p\in\mathbb{N}} \sum_{j=1}^{m\cdot 2^p} \mathbb{P}^\varepsilon \Big\{ \sup_{r\in I_j} \big| M_r(\omega) - M_{(j-1)\cdot 2^{-p}}(\omega) \big| > \sqrt{2\log 2mp2^{-p}} \Big\} \\ &\leq \sum_{m\geq N} \sum_{p\in\mathbb{N}} \sum_{j=1}^{m\cdot 2^p} 2^{1-mp} \to 0, \end{aligned}$$

as $N \to \infty$ and uniformly in ε, and so we have proved that $\forall \alpha \in (0,1)$ there is a $N(\alpha)$ such that $\mathbb{P}^\varepsilon(L_{N(\alpha)}) \geq 1-\alpha$ for every $\varepsilon \in (0,1)$. Now observe that the same result holds true if we replace $L_{N(\alpha)}$ with:

$$L^0_{N(\alpha)} \coloneqq L_{N(\alpha)} \cap \{\omega \in \Omega : \omega_0 = 0\},$$

being the latter a set of full measure $\forall \varepsilon \in (0,1)$. We want to show that $L^0_{N(\alpha)}$ is compact in Ω, so that Prokhorov Theorem applies. Fixed any compact K of $\mathbb{R}^+$, the family of functions $L^0_{N(\alpha)}$ is equicontinuous and equibounded when restricted to K (here we use the fact that b is bounded), hence by Ascoli-Arzelà Theorem $L^0_{N(\alpha)}$ is relatively compact with respect to uniform convergence on compact sets; being both the conditions that define $L^0_{N(\alpha)}$ closed under uniform convergence on compact sets, $L^0_{N(\alpha)}$ is closed in Ω and so it is compact, and this completes the proof. □

The previous proposition guarantees at least that a zero-noise limit for (1) exists in a weak sense, so that we do have something to talk about. The following proposition is crucial in relating the family $\{\mathbb{P}^\varepsilon\}_{\varepsilon\in(0,1)}$ to the ODE obtained by imposing $\varepsilon = 0$ in (1).

Proposition 2. Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ be a sequence converging to zero and suppose that $\mathbb{P}^{\varepsilon_n} \to \mathbb{P}$ weakly as $n \to \infty$. Then the measure $\mathbb{P}$ only charges solutions of the ODE:

$$X_t = \int_0^t b(X_s)\, ds. \tag{2}$$

Proof. Let B be the Brownian motion driving the SDEs (1) and let $\mathbb{W}$ be its law on $(\Omega, \mathcal{F})$. Then it is easy to check that $\mathbb{P}^{\varepsilon_n} \otimes \mathbb{W}$ converges weakly to $\mathbb{P} \otimes \mathbb{W}$ as measures on $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F})$, hence by Skorokhod Theorem there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $\tilde{X}^{(\varepsilon_n)}, \tilde{X}, \tilde{B}^{(\varepsilon_n)}, \tilde{B}$ such that:

1. $(\tilde{X}^{(\varepsilon_n)}, \tilde{B}^{(\varepsilon_n)})$ has law $\mathbb{P}^{\varepsilon_n} \otimes \mathbb{W}$ for every $n \in \mathbb{N}$;
2. $(\tilde{X}, \tilde{B})$ has law $\mathbb{P} \otimes \mathbb{W}$;
3. $(\tilde{X}^{(\varepsilon_n)}, \tilde{B}^{(\varepsilon_n)})$ converges to $(\tilde{X}, \tilde{B})$ $\tilde{\mathbb{P}}$-a.s. as $n \to \infty$, i.e. for almost every $\tilde{\omega} \in \tilde{\Omega}$ the function $t \mapsto (\tilde{X}_t^{(\varepsilon_n)}(\tilde{\omega}), \tilde{B}_t^{(\varepsilon_n)}(\tilde{\omega}))$ converges uniformly on compact subsets towards $t \mapsto (\tilde{X}_t(\tilde{\omega}), \tilde{B}_t(\tilde{\omega}))$, being the pointwise convergence in $\Omega \times \Omega$ given by the metric D_∞^{loc}.

Observe now that since the law of $(\tilde{X}^{(\varepsilon_n)}, \tilde{B}^{(\varepsilon_n)})$ is $\mathbb{P}^{\varepsilon_n} \otimes \mathbb{W}$, the process

$$t \mapsto \tilde{X}_t^{(\varepsilon_n)} - \int_0^t b(\tilde{X}_s^{(\varepsilon_n)})\, ds - \varepsilon_n \int_0^t d\tilde{B}_s^{(\varepsilon_n)}$$

must have the same law of the process constantly equal to zero, thus we have the following identity of stochastic processes

$$\tilde{X}_\cdot^{(\varepsilon_n)} - \varepsilon_n \int_0^\cdot d\tilde{B}_s^{(\varepsilon_n)} = \int_0^\cdot b(\tilde{X}_s^{(\varepsilon_n)})\, ds, \qquad \tilde{\mathbb{P}}\text{-a.s.}. \tag{3}$$

By 3, the LHS of equation (3) converges $\tilde{\mathbb{P}}$-a.s. to $\tilde{X}$, while the RHS converges $\tilde{\mathbb{P}}$-a.s. to $\int_0^\cdot b(\tilde{X}_s)\, ds$, by continuity of b. Hence $\tilde{\mathbb{P}}$-a.s. we must have $\tilde{X}_t = \int_0^t b(\tilde{X}_s)\, ds$ for every $t \geq 0$ and since $\tilde{X}_\#(\tilde{\mathbb{P}}) = \mathbb{P}$ this means that $\mathbb{P}\{\, \omega \in \Omega : \omega_t = \int_0^t b(\omega_s)\, ds\ \forall t \geq 0 \,\} = 1$, that is the desired result. □

Observe that the previous proposition completely solves the zero-noise problem when uniqueness holds for (2): in fact, any cluster point for $\varepsilon \to 0$ of $\{\mathbb{P}^\varepsilon\}_{\varepsilon \in (0,1)}$ must be the Dirac measure of the unique solution of (2). On the other hand, it opens up an entire category of problems when uniqueness does not hold, namely that of identifying any possible weak limit of $\{\mathbb{P}^\varepsilon\}_{\varepsilon \in (0,1)}$ among all the probability measures on Ω that give full measure to the set of solutions of (2). This motivates the study carried out in the next section, where solutions of (2) are described and the concept of exit time from an interval is introduced.

2 Description of solutions to the zero-noise ODE

In this section a brief descrition of solutions of (2) will be given. No difficulty arises from considering any initial condition different from zero, but we restrict ourselves to this case to lighten the discussion. Being mostly interested in the case of non uniqueness, we will assume $b(0) = 0$ (if not, local uniqueness holds) and also that the point 0 is an isolated zero of b.

Proposition 3. Consider the problem (2), where $b \in C_b(\mathbb{R})$ and 0 is an isolated zero of b: then there are local solutions to (2) different from the constant one if and only if there exists $r > 0$ such that at least one of the following holds:

1. $b(x) > 0$ for $x \in (0, r)$ and $\int_0^r \frac{dy}{b(y)} < \infty$;

2. $b(x) < 0$ for $x \in (-r, 0)$ and $\int_0^{-r} \frac{dy}{b(y)} < \infty$.

In this case, denoting $H(x) \coloneqq \int_0^x \frac{dy}{b(y)}$:

1. if $b(x) \geq 0$ or $b(x) \leq 0$ in $(-r, r)$, then there exists $t_0 \geq 0$ such that locally $X_t = H^{-1}((t - t_0)^+)$;

2. if $xb(x) \geq 0$ in $(-r, r)$ and $H(-r) = \infty, H(r) < \infty$ (respectively $H(-r) < \infty, H(r) = \infty$), then the local solutions are the same as the previous case with $b(x) \geq 0$ (respectively $b(x) \leq 0$);

3. if $xb(x) \geq 0$ in $(-r, r)$ and $H(-r) < \infty, H(r) < \infty$, then there exists $t_0 \geq 0$ such that locally either $X_t = (H\big|_{\mathbb{R}^+})^{-1}((t - t_0)^+)$ or $X_t = (H\big|_{\mathbb{R}^-})^{-1}((t - t_0)^+)$.

We will call in the following $(H\big|_{\mathbb{R}^+})^{-1}((t - t_0)^+)$ the upper extremal solution and $(H\big|_{\mathbb{R}^-})^{-1}((t - t_0)^+)$ the lower extremal solution to (2). Observe that the upper extremal solution is the one which minimizes the exit time from $(-r, r)$ among all the solutions which leave $(-r, r)$ at the point r: in this case the exit time is equal to $H(r)$. Similarly, the lower extremal solution is the one which minimizes the exit time from $(-r, r)$ among all the solutions which leave $(-r, r)$ at the point $-r$: in this case the exit time is equal to $H(-r)$. These minimality results will be fundamental in the following analysis.

3 Exit time from the interval $(-r, r)$

In this section we will follow the approch proposed by Bafico and Baldi in [1]. First of all, we develop some technical tools that will be very useful in the study of zero-noise limit of (1). Our investigation will be concerning the function

$$\begin{aligned} \tau : \Omega &\to \mathbb{R}^+ \cup \{+\infty\} \\ \omega &\mapsto \inf\{\, t \geq 0 : |\omega_t| \geq r \,\}, \end{aligned}$$

r being a positive real such that analysis of section 2 applies to our problem. Here we have some nice properties that the function τ enjoys.

Proposition 4. $\tau : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is lower semicontinuous. In particular, τ is a random variable.

Proof. It suffices to show that for every $a \in \mathbb{R}^+$ the set

$$\{\, \omega \in \Omega : \tau(\omega) > a \,\}$$

is open in Ω with respect to the topology induced by D_∞^{loc}. Take $\omega \in \{\, \tau > a \,\}$; we have that $\sup_{t \le a} |\omega_t| < r$, and by continuity of ω and compactness of $[0, a]$ it exists $\varepsilon > 0$ such that $\sup_{t \le a} |\omega_t| \le r - \varepsilon$. Now take $n \in \mathbb{N}$ greater than a; we have

$$D_\infty^{loc}(\omega, \omega') \ge 2^{-n} \frac{\sup_{t \in [0,n]} |\omega_t - \omega'_t|}{1 + \sup_{t \in [0,n]} |\omega_t - \omega'_t|},$$

thus for $\delta = \delta(\varepsilon, n)$ small enough the following implication holds:

$$D_\infty^{loc}(\omega, \omega') < \delta \Longrightarrow \sup_{t \in [0,n]} |\omega_t - \omega'_t| < \varepsilon,$$

which means that $\omega' \in \{\, \omega \in \Omega : \tau(\omega) > a \,\}$. □

Proposition 5. If $\mathbb{P}^{\varepsilon_n} \to \mathbb{P}$ weakly as $n \to \infty$, then

$$\liminf_{n \to \infty} \mathbb{E}^{\mathbb{P}^{\varepsilon_n}} [\tau] \ge \mathbb{E}^{\mathbb{P}} [\tau] .$$

Proof. Let define for every $k \in \mathbb{N}$ the function

$$\tau_k(\omega) \coloneqq \inf_{\omega' \in \Omega} \{\, \tau(\omega') + k \cdot D_\infty^{loc}(\omega, \omega') \,\}.$$

Every τ_k is bounded continous and the sequence $\{\tau_k\}_{k \in \mathbb{N}}$ is clearly increasing. By choosing $\omega' = \omega$ we get $\tau_k \le \tau$ and by lower semicontinuity of τ the sequence $\{\tau_k\}_{k \in \mathbb{N}}$ converges to τ. Hence for every $n \in \mathbb{N}$ and every $k \in \mathbb{N}$:

$$\mathbb{E}^{\mathbb{P}^{\varepsilon_n}} [\tau] \ge \mathbb{E}^{\mathbb{P}^{\varepsilon_n}} [\tau_k]$$

Taking $n \to \infty$ we obtain:

$$\liminf_{n \to \infty} \mathbb{E}^{\mathbb{P}^{\varepsilon_n}} [\tau] \ge \limsup_{n \to \infty} \mathbb{E}^{\mathbb{P}^{\varepsilon_n}} [\tau_k] = \mathbb{E}^{\mathbb{P}} [\tau_k] .$$

We conclude by taking the limit for $k \to \infty$ and by applying the Monotone Convergence Theorem. □

Here we present the main idea of [1], which allows us to explicitely compute some quantities related to τ giving lots of information about the limit points of $\{\mathbb{P}^\varepsilon\}_{\varepsilon\in(0,1)}$ for $\varepsilon \to 0$. Let $r > 0$ be as in section 2 and consider the boundary value problem:

$$\begin{cases} \frac{\varepsilon^2}{2}\phi''_\varepsilon(x) + b(x)\phi'_\varepsilon(x) &= -1, \quad x \in (-r, r) \\ \phi_\varepsilon(-r) = \phi_\varepsilon(r) &= 0 \end{cases} \tag{4}$$

Let ϕ_ε be a C^2 solution to (4). Let $X^{(\varepsilon)}$ on the space $(E, \mathcal{E}, \mathbb{Q})$ be the solution to (1) and define $\tau^\varepsilon := \inf\{t \geq 0 : |X_t^{(\varepsilon)}| \geq r\}$. Itô Formula applied to the semimartingale $X^{(\varepsilon)}$ between time 0 and time $t \wedge \tau^\varepsilon$ gives:

$$\begin{aligned} \phi_\varepsilon(X_{t\wedge\tau^\varepsilon}^{(\varepsilon)}) = \phi_\varepsilon(X_0^{(\varepsilon)}) \; &+ \int_0^{t\wedge\tau^\varepsilon} \left(b(X_s^{(\varepsilon)})\phi'_\varepsilon(X_s^{(\varepsilon)}) + \frac{\varepsilon^2}{2}\phi''_\varepsilon(X_s^{(\varepsilon)}) \right) ds \\ &+ \int_0^{t\wedge\tau^\varepsilon} \varepsilon\phi'_\varepsilon(X_s^{(\varepsilon)})\, dB_s. \end{aligned}$$

Using that ϕ_ε solves (4) and taking expectation with respect to the probability $\mathbb{Q}$ we get:

$$\mathbb{E}^\mathbb{Q}\left[\phi_\varepsilon(X_{t\wedge\tau^\varepsilon}^{(\varepsilon)})\right] = \phi_\varepsilon(0) - \mathbb{E}^\mathbb{Q}\left[t \wedge \tau^\varepsilon\right].$$

Being ϕ_ε bounded, the preceding equation implies $\mathbb{Q}\{\tau^\varepsilon = +\infty\} = 0$. Hence taking $t \to \infty$ we obtain that the LHS tends to zero by Dominated Convergence, while the RHS tends to $\phi_\varepsilon(0) - \mathbb{E}^\mathbb{Q}[\tau^\varepsilon]$ by Monotone Convergence. Thus we have $\phi_\varepsilon(0) = \mathbb{E}^\mathbb{Q}[\tau^\varepsilon]$ and since the law of $X^{(\varepsilon)}$ is $\mathbb{P}^\varepsilon$ we get the fundamental identity:

$$\phi_\varepsilon(0) = \mathbb{E}^{\mathbb{P}^\varepsilon}[\tau].$$

Now observe that the general solution to (4) is $\phi_\varepsilon(x) = c_1 + c_2 A_\varepsilon(x) - B_\varepsilon(x)$, where:

$$A_\varepsilon(x) = \int_0^x \exp\left(-\frac{2}{\varepsilon^2}\int_0^y b(u)\, du\right) dy,$$

$$B_\varepsilon(x) = \frac{2}{\varepsilon^2}\int_0^x \int_y^x \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\, dz\right) du\, dy,$$

and the constants c_1 and c_2 are given by:

$$c_1 = c_1(\varepsilon) = B_\varepsilon(r)\frac{-A_\varepsilon(-r)}{A_\varepsilon(r) - A_\varepsilon(-r)} + B_\varepsilon(-r)\frac{A_\varepsilon(r)}{A_\varepsilon(r) - A_\varepsilon(-r)},$$

$$c_2 = c_2(\varepsilon) = \frac{B_\varepsilon(r) - B_\varepsilon(-r)}{A_\varepsilon(r) - A_\varepsilon(-r)}.$$

Being $A_\varepsilon(0) = B_\varepsilon(0) = 0$ we deduce that $\mathbb{E}^{\mathbb{P}^\varepsilon}[\tau] = c_1(\varepsilon)$. In a similar fashion, if we consider instead the problem:

$$\begin{cases} \frac{\varepsilon^2}{2}\phi_\varepsilon''(x) + b(x)\phi_\varepsilon'(x) & = 0, \quad x \in (-r, r) \\ \phi_\varepsilon(-r) & = 0 \\ \phi_\varepsilon(r) & = 1 \end{cases}$$

we obtain:

$$\mathbb{P}^\varepsilon\Big\{\, \omega \in \Omega : \tau(\omega) < +\infty,\, \omega_\tau = r \,\Big\} = \frac{-A_\varepsilon(-r)}{A_\varepsilon(r) - A_\varepsilon(-r)},$$

and in the same way, interchanging boundary conditions:

$$\mathbb{P}^\varepsilon\Big\{\, \omega \in \Omega : \tau(\omega) < +\infty,\, \omega_\tau = -r \,\Big\} = \frac{A_\varepsilon(r)}{A_\varepsilon(r) - A_\varepsilon(-r)}.$$

We conclude this section with a proposition which allows to transfer information about $\mathbb{P}^\varepsilon$ into information about any weak limit of $\mathbb{P}^\varepsilon$ for $\varepsilon \to 0$.

Proposition 6. Let $\varepsilon_n \to 0$ as $n \to \infty$ and suppose that $\mathbb{P}^{\varepsilon_n} \to \mathbb{P}$ weakly; then:

$$\liminf_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\} \geq \mathbb{P}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\},$$
$$\liminf_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = -r \,\Big\} \geq \mathbb{P}\Big\{\, \tau < +\infty,\, \omega_\tau = -r \,\Big\}.$$

If moreover $\mathbb{P}\{\tau = +\infty\} = 0$ one can replace the inequalities by equalities and $\liminf$ by $\lim$.

Proof. Using the explicit descritption of section 2, it can be verified that $1_{\{\tau<+\infty,\omega_\tau=r\}}$ and $1_{\{\tau<+\infty,\omega_\tau=-r\}}$ are lower semicontinuous at every point $\omega \in \Omega$ solution to (2), then $\mathbb{P}$-a.s. and the same proof of proposition 5 applies. If moreover $\mathbb{P}\{\tau = +\infty\} = 0$ then $1_{\{\tau<+\infty,\omega_\tau=r\}} + 1_{\{\tau<+\infty,\omega_\tau=-r\}} = 1$ $\mathbb{P}$-a.s. and thus:

$$\begin{aligned} \mathbb{P}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\} &\leq \liminf_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\} \\ &\leq \limsup_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\} \\ &= 1 - \liminf_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = -r \,\Big\} \\ &\leq 1 - \mathbb{P}\Big\{\, \tau < +\infty,\, \omega_\tau = -r \,\Big\} \\ &= \mathbb{P}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\}. \end{aligned}$$

and the thesis follows from the equality:

$$\liminf_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\} = \limsup_{n\to\infty} \mathbb{P}^{\varepsilon_n}\Big\{\, \tau < +\infty,\, \omega_\tau = r \,\Big\}.$$

□

4 Main theorem

In this section the main theorem of [1] is presented. It deals with the case 3 of proposition 3, where a twofold family of solutions to (2) exists. This analysis relies on the description of solutions to (2) given in section 2 and estimates on the expected value of τ following by the theory developed in section 3. We will see how in the zero-noise limit the upper extremal solution and the lower extremal solution to (2) are selected, due to their minimality property. Explicit weights will be identified, depending on the behaviour of $\mathbb{P}^\varepsilon\{\,\tau < +\infty, \omega_\tau = r\,\}$ as $\varepsilon \to 0$. We begin with the following:

Lemma 7. Suppose that $b(x) > 0$ for $x \in (0, r)$ (respectively $b(x) < 0$ for $x \in (-r, 0)$) and that for some $\delta > 0$ the function

$$h(x) = \min_{y\in[x,x+\delta]} b(y), \qquad \left(\text{resp.}\, k(x) = \max_{y\in[x-\delta,x]} b(y)\right)$$

is such that

$$\int_0^r \frac{dx}{h(x)} < \infty, \qquad \left(\text{resp.} \int_0^{-r} \frac{dx}{k(x)} < \infty\right).$$

Then

$$\lim_{\varepsilon\to 0} B_\varepsilon(r) = \int_0^r \frac{dx}{b(x)} \qquad \left(\text{resp.} \lim_{\varepsilon\to 0} B_\varepsilon(-r) = \int_0^{-r} \frac{dx}{b(x)}\right).$$

Proof. We only consider the case $b(x) > 0$ for $x \in (0, r)$, being the proof identical in the other one. First of all, note that if the hypothesis of the theorem hold with $\delta = \delta_0$, then they also hold with every positive $\delta < \delta_0$. We want to study the behaviour as $\varepsilon \to 0$ of

$$\begin{aligned}
B_\varepsilon(r) &= \frac{2}{\varepsilon^2}\int_0^r \int_y^r \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du\,dy \\
&= \frac{2}{\varepsilon^2}\int_0^r \int_y^{y+\delta} \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du\,dy \\
&\quad + \frac{2}{\varepsilon^2}\int_0^r \int_{y+\delta}^r \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du\,dy
\end{aligned}$$

The latter integral tends to zero as $\varepsilon \to 0$ because the integrand tends uniformly to zero; regarding the first one:

$$\begin{aligned}
&\int_0^r \int_y^{y+\delta} \frac{2}{\varepsilon^2} \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du \\
&= \int_0^r \int_y^{y+\delta} \frac{1}{b(u)}\frac{2b(u)}{\varepsilon^2} \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du.
\end{aligned}$$

Since

$$\int_0^r \int_y^{y+\delta} \frac{2b(u)}{\varepsilon^2} \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du$$
$$= \exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\Bigg|_{u=y+\delta}^{u=y} = 1 + o(1)_{\varepsilon\to 0},$$

we have the following chain of inequalities:

$$\begin{aligned}\left(\max_{u\in[y,y+\delta,]} b(u)\right)^{-1} &\le \liminf_{\varepsilon\to 0}\int_y^{y+\delta}\frac{2}{\varepsilon^2}\exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du\\ &\le \limsup_{\varepsilon\to 0}\int_y^{y+\delta}\frac{2}{\varepsilon^2}\exp\left(-\frac{2}{\varepsilon^2}\int_y^u b(z)\,dz\right)\,du\\ &\le \left(\min_{u\in[y,y+\delta,]} b(u)\right)^{-1}\end{aligned}$$

and the result follows by taking $\delta \to 0$ and by Dominated Convergence (here the hypothesis of integrability play a role in giving a domination for the integrand). □

Theorem 8. Suppose that $xb(x) \ge 0$ in $(-r, r)$ and $H(-r) < \infty, H(r) < \infty$; in addition, suppose also that it exists $\delta > 0$ such that the functions

$$h(x) = \min_{y\in[x,x+\delta]} b(y), \qquad k(x) = \max_{y\in[x-\delta,x]} b(y)$$

are such that

$$\int_0^r \frac{dx}{h(x)} < \infty, \qquad \int_0^{-r} \frac{dx}{k(x)} < \infty.$$

Then every limit value of $\mathbb{P}^\varepsilon$, $\varepsilon \to 0$, is concentrated on the upper extremal solution and the lower extremal solution to (2) for a small time interval.

Proof. Observe that by Lemma 7 $\mathbb{E}^{\mathbb{P}^{\varepsilon_n}}[\tau]$ is bounded and thus by proposition 5 we have $\mathbb{P}\{\tau = +\infty\} = 0$. Therefore by proposition 6:

$$\mathbb{P}\{\omega_\tau = r\} = \lim_{n\to\infty} \mathbb{P}^{\varepsilon_n}\{\omega_\tau = r\} = \lim_{n\to\infty} \frac{-A_{\varepsilon_n}(-r)}{A_{\varepsilon_n}(r) - A_{\varepsilon_n}(-r)} = p \in [0, 1].$$

Hence proposition 5 and Lemma 7 give:

$$\mathbb{E}^{\mathbb{P}}[\tau] \le pH(r) + (1-p)H(-r).$$

Now observe that minimality property of upper and lower extremal solutions implies that every probability $\mathbb{P}$ concentrated on the solutions of (2) and such that $\mathbb{P}\{\omega_\tau = r\} = p$ and $\mathbb{P}\{\omega_\tau = -r\} = 1-p$ must necessarily satisfy the opposite inequality, i.e.

$$\mathbb{E}^{\mathbb{P}}[\tau] \geq pH(r) + (1-p)H(-r).$$

The equality is possible only if, when restricted to a small time interval, the probability $\mathbb{P}$ gives mass p to the upper extremal solution and mass $1-p$ to the lower extremal solution, which proves our claim. $\square$

5 Some examples

In this section we are going to study some zero-noise limit as an application of the results developed so far. By Theorem 8 we will need to compute certain limits for $\varepsilon \to 0$ in order to completely characterize the probability for upper and lower extremal solution to be selected, so the following Lemma will be useful.

Lemma 9. Let B and G be strictly increasing functions on $[0, r]$ with $B(0) = G(0) = 0$ and $\lim_{x\to 0} B(x)/G(x) = 0$. Suppose that there exists a function h such that $\lim_{\delta\to 0} h(\delta) = 0$ and that for every $\delta > 0$ and for every x in a neighborhood of 0 (depending on δ) the inequality $\delta G(x) \leq G(h(\delta)x)$ holds. Then:

$$\lim_{\varepsilon\to 0} \frac{\int_0^r \exp\left(-\frac{2}{\varepsilon^2}B(x)\right)\, dx}{\int_0^r \exp\left(-\frac{2}{\varepsilon^2}G(x)\right)\, dx} = +\infty. \tag{5}$$

Proof. Since B and G are strictly increasing, by uniform convergence we have:

$$\lim_{\varepsilon\to 0} \frac{\int_0^r \exp\left(-\frac{2}{\varepsilon^2}B(x)\right)\, dx}{\int_0^r \exp\left(-\frac{2}{\varepsilon^2}G(x)\right)\, dx} = \lim_{\varepsilon\to 0} \frac{\int_0^{r'} \exp\left(-\frac{2}{\varepsilon^2}B(x)\right)\, dx}{\int_0^{r''} \exp\left(-\frac{2}{\varepsilon^2}G(x)\right)\, dx}$$

for any $r' < r$ and $r'' < r$. Fix now $\delta > 0$ and $r' < r$ such that for every $x < r'$ the inequality $B(x) \leq \delta G(x)$ holds (this is possible because the hypothesis on the limit of the ratio $B(x)/G(x)$), so that we get:

$$\int_0^{r'} \exp\left(-\frac{2}{\varepsilon^2}B(x)\right)\, dx \geq \int_0^{r'h(\delta)} \exp\left(-\frac{2}{\varepsilon^2}G(y)\right)\, \frac{dy}{h(\delta)},$$

by a change of variable argument. Thus the limit in (5) is greater than $1/h(\delta)$ for every $\delta > 0$ and is therefore equal to $+\infty$. $\square$

The same idea (and calculations) gives the following:

Lemma 10. Let B and G be strictly increasing functions on $[0, r]$ with $B(0) = G(0) = 0$ and $\lim_{x\to 0} B(x)/G(x) = M \in \mathbb{R}$. Suppose that there exists a continuous function h such that for every $K > 0$ there exists $\delta > 0$ and two functions k_1 and k_2, depending on K and δ, such that $\lim_{x\to 0} k_1(x) = \lim_{x\to 0} k_2(x) = 1$ and for every $x \in [0, \delta]$ the inequality $k_2(x)G(h(K)x) \leq KG(x) \leq k_1(x)G(h(K)x)$ holds. Then:

$$\lim_{\varepsilon\to 0} \frac{\int_0^r \exp\left(-\frac{2}{\varepsilon^2} B(x)\right) dx}{\int_0^r \exp\left(-\frac{2}{\varepsilon^2} G(x)\right) dx} = \frac{1}{h(M)}.$$

Now we have all the machinery and we are ready for some concrete example.

Example 1.

$$b(x) = \begin{cases} c_1 x^\alpha & \text{if } x \geq 0; \\ -c_2(-x)^\alpha & \text{if } x < 0, \end{cases}$$

with $c_1, c_2 > 0$ and $\alpha < 1$. By Theorem 8 in the zero-noise limit extremal solutions are selected, and weights are determined by the limit:

$$\lim_{\varepsilon\to 0} \frac{-A_\varepsilon(-r)}{A_\varepsilon(r) - A_\varepsilon(-r)} = \lim_{\varepsilon\to 0} \frac{\int_{-r}^0 \exp\left(-\frac{2}{\varepsilon^2}\int_0^y b(u)\,du\right) dy}{\int_{-r}^r \exp\left(-\frac{2}{\varepsilon^2}\int_0^y b(u)\,du\right) dy} = p.$$

Since:

$$\int_0^y b(u)\,du = \begin{cases} \frac{c_1}{1+\alpha} y^{1+\alpha} & \text{if } y \geq 0; \\ \frac{c_2}{1+\alpha}(-y)^{1+\alpha} & \text{if } y < 0, \end{cases}$$

we can rewrite p by a change of variable as:

$$\frac{1}{p} = 1 + \lim_{\varepsilon\to 0} \frac{\int_0^r \exp\left(-\frac{2}{\varepsilon^2}\frac{c_1}{1+\alpha} y^{1+\alpha}\right) dy}{\int_0^r \exp\left(-\frac{2}{\varepsilon^2}\frac{c_2}{1+\alpha} y^{1+\alpha}\right) dy}$$

and Lemma 10 finally gives that the probability p for the upper extremal solution to be selected in the zero-noise limit equals:

$$p = \frac{c_1^{\frac{1}{1+\alpha}}}{c_1^{\frac{1}{1+\alpha}} + c_2^{\frac{1}{1+\alpha}}}.$$

Example 2.

$$b(x) = \begin{cases} -x^{\frac{1}{2}} \log(x) & \text{if } x \geq 0; \\ (-x)^{\frac{1}{2}} \log(\sin x^2) & \text{if } x < 0, \end{cases}$$

Here the upper extremal solution is selected with probability:

$$p = \frac{1}{1 + 2^{\frac{2}{3}}}.$$

Example 3.

$$b(x) = \begin{cases} c_1 x^\alpha & \text{if } x \geq 0; \\ -c_2(-x)^\beta & \text{if } x < 0, \end{cases}$$

with $c_1, c_2 > 0$ and $\alpha < \beta < 1$. Here the same calculations and Lemma 9 give $p = 1$, regardless of the choice of coefficients c_1 and c_2.

Example 4.

$$b(x) = \begin{cases} x^\alpha & \text{if } x \geq 0; \\ 0 & \text{if } x < 0, \end{cases}$$

with $\alpha < 1$. Here Theorem 8 does not apply, but comparison criterion for SDEs gives that, when the noise ε is stricly positive, the unique solution to this equation is greater or equal to the unique solution to the equation with drift equal to (for instance):

$$\tilde{b}(x) = \begin{cases} x^\alpha & \text{if } x \geq 0; \\ -(-x)^{\frac{1+\alpha}{2}} & \text{if } x < 0, \end{cases}$$

hence necessary also in this case the upper extremal solution is selected in the zero-noise limit with probability $p = 1$.

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