An Introduction to Random Dynamical Systems for Climate

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0.1 Preface

These notes are the result of a Master thesis and a series of lectures given at Institute Henri Poincaré on the occasion of the intensive research period "The Mathematics of Climate and the Environment".

Chapter 1

Non-Autonomous Dynamical Systems

1.1 Introduction

In the following, to avoid misunderstandings, we have to always keep in mind the distinction between *climate* and *climate models*. When we say that climate is non-autonomous we do not criticize the several autonomous climate models, which may be perfectly justified by reasons of simplicity, idealization, interest in particular phenomena and so on.

Climate is non-autonomous. Solar radiation is the most obvious non-autonomous external input. It has a daily component of variability and an annual component. The annual component can be considered periodic; the daily one also if the random effect of clouds is not included in it, otherwise could be considered as a random rapidly varying input. Another fundamental non-autonomous input is produced by biological systems, in particular the CO2 emission by humans in the last 150 years. The latter input is non-periodic, it has an increasing trend with some small variations. Another non-autonomous input, this time at the level of models instead of climate itself, is the randomization of sub-grid processes, more difficult to explain in a few words but fundamental in modern numerical codes; this randomization has usually a stationary character. We have thus exemplified non-autonomous inputs of periodic, increasing and random type.

We thus have, in general, a multidimensional PDE system of the form

$$\partial_t u = Au + B\left(u\right) + C\left(u, q\left(t\right)\right) + D\left(u\right)\dot{\omega}\left(t\right)$$

(other generic descriptions are equally meaningful at this informal level). We collected in $u\left(t\right)$ the state variables (wind velocity, temperature etc.) and in $q\left(t\right), \dot{\omega}\left(t\right)$ the time-varying inputs, separting them, for conceptual convenience, in deterministic slowly varying terms $q\left(t\right)$ (anthropogenic factors, annual variation of solar radiation) and stochastic fast varying terms $\dot{\omega}\left(t\right)$ assumed to be a - possibly state dependent - white noise in time (delay terms, relevant, will be assumed of the form that allows to write the previous system enlarging the set of variables $u(t)$).

It is then clear that we shall spend considerable effort to keep into account some degree of non-autonomy of the dynamics. We shall discuss different levels, concentrating on the one typical of autonomous Random Dynamical Systems (RDS) for the development of rigorous results, but with the hope that this more general presentation and insistence on the role of non-autonomous inputs will trigger more research on non-autonomous RDS.

1.2 Non autonomous dynamical systems: definition and examples

A relatively general mathematical scheme is made of a metric space (X, d) and a family of maps

$$U(s,t) : X \to X, \qquad s \leq t$$

(to fix the ideas we assume for simplicity that time parameters s, t vary in $\mathbb{R}$, but different time sets can be considered), with the two properties $U(s,s) = Id$ and $U(r,t) \circ U(s,r) = U(s,t)$, $r \leq s \leq t$ and various possible regularity properties, like $U(s,t)$ continuous in X, for every $s \leq t$.We shall always assume continuity. Thus let us formalize as a definition:

Definition 1 *A (forward in time) Non Autonomous Dynamical System (NADS) (also called an evolution operator) is a family of continuous maps indexed by two times*

$$U(s,t) : X \to X \qquad s \leq t$$

with the rules

$$\begin{aligned} U(s,s) &= Id \qquad \text{for all } s \\ U(r,t) \circ U(s,r) &= U(s,t) \qquad \text{for all } s \leq r \leq t. \end{aligned}$$

Remark 2 *Sometimes the definition of $U(s,t)$ and the rules are required also for $s > t$; in that case we have a NADS with forward and backward properties. This is the case of finite dimensional systems and usually of hyperbolic problems. Since, however, we shall deal most often with parabolic problems, we restrict the exposition to the forward-in-time case.*

Remark 3 *The case, for $T = [0, \infty)$, of a family depending on a single time parameter*

$$\mathcal{U}(t) : X \to X \qquad t \geq 0$$

with the rules

$$\begin{aligned} \mathcal{U}(0) &= Id \\ \mathcal{U}(t) \circ \mathcal{U}(s) &= \mathcal{U}(t+s) \qquad \text{for all } s, t \geq 0 \end{aligned}$$

is a particular case, called simply a Dynamical System (DS). The correspondence with the more general concept is

$$U(s,t) = \mathcal{U}(t-s)$$

(check that the properties of $U(s,t)$ are satisfied).

Example 4 *Let $b : T \times \mathbb{R}^d \to \mathbb{R}^d$ continuous and satisfying*

$$|b(t,x) - b(t,y)| \leq L|x-y| \qquad \text{for all } t \in T \text{ and } x, y \in \mathbb{R}^d.$$

For every $s \in T$, the Cauchy problem

$$\begin{aligned} X'(t) &= b(t, X(t)) \qquad t \geq s \\ X(s) &= x \end{aligned}$$

has a unique global solution of class $C^1(T;\mathbb{R}^d)$ (classical Cauchy-Lipschitz theorem). Denote it by $X^{s,x}(t)$. The map

$$x \mapsto X^{s,x}(t)$$

is Lipschitz continuous (from Gronwall lemma). The family of maps

$$U(s,t) : \mathbb{R}^d \to \mathbb{R}^d \qquad s \leq t$$

defined as

$$U(s,t)(x) = X^{s,x}(t)$$

is a Lipschitz continuous NADS. Verification is a particular case of computations similar to those of Proposition 42 below.

Example 5 *Consider the stochastic equation in $\mathbb{R}^d$*

$$dX_t = b(X_t)\,dt + dW_t$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and W_t is a Brownian motion (BM) in $\mathbb{R}^d$. Let $C_0(\mathbb{R};\mathbb{R}^d)$ be the space of continuous functions null at $t = 0$. Let us first define the two-sided Wiener measure. Take, on some probability space, two independent copies of the BM W, say $W_t^{(i)}$, $i = 1,2$; define the two-sided BM:

$$W_t = W_t^{(1)} \text{ for } t \geq 0,\ W_t = W_{-t}^{(2)} \text{ for } t \leq 0;$$

and call $\mathbb{P}$ its law, on Borel sets of $C_0(\mathbb{R};\mathbb{R}^d)$; this is the two-sided Wiener measure.

Now consider the canonical space $\Omega = C_0(\mathbb{R};\mathbb{R}^d)$ with Borel σ-field $\mathcal{F}$ and two-sided Wiener measure $\mathbb{P}$. On $(\Omega, \mathcal{F}, \mathbb{P})$ consider the canonical two-sided BM defined as

$$W_t(\omega) = \omega(t) \qquad \omega \in C_0\left(\mathbb{R};\mathbb{R}^d\right).$$

Let us now interpret the stochastic equation in integral form on the canonical space:

$$X_t(\omega) = x + \int_s^t b(X_r(\omega))\, dr + \omega(t) - \omega(s).$$

Given $\omega \in C_0(\mathbb{R}; \mathbb{R}^d)$, we may solve uniquely this equation by the contraction principle, first locally but then with global prolungation step by step, and construct a solution $X_t^{s,x}(\omega)$ defined for all t, continuously dependent (in a Lipschitz way) on x. Again we set

$$U(s,t)(x) = X_t^{s,x}(\omega).$$

In this example the dynamical system is parametrized by ω, feature that we shall discuss more extensively in later sections.

Example 6 *On the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (just for simplicity) consider the Navier-Stokes equations*

$$\partial_t u(t,x) + u(t,x) \cdot \nabla_x u(t,x) + \nabla_x p(t,x) = \nu \Delta_x u(t,x) + f(t,x) \qquad (t,x) \in T \times \mathbb{T}^2$$
$$\operatorname{div}_x u(t,x) = 0 \qquad (t,x) \in T \times \mathbb{T}^2$$

with periodic boundary conditions, zero average, and the "initial" condition

$$u(s,x) = \varphi(x).$$

Let H be the Hilbert space obtained as the closure in the L^2-topology of C^∞ periodic zero average divergence free vector fields on $\mathbb{T}^2$. If

$$f \in L^2_{loc}(T; H) \qquad \varphi \in H$$

then there exists a unique weak solution, denoted by $u^{s,\varphi}(t,x)$, with the property $u^{s,\varphi}(t,\cdot) \in H$ and several others. The concept of weak solution of the 2D Navier-Stokes equations will be explained later. The family of maps

$$U(s,t) : H \to H \qquad s \leq t$$

defined as

$$U(s,t)(\varphi) = u^{s,\varphi}(t,\cdot)$$

is a Lipschitz continuous NADS.

Example 7 *On a smooth bounded open domain $D \in \mathbb{R}^3$ consider the diffusion equation*

$$\partial_t \theta(t,x) + u(t,x) \cdot \nabla_x \theta(t,x) = \kappa \Delta_x \theta(t,x) + f(\theta(t,x)) + Q(t,x) \qquad (t,x) \in T \times D$$

where u is a velocity field, for instance solution of equations of Navier-Stokes type, f is a nonlinear function, for instance

$$f(\theta) = -C\theta^4$$

and $Q(t,x)$ is a time-dependent heat exchange, e.g. corresponding to solar radiation. Also this equation generates a NADS in suitable function spaces. More details a due time.

Can we introduce interesting dynamical concepts for such general objects? We discuss (more or less everywhere in these lectures) mainly two concepts: attractors and measures with suitable invariance properties. Behind these concepts there is *invariance.*

1.3 Invariance. Criticalities of the concept for NADS

In the autonomous case we say that a set $A \subset X$ is invariant if

$$\varphi_t (A) = A \qquad \text{for all } t$$

(also the simpler property of positive invariance, $\varphi_t (A) \subset A$, sometimes is already quite useful). To avoid trivialities (like $\varphi_t (X) = X$), one usually add the requirement of boundedness or compactness of invariance sets. This identifies a notion, for instance compact invariant set, which is already quite useful and interesting. A probability measure μ is invariant if

$$(\varphi_t)_\sharp \mu = \mu \qquad \text{for all } t$$

where $\sharp$ stands for the push-forward ($(\varphi_t)_\sharp \mu$ is the probability measure defined by $\left((\varphi_t)_\sharp \mu\right)(B) = \mu\left(\varphi_t^{-1}(B)\right)$ for all Borel sets B). Being a probability measure implies it is almost supported on compact sets (in the sense of tightness), hence it incorporates a requirement similar to the boundedness or compacteness of invariant sets.

In the non-autonomous case, invariance can be defined (presumably) only in the following way (we add compactness to go faster to the main points):

Definition 8 *A family of sets $\{A_t \subset X, t \in \mathbb{R}\}$ is called a compact invariant set for $U(s,t)$ if A_t is compact for every t and*

$$U(s,t) A_s = A_t \qquad \textit{for all } s \leq t.$$

A family of Borel probability measures $\{\mu_t, t \in \mathbb{R}\}$ is invariant for $U(s,t)$ if

$$U(s,t)_\sharp \mu_s = \mu_t \qquad \textit{for all } s \leq t.$$

Apparently they look similar to the autonomous case but they are almost empty concepts, satisfied by trivial object of no long-time interest: assume the dynamics can be run both forward and backward globally in time - as it is for usual ordinary differential equations) and let

$$x(t), \qquad t \in \mathbb{R}$$

be a trajectory. It satisfies

$$U(s,t) x(s) = x(t)$$

and thus the singleton $\{x(t)\}$ is a compact invariant set; the same for any suitable bunches of trajectories. Let $\{\mu_t, t \in \mathbb{R}\}$ be simply defined by

$$\mu_t = \delta_{x(t)}.$$

It is an invariant measure. This genericity is very far from the specificity of invariant sets and measures of the autonomous case!

Example 9 *Consider the equation*

$$\begin{aligned} x'(t) &= -x(t) + f(t), \qquad t \geq s \text{ and } t < s \\ x(s) &= x_s^0 \end{aligned}$$

with a given initial condition x_s^0 and a continuous function f. Its solution is

$$\begin{aligned} x(t) &= e^{-(t-s)} x_s^0 + \int_s^t e^{-(t-r)} f(r)\, dr \\ &=: e^{-(t-s)} x_s^0 + x_s(t). \end{aligned}$$

This is an example of full-time trajectory, hence $\{x(t)\}$ is an invariant set, supporting a delta Dirac invariant measure $\mu_t = \delta_{x(t)}$; and this independently of the choice of s and x_s^0.

Example 10 *There is however a special case of the following example: the function*

$$x_{-\infty}(t) := \int_{-\infty}^t e^{-(t-r)} f(r)\, dr$$

when defined, for instance when f has at most polynomial growth at $-\infty$. This function satisfies the differential equation, the singleton $\{x_{-\infty}(t)\}$ is an invariant set but it has something special, intuitively speaking. For instance: assume $f = 1$ for simplicity; then the function $x_{-\infty}(t) = 1$, it is also bounded, but the other solutions above are all unbounded as $t \to -\infty$ except when $x_s^0 = 1$ (which is the case of $x_{-\infty}(t)$).

The way to focus on interesting objects is to require, as in the previous example, that they come from bounded elements at "time $-\infty$". One can formulate several conditions; let us focus first on the concept of global attractors.

Remark 11 *Also in the autonomous case it happens for many examples that invariant sets and invariant measures are non unique but this corresponds to interesting different long time objects of the dynamics. In the example above the non-uniqueness is simply related to different initial conditions, in spite of the fact that all trajectories tend to approach each other when time increases. Thus it is an artificial non-uniqueness due to a drawback of the concept.*

1.4 Global attractor and omega-limit sets

What does it mean that a set attracts others in the non-autonomous case?

Definition 12 *We say that $A(t)$ attracts B at time t if for every $\epsilon > 0$ there exists $s_0 < 0$ such that for all $s < s_0$ we have*

$$U(s,t)(B) \subset \mathcal{U}_\epsilon(A(t)).$$

We say that a family of sets $\{A(t), t \in \mathbb{R}\}$ attracts B if the set $A(t)$ attracts B at time t, for every $t \in \mathbb{R}$.

This definition can be formulated by means of the non-symmetric distance between sets. Given $A, B \subset X$ define

$$d(B,A) = \sup_{x \in B} d(x,A)$$
$$\text{where } d(x,A) = \inf_{y \in A} d(x,y).$$

Then $A(t)$ attracts B if

$$\lim_{s \to -\infty} d(U(s,t)(B), A(t)) = 0.$$

Definition 13 *We call pull-back omega-limit set of B at time t the set*

$$\begin{aligned}\Omega(B,t) &= \overline{\bigcap_{s_0 \geq 0} \bigcup_{s \leq s_0} U(s,t)(B)} \\ &= \{y \in X : \exists x_n \subset B, s_n \to -\infty, U(s_n,t)(x_n) \to y\}.\end{aligned}$$

Notice that obviously it can be an empty set.

Proposition 14 *$A(t)$ attracts B if and only if $\Omega(B,t) \subset A(t)$.*

Proof. Let us prove that if $A(t)$ attracts B then $\Omega(B,t) \subset A(t)$. Take $y \in \Omega(B,t)$ and $x_n \in B$, $s_n \to -\infty$ such that

$$U(s_n,t)(x_n) \to y.$$

We have $U(s_n,t)(x_n) \in \mathcal{U}_\epsilon(A(t))$ eventually, hence y is in the closure of $\mathcal{U}_\epsilon(A(t))$. By arbitrariety of ϵ, $y \in A(t)$.

Let us prove the converse statement by contradiction. Assuming that $A(t)$ does not attract B means that there exists $\epsilon > 0$ such that, for every $s_0 < 0$ there exists $s < s_0$ and $x \in B$ such that $U(s,t)(x) \notin \mathcal{U}_\epsilon(A(t))$. We can thus construct a sequence with this property so that a point $y \in \Omega(B,t)$ does not belong to $\mathcal{U}_\epsilon(A(t))$. This contradicts the assumption. ■

Below we give the definition of compact absorbing family; we anticipate here for compactness of exposition a criterium partially based on such a concept.

Proposition 15 *In general,*

$$U(s,t)\,\Omega(B,s) \subset \Omega(B,t).$$

If there is a compact absorbing family, then

$$U(s,t)\,\Omega(B,s) = \Omega(B,t).$$

Proof. Take $y \in \Omega(B,t)$ and $x_n \in B$, $s_n \to -\infty$ such that

$$U(s_n,s)(x_n) \to y.$$

Then

$$U(s_n,t)(x_n) = U(s,t)\,U(s_n,s)(x_n) \to U(s,t)\,y$$

namely $U(s,t)\,y \in \Omega(B,t)$, i.e. $U(s,t)\,\Omega(B,s) \subset \Omega(B,t)$. Conversely, take $z \in \Omega(B,t)$ and $x_n \in B$, $s_n \to -\infty$ such that

$$U(s_n,t)(x_n) \to z.$$

Then

$$U(s,t)\,U(s_n,s)(x_n) \to z.$$

The existence of a compact absorbing set implies that $U(s_n,s)(x_n)$ is included, eventually, in a compact set, hence there is a convergent subsequence $U(s_{n_k},s)(x_{n_k}) \to y$, hence $y \in \Omega(B,s)$ and $z = U(s,t)\,y$, therefore $\Omega(B,t) \subset U(s,t)\,\Omega(B,s)$. ■

Definition 16 *Given the NADS*

$$U(s,t) : X \to X \qquad s \le t, s,t \in \mathbb{R}$$

we say that a family if sets

$$A(t) \qquad t \in \mathbb{R}$$

is a pull-back global compact attractor (PBCGA) if:
i) $A(t)$ is compact for every $t \in \mathbb{R}$
ii) $A(\cdot)$ is invariant: $U(s,t)(A(s)) = A(t)$ for every $s \le t, s,t \in \mathbb{R}$
iii) $A(\cdot)$ pull-back attracts bounded sets:

$$\Omega(B,t) \subset A(t)$$

for all bounded set $B \subset X$.

Remark 17 *The climate we observe today is the result of a long time-evolution of some (unknown) initial condition. We thus observe configurations in the pull-back attractor. This is why the notion is at the core of conceptual climatology.*

Example 18 *Consider again Example 9, with f bounded (one can consider a more general case). The singleton $\{x_{-\infty}(t)\}$ is a compact global attractor, while each other invariant set of the form $\{x(t)\}$ (or compact unions of such sets not containing $\{x_{-\infty}(t)\}$, see Section 1.6) is not. What distinguishes the two cases is only property (iii). Take a bounded set B. Then*

$$U(s,t)(B) = \left\{ e^{-(t-s)} x_s^0 + x_s(t) ; x_s^0 \in B \right\}$$

where

$$x_s(t) = \int_s^t e^{-(t-r)} f(r)\, dr$$

We have

$$\begin{aligned} d(U(s,t)(B), \{x_{-\infty}(t)\}) &= \sup_{x_s^0 \in B} d\left(e^{-(t-s)} x_s^0 + x_s(t), x_{-\infty}(t)\right) \\ &\leq e^{-(t-s)} \sup_{x_s^0 \in B} \left|x_s^0\right| + |x_s(t) - x_{-\infty}(t)| \end{aligned}$$

which is easily seen to go to zero as $s \to -\infty$. On the contrary, taken $x_0^0 \neq x_{-\infty}(0)$, we have

$$\begin{aligned} d\left(U(s,t)(B), \left\{e^{-t} x_0^0 + x_0(t)\right\}\right) &= \sup_{x_s^0 \in B} d\left(e^{-(t-s)} x_s^0 + x_s(t), e^{-t} x_0^0 + x_0(t)\right) \\ &= \sup_{x_s^0 \in B} \left| e^{-(t-s)} x_s^0 + x_s(t) - e^{-t} x_0^0 - x_0(t) \right| \\ &= \sup_{x_s^0 \in B} \left| e^{-(t-s)} x_s^0 + (x_s(t) - x_{-\infty}(t)) - e^{-t}\left(x_0^0 - x_{-\infty}(0)\right) \right| \end{aligned}$$

where we have used

$$x_{-\infty}(t) = e^{-t} x_{-\infty}(0) + x_0(t).$$

The quantities $\sup_{x_s^0 \in B} \left|e^{-(t-s)} x_s^0\right|$ and $|x_s(t) - x_{-\infty}(t)|$ converge to zero as $s \to -\infty$ while the quantity $e^{-t}\left(x_0^0 - x_{-\infty}(0)\right)$ is not zero and independent of s. Hence the distance above does not tend to zero as $s \to -\infty$.

1.5 A criterium for existence of global attractor

Definition 19 *A family of sets $D(t)$, $t \in \mathbb{R}$ is called a bounded (resp. compact) pull-back absorbing family if:*

i) $D(t)$ is bounded (resp. compact) for every $t \in \mathbb{R}$

ii) for every $t \in \mathbb{R}$ and every bounded set $B \subset X$ there exists $t_B < t$ such that

$$U(s,t)(B) \subset D(t) \qquad \text{for every } s < t_B.$$

Theorem 20 *Let $U(s,t)$ be a continuous NADS. Assume that there exists a compact pull-back absorbing family. Then a PBCGA exists.*

In infinite dimensional examples of parabolic type the existence of a compact absorbing family is usually proved by means of the following lemma; hence what we usually apply in those examples is the next corollary.

Lemma 21 *Let $U(s,t)$ be a continuous NADS. Assume that:*

i) (compact NADS) for every t and bounded set $B \subset X$, the set $\overline{U(t,t+1)(B)}$ is compact

ii) there exists a pull-back absorbing family.

Then there exists a compact pull-back absorbing family.

Proof. Call $D(t)$ the (bounded) absorbing family. Set $D'(t) = \overline{U(t-1,t)(D(t-1))}$. It is easy to check that this is a compact absorbing family. ■

Remark 22 *In the compactness assumption for $U(t,t+1)$ the time lag 1 is obviously arbitrary. Over this remark and the peculiarities of certain examples, the literature developed a more general criterium based on asymptotic compactness.*

Corollary 23 *Let $U(s,t)$ be a continuous NADS. Assume that:*

i) (compact NADS) for every t and bounded set $B \subset X$, the set $\overline{U(t,t+1)(B)}$ is compact

ii) there exists a pull-back absorbing family.

Then a PBCGA exists.

Example 24 *We show by a simple example that the property of pull-back absorbing set can be verified in examples. Consider, over all $t \in \mathbb{R}$, the equation*

$$X'(t) = f(t)\,X(t) - X^3(t)$$

where $f(t)$ is a given bounded function. In the following arguments we can think that we work on a "pull-back" interval $[s,t]$ or equivalently on a standard forward interval $[0,T]$: the bounds are the same, depending only on $\|f\|_\infty$.

We prove now that a pull-back absorbing family exists, made of a single bounded set (hence backward frequently bounded); in finite dimensions the closure of a bounded set is compact, hence a PBCGA exists.

Let us investigate the time-evolution of the "energy" of a solution:

$$\frac{d}{dt}X^2 = 2XX' = -2f(t)\,X^2 - 2X^4 \leq 2\,\|f\|_\infty X^2 - 2X^4.$$

What counts is the strong dissipation for large values of $|X|$ provided by the term $-2X^4$; the term $2\|f\|_\infty X^2$ may complicate the dynamics in a bounded region around the origin but not "at infinity". One way to capture rigorously these features is to estimate

$$2\|f\|_\infty X^2 - 2X^4 \leq C - X^4$$

for some $C = C(\|f\|_\infty) > 0$, then reducing the inequality to

$$\frac{d}{dt}X^2 \leq C - X^4$$

(this step is not really necessary, one can work with the original inequality). Let $y(t)$ (for us $= X^2(t)$) be a non negative differentiable functions which satisfies, for every $t \geq 0$,

$$y'(t) \leq C - y^2(t).$$

A simple picture immediately clarifies the result. Since the function $y \mapsto C - y^2(t)$, for $y \geq 0$, is positive in the interval $\left(0, \sqrt{C}\right)$, negative for $y > \sqrt{C}$, the same happens to $y'(t)$. Therefore, if $y(0) \in \left[0, \sqrt{C}\right]$, we can show that $y(t) \in \left[0, \sqrt{C}\right]$ for every $t \geq 0$. If $y(0) > \sqrt{C}$, $y(t)$ decreases until $y(t) > \sqrt{C}$. More precisely, if $y(0) \in \left[0, \sqrt{C}+1\right]$, then $y(0) \in \left[0, \sqrt{C}+1\right]$ for every $t \geq 0$; if $y(0) > \sqrt{C}+1$, in a finite time depending on $y(0)$ the function $y(t)$ enters $\left[0, \sqrt{C}+1\right]$ and then (for what already said) it never leaves it. Translated to $X(t)$: the ball $B\left(0, \sqrt{C}+1\right)$ is an absorbing set.

Proof. (Proof of Theorem 20) Set

$$A(t) = \overline{\bigcup_{B \text{ bounded}} \Omega(B,t)}.$$

Let us prove it fulfills all properties of a PBGCA. By definition, $\Omega(B,t) \subset A(t)$, hence we have pull-back attraction.

From property (ii) of absorbing set,

$$\Omega(B,t) \subset D(t)$$

for every t and every bounded set B. In particular,

$$\overline{\bigcup_{B \text{ bounded}} \Omega(B,t)} \subset \overline{D(t)} = D(t).$$

Hence $A(t) \subset D(t)$ namely it is compact (being closed subset of a compact set).

Let us prove forward invariance. It will be an easy consequence of the forward invariance of omega-limit sets: $U(s,t)\,\Omega(B,s) \subset \Omega(B,t)$. It implies

$$\overline{\bigcup_{B \text{ bounded}} U(s,t)\,\Omega(B,s)} \subset \overline{\bigcup_{B \text{ bounded}} \Omega(B,t)} = A(t).$$

We always have the set-theoretical property

$$U(s,t)\left(\bigcup_{B \text{ bounded}} \Omega(B,s)\right) \subset \bigcup_{B \text{ bounded}} U(s,t)\,\Omega(B,s)$$

hence we have proved that

$$\overline{U(s,t)\left(\bigcup_{B \text{ bounded}} \Omega(B,s)\right)} \subset A(t).$$

But, by continuity of $U(s,t)$ and definition of $A(s)$ we have

$$U(s,t)\,A((s)) = U(s,t)\overline{\left(\bigcup_{B \text{ bounded}} \Omega(B,s)\right)} \subset \overline{U(s,t)\left(\bigcup_{B \text{ bounded}} \Omega(B,s)\right)}$$

which implies

$$U(s,t)\,A((s)) \subset A(t).$$

The opposite inclusion

$$A(t) \subset U(s,t)\,A((s))$$

is the most tricky step of the proof. Let $z \in A(t)$. There exists a sequence of bounded sets B_n and points $z_n \in \Omega(B_n,t)$ such that

$$z = \lim_{n\to\infty} z_n.$$

Each z_n is equal to

$$z_n = \lim_{k\to\infty} U(s_k^n,t)(x_k^n)$$

where $\lim_{k\to\infty} s_k^n = -\infty$ and $(x_k^n)_{k\in\mathbb{N}} \subset B_n$. Hence

$$z_n = \lim_{k\to\infty} U(s,t)\left(U(s_k^n,s)(x_k^n)\right) = \lim_{k\to\infty} U(s,t)(y_k^n)$$
$$y_k^n := U(s_k^n,s)(x_k^n).$$

The existence of a compact absorbing set $D(s)$ implies that there exists a subsequence $\left(y_{k_m^n}^n\right)_{m\in\mathbb{N}}$ with a limit

$$y^n = \lim_{m\to\infty} y_{k_m^n}^n$$
$$y^n \in D(s).$$

Then, by continuity of $U(s,t)$,

$$z_n = U(s,t)\, y^n.$$

Again by compactness of $D(s)$ there is a subsequence $(y^{n_j})_{j\in\mathbb{N}}$ with a limit $y \in D(s)$. By continuity of $U(s,t)$,

$$z = U(s,t)\, y.$$

It remains to check that $y \in A(s)$. Due to the closure in the definition of $A(s)$, it is sufficient to prove that $y^{n_j} \in A(s)$. Looking at the definition of $y^n_{k^n_m}$, we see that its limit (in m) y^n is in $\Omega(B_n, s)$, hence in $A(s)$. ■

1.6 On the uniqueness of the global attractor

The attraction property has a strong power of identification, with respect to the poor property of invariance. However, given a global attractor $A(t)$ we may always add to it trajectories $\{x(t)\}$ and still have all properties, because compactness and invariance are satisfied and attraction continue to hold when we enlarge the attracting sets. One way to escape this artificial non-uniqueness is by asking a property of minimality. We call it so because of the use in the literature of attractors, although minimal in set theory is a different notion.

Definition 25 *We say that a global compact attractor $A(t)$ is minimal if any other global compact attractor $A'(t)$ satisfies*

$$A(t) \subset A'(t)$$

for all $t \in \mathbb{R}$.

When it exists, the minimal global compact attractor is obviously unique.

Proposition 26 *Under the assumptions of Theorem 20, the attractor*

$$A(t) = \overline{\bigcup_{B \text{ bounded}} \Omega(B,t)}$$

is minimal.

Proof. If $A'(t)$ is a global attractor, it includes all omega-limit sets, hence it includes (being closed) $A(t)$. ■

The previous solution of the uniqueness problem is very simple and efficient under the assumptions of Theorem 20, which are those we verify in examples. However, at a more conceptual level one could ask whether there are cses when we can establish uniqueness from the definition itself. Indeed, in the autonomous case (see for instance the first chapter of [49]) it is known that the analogous concept is unique as an immediate consequence of the definition, because the attractor is itself a bounded set and the attraction property of bounded sets easily implies uniqueness.

We thus mimic the autonomous case by a definition and a simple criterium.

Definition 27 *A family of compact sets $A(t)$, $t \in \mathbb{R}$, is called backward frequently bounded if there is a bounded set B and a sequence $s_n \to -\infty$ such that $A(s_n) \subset B$ for every $n \in \mathbb{N}$.*

Proposition 28 *In the category of backward frequently bounded families, the PBCGA is unique, when it exists.*

Proof. Assume $A(\cdot)$ and $\widetilde{A}(\cdot)$ are two PBCGA, both backward frequently bounded and let B be a bounded set and $s_n \to -\infty$ be a sequence such that $A(s_n) \subset B$ for every $n \in \mathbb{N}$. Take $z \in A(t)$ and, thanks to property $U(s_n, t) A(s_n) = A(t)$, let $x_n \in A(s_n)$ be such that $U(s_n, t) x_n = z$. We have $x_n \in B$. Hence we have

$$z \in \{y \in X : \exists x_n \subset B, s_n \to -\infty, U(s_n, t)(x_n) \to y\}.$$

This proves

$$A(t) \subset \Omega(B, t).$$

But

$$\Omega(B, t) \subset \widetilde{A}(t)$$

hence

$$A(t) \subset \widetilde{A}(t).$$

The converse is also true, hence the two families coincide. ■

Example 29 *In Example 9 with f bounded, both $\{x_{-\infty}(t)\}$ and any compact invariant set containing $\{x_{-\infty}(t)\}$ is a compact global attractor. But only $\{x_{-\infty}(t)\}$ is backward frequently bounded.*

1.7 Invariant measures: attempts of qualifications to reduce non-uniqueness

For invariant measures it is more difficult to remedy the criticalities in the definition of invariance. Already in the case of autonomous systems, the global compact attractor is unique but invariant measures are not - without this fact being a pathology; for instance any system with multiple equilibria has a dirac Delta invariant measure for each equilibrium, plus their convex combinations. Therefore it is necessary to accept some degree of non-uniqueness, intrinsic of the dynamics; but we would like to eliminate the artificial non-uniqueness caused by the poor definition. How to separate these two different sources of non-uniqueness?

The idea, supported by physical meaning, is always looking for objects, at time t, which come from visible objects at time $-\infty$. Call $\mathcal{P}(X)$ the set of Borel probability measures on X.

Definition 30 *We say that a time-dependent probability measure μ_t, invariant for $U(s,t)$, is the time-development of $\rho \in \mathcal{P}(X)$ if*

$$\lim_{s\to-\infty} U(s,t)_\sharp \rho = \mu_t \qquad \text{for all } t \in \mathbb{R}.$$

We say it is the average-time-development if

$$\lim_{T\to+\infty} \frac{1}{T}\int_{t-T}^{t} U(s,t)_\sharp \rho ds = \mu_t \qquad \text{for all } t \in \mathbb{R}.$$

When one or the other of these concepts hold only up to a subsequence, we say that μ_t is a (average-) time-development up to a subsequence.

In the previous definitions the limits are understood in the weak sense: for every $\phi \in C_b(X)$ we have

$$\lim_{s\to-\infty} \int_X \phi(U(s,t)x)\rho(dx) = \int_X \phi(y)\mu_t(dy)$$

in the first case (time-development) and

$$\lim_{T\to+\infty} \frac{1}{T}\int_{t-T}^{t}\int_X \phi(U(s,t)x)\rho(dx)\,ds = \int_X \phi(y)\mu_t(dy)$$

in the second case (average-time-development); we have also used the theorem on transformation of integrals on the left-hand-side of these reformulations.

In a later chapter we shall also discuss the very important concept of *physical measure*, which relates to the concept of time-development of a measure ρ, but with a further specification about ρ. Time-developments and physical measures are the most important ones, those which really capture essential features of the dynamics.

Remark 31 *It is possible to formulate general existence theorems for invariant measures, based on the existence of compact global attractors (als in the non autonomous case). However, this is a very poor information opposite to attractor, because of the non-uniqueness just mentioned. Using the concept of compact absorbing set, there is a chance to prove a general theorem about existence of average-time-development measures up to subsequences, in the spirit of Krylov-Bogoliubov argument, for every a priori given $\rho \in \mathcal{P}(X)$. However, stronger results are missing in full generality on the NADS. The additional structure of RDS illustrated in the next sections will be fundamental to reach a first relevant result on time-development of suitable measures $\rho \in \mathcal{P}(X)$.*

1.8 Separation between non-autonomous input and autonomous dynamics: cocycles

In all examples of non-autonomous sources described above for climate - perhaps with the exception of certain sub-grid models, but this is not very clear - the non-autonomy is not in the structural aspect of the dynamics itself, but it is due to some external time-varying input. A way to account for this fact is to introduce a separate structure for the input and *a sort of* autonomous dynamical system for the dynamics.

The time-varying input is described either by a single function

$$\omega : \mathbb{R} \to Y$$

(Y is a set) and all its translations

$$\Omega = \{\theta_t \omega; t \in \mathbb{R}\}$$

where $\theta_t \omega : \mathbb{R} \to Y$ is defined as

$$(\theta_t \omega)(r) = \omega(t + r)$$

or more generally by a set Ω and a group of transformations $\theta_t : \Omega \to \Omega$,

$$\begin{aligned} \theta_0 &= Id \\ \theta_{t+s} &= \theta_t \circ \theta_s. \end{aligned}$$

The dynamics is described by a family of continuous maps $\varphi_t(\omega) : X \to X$ parametrized by elements $\omega \in \Omega$.

Definition 32 *A dynamical system over the structure $\left(\Omega, (\theta_t)_{t\in\mathbb{R}}\right)$ is a family of continuous maps*

$$\varphi_t(\omega) : X \to X$$

satisfying the cocycle properties

$$\begin{aligned} \varphi_{t+s}(\omega) &= \varphi_s(\theta_t \omega) \circ \varphi_t(\omega) \\ \varphi_0(\omega) &= Id \end{aligned}$$

for all values of the parameters.

When $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, θ_t are measurable transformations and $\mathbb{P}$ is invariant under θ_t (in such a case we say that $\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ is a metric dynamical system), and φ is a strongly measurable map, and the previous identities are asked to hold for $\mathbb{P}$-a.e. ω, we say that $\varphi_t(\omega)$ is a Random Dynamical System (RDS).

We have been intentionally vague in qualifying whether the identities hold $\mathbb{P}$-a.s. uniformly in the time parameters or not. The easy case for the development of the theory is when they hold uniformly, which is however more difficult to check in examples; many results may be extended to the case when the null sets may depend on parameters. We address Chapter 3 for precise results.

A basic question is whether the two schemes, NADS and cocycles, are equivalent. The answer is that cocycles are richer structures.

Proposition 33 *Given a dynamical system $\varphi_t(\omega)$ over the structure $\left(\Omega, (\theta_t)_{t\in\mathbb{R}}\right)$, given $\omega \in \Omega$, the family of maps*

$$U_\omega(s,t) : X \to X \qquad s \leq t$$

defined by

$$U_\omega(s,t) = \varphi_{t-s}(\theta_s\omega)$$

is a non-autonomous dynamical system.

Proof. Clearly continuity holds as well as $U_\omega(s,s) = Id$. For $s \leq r \leq t$ we have

$$\begin{aligned} U_\omega(r,t) \circ U_\omega(s,r) &= \varphi_{t-r}(\theta_r\omega) \circ \varphi_{r-s}(\theta_s\omega) \\ &= \varphi_{t-r}(\theta_{r-s}\theta_s\omega) \circ \varphi_{r-s}(\theta_s\omega) \\ &= \varphi_{t-s}(\theta_s\omega). \end{aligned}$$

■

Conversely, given a NADS, let us see whether we can define a cocycle. The definition of $\varphi_t(\omega)$ is obvious:

$$\varphi_t(\omega) = U_\omega(0,t).$$

However, the family $U_\omega(s,t)$ does not incorporate information about $(\theta_t)_{t\in\mathbb{R}}$. The group must be given; but then it is necessary to ask a compatibility rule with $U_\omega(s,t)$ to deduce the cocycle property.

Proposition 34 *Let $U_\omega(s,t)$ be a NADS parametrized by ω in a structure $\left(\Omega, (\theta_t)_{t\in\mathbb{R}}\right)$. Assume*

$$U_{\theta_t\omega}(0,s) = U_\omega(t,t+s).$$

Then $\varphi_t(\omega) := U_\omega(0,t)$ satisfies the cocycle property. The condition is also necessary; more generally it holds

$$U_{\theta_r\omega}(s,t) = U_\omega(s+r,t+r). \tag{1.1}$$

It follows from the identities

$$\begin{aligned} \varphi_s(\theta_t\omega) \circ \varphi_t(\omega) &= U_{\theta_t\omega}(0,s) \circ U_\omega(0,t) \\ \varphi_{t+s}(\omega) &= U_\omega(0,t+s). \end{aligned}$$

Formula (1.1) follows from

$$U_{\theta_r\omega}(s,t) = \varphi_{t-s}(\theta_s\theta_r\omega) = \varphi_{t+r-(s+r)}(\theta_{s+r}\omega) = U_\omega(s+r,t+r).$$

Therefore the cocycle structure is richer than NADS and may produce stronger results. We have to check it holds in examples and, as already said, it holds when the source of non-autonomy is external to an autonomous dynamics, as it is in all our examples.

Let us make two different examples of metric dynamical system $\left(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\right)$. In the next section we write a detailed example of random dynamical system $\varphi_t(\omega)$.

Example 35 *On the canonical two-sided Wiener space defined in Example 5 consider the "shift"*

$$\theta_t(\omega) = \omega(t+\cdot) - \omega(t)$$

for every $\omega \in C_0\left(\mathbb{R};\mathbb{R}^d\right)$. By known properties of Brownian motion it follows $(\theta_t)_\sharp \mathbb{P} = \mathbb{P}$. Indeed, if $(X_t)_{t\in\mathbb{R}}$ is a two-sided BM with law $\mathbb{P}$, we have

$$\theta_t(X_\cdot)(r) = X_{t+r} - X_t$$

which is a new two-sided Brownian motion.

Example 36 *On $C\left(\mathbb{R};\mathbb{R}^d\right)$ consider the ordinary shift*

$$\theta_t(\omega) = \omega(t+\cdot).$$

Given a two-sided Brownian motion $(X_t)_{t\in\mathbb{R}}$ and two strictly positive numbers λ,σ, define $(Y_t)_{t\in\mathbb{R}}$ as

$$Y_t = \int_{-\infty}^t e^{-\lambda(t-s)}\sigma dX_s$$

which can be also defined by integration by parts, to avoid stochastic integrals. One can prove it is pathwise well defined and it is a stationary Gaussian process, solution of

$$dY_t = -\lambda Y_t dt + \sigma dX_t.$$

Call $\mathbb{P}$ its law; by stationarity, we have $(\theta_t)_\sharp \mathbb{P} = \mathbb{P}$ for every $t \in \mathbb{R}$.

1.9 Improvement of the invariance concept for cocycles

It is apparently a simple exercise to reformulate invariance and attraction in this new language. However, there is a surprise. Let us start with invariant set, parametrizing by ω: a family of sets $\{A_t(\omega) \subset X, t \in \mathbb{R}\}$ such that

$$U_\omega(s,t)A_s(\omega) = A_t(\omega) \qquad \text{for all } s \le t.$$

Therefore the reformulation is

$$\varphi_{t-s}(\theta_s\omega) A_s(\omega) = A_t(\omega) \qquad \text{for all } s \leq t.$$

namely, changing times,

$$\varphi_t(\theta_s\omega) A_s(\omega) = A_{t+s}(\omega) \qquad \text{for all } t \geq 0 \text{ and all } s;$$

in particular,

$$\varphi_t(\omega) A_0(\omega) = A_t(\omega) \qquad \text{for all } t \geq 0.$$

Until now nothing has changed. But now we can ask more: that

$$A_t(\omega) = A_0(\theta_t\omega) \tag{1.2}$$

in which case the condition becomes

$$\varphi_t(\omega) A_0(\omega) = A_0(\theta_t\omega) \qquad \text{for all } t \geq 0.$$

The new condition (1.2) is not automatically satisfied. It is a sort of θ_t-stationarity of the random set $A_0(\omega)$.

Definition 37 *Given a dynamical system $\varphi_t(\omega)$ over the structure $\left(\Omega, (\theta_t)_{t\in\mathbb{R}}\right)$, we say that a set $A(\omega)$ parametrized by $\omega \in \Omega$ is invariant if*

$$\varphi_t(\omega) A(\omega) = A(\theta_t\omega) \qquad \textit{for all } t \geq 0.$$

Moreover, we say that a Borel probability measure $\mu(\omega)$ parametrized by $\omega \in \Omega$ is invariant if

$$\varphi_t(\omega)_\sharp \mu(\omega) = \mu(\theta_t\omega) \qquad \textit{for all } t \geq 0.$$

Proposition 38 *If $A(\omega)$ (resp. $\mu(\omega)$) is invariant for $\varphi_t(\omega)$, then*

$$A_t(\omega) := A(\theta_t\omega)$$

(resp. $\mu_t(\omega) := \mu(\theta_t\omega)$) is invariant for $U_\omega(s,t)$.

Proof. We have, for all $s \leq t$,

$$\begin{aligned} U_\omega(s,t) A_s(\omega) &= \varphi_{t-s}(\theta_s\omega) A(\theta_s\omega) \\ &= A(\theta_{t-s}\theta_s\omega) \\ &= A_t(\omega). \end{aligned}$$

■

Example 39 *Consider the random dynamical system φ of example 57 below, associated to the stochastic equation*

$$dX_t = -\alpha X_t dt + \sigma dW_t.$$

For almost every ω, the integral

$$x_0(\omega) = -\int_{-\infty}^{0} \alpha e^{\alpha s} \sigma \omega(s) ds$$

is convergent, as a consequence of the fact that, for almost every ω, $\frac{W_t}{t}$ converges to 0 as t goes to infinity (see for example [32], problem 9.3). We show that $\{x_0(\omega)\}$ is an invariant set, and then the random measure $\delta_{x_0(\omega)}$ is invariant for φ. Comparing with example 18, we see that the new notion of invariance capture the same property of attractors, at least in this particular case. We have

$$\begin{aligned}
\varphi(t,\omega)x_0(\omega) &= e^{-\alpha t}x_0(\omega) + \sigma\omega(t) - \int_0^t \alpha e^{-\alpha(t-s)}\sigma\omega(s)ds \\
&= -\int_{-\infty}^{0} \alpha e^{-\alpha(t-s)}\sigma\omega(s)ds + \sigma\omega(t) - \int_0^t \alpha e^{-\alpha(t-s)}\sigma\omega(s)ds \\
&= -\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}\sigma\omega(s)ds + \sigma\omega(t) \\
&= -\int_{-\infty}^{0} \alpha e^{\alpha s}\sigma\omega(t+s)ds + \sigma\omega(t)\left(\int_{-\infty}^{0} \alpha e^{\alpha s}ds\right) \\
&= -\int_{-\infty}^{0} \alpha e^{\alpha s}\sigma(\omega(t+s) - \omega(t))ds = x_0(\theta_t\omega).
\end{aligned}$$

The compact global attractor constructed by Theorem 20 is invariant in this new sense, when the NADS comes from a cocycle.

Theorem 40 *Let $\varphi_t(\omega)$ be a cocycle over $\left(\Omega, (\theta_t)_{t\in\mathbb{R}}\right)$, and let*

$$U_\omega(s,t) = \varphi_{t-s}(\theta_s\omega)$$

be the associated NADS. Let $A_t(\omega)$ be the minimal global compact attractor associated to $U_\omega(s,t)$. Then it holds (1.2), which implies that $A_0(\omega)$ is invariant for the cocycle, in the sense of Definition **??**.

Proof. We prove the analog of (1.2) for the omega-limit sets:

$$\Omega_t(B,\omega) = \Omega_0(B,\theta_t\omega).$$

By definition of $A_t(\omega)$, it follows (1.2). The identity of omega-limit sets is due to the following fact. The set $\Omega_t(B,\omega)$ is made of points y such that there exist $x_n \in B$, $s_n \to -\infty$

such that $U_\omega(s_n, t)\, x_n \to y$. The set $\Omega_0(B, \theta_t\omega)$ is made of points y' such that there exist $x'_n \in B$, $s'_n \to -\infty$ such that $U_{\theta_t\omega}(s'_n, 0)\, x'_n \to y'$. But the latter formula, from (1.1), can be rewritten as $U_\omega(s'_n + t, t)\, x'_n \to y'$. Thus (renaming $s_n = s'_n + t$) we see that the two conditions are equivalent, they define the same set. ■

The theory in the case of RDS will be explicitly developed in Chapter 3. In the next section we describe a non-autonomous version. Since the examples we have in mind allow to use the description based on the group θ_t, we use it having verified that it augments the richness of the theory.

1.10 Non-autonomous RDS (naRDS)

An RDS as defined above is a non-autonomous system. However, we shall call it an autonomous RDS. The attribute of autonomous refers to the fact that φ_t is time-independent. For applications to climatology the most interesting scheme would be to put into ω the fast varying sources of time-variation, stationary or with stationary increments, but considering also slowly varying ones, which may be periodic (annual solar radiation) or have a trend (CO2). Thus we need a concept of *non-autonomous Random Dynamical Systems* (naRDS). The generic equation we have in mind has the form

$$\partial_t u = Au + B(u) + C(u, q(t)) + D(u)\,\dot{\omega}(t)$$

where q is the slowly varying term and $\dot{\omega}$ is white noise.

Definition 41 *A non-autonomous Random Dynamical System (naRDS) is made of two input structures and a cocycle over their product. Precisely, the two input structures are*

$$\left(\mathcal{Q}, (\vartheta_t)_{t\in\mathbb{R}}\right), \left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\right)$$

where $\mathcal{Q}$, Ω are sets, $(\vartheta_t)_{t\in\mathbb{R}}$ is a group of transformations of $\mathcal{Q}$, $\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ is a metric dynamical system, to which we associate the product space

$$\Gamma = \mathcal{Q} \times \Omega \qquad \gamma = (q, \omega) \in \mathcal{Y}$$

and the group of transformations

$$\Theta_t : \Gamma \to \Gamma \qquad \Theta_t(q, \omega) = (\vartheta_t q, \theta_t \omega).$$

The cocycle over $\left(\Gamma, (\Theta_t)_{t\in\mathbb{R}}\right)$ is a family of maps on a metric space (X, d)

$$\varphi_t(\gamma) : X \to X$$

such that

$$\varphi_0 = Id$$
$$\varphi_{t+s}(\gamma) = \varphi_s(\Theta_t\gamma) \circ \varphi_t(\gamma)$$

for all values of the parameters.

In the finite dimensional case, this is a quite generic structure. Using the theory of stochastic flows, it is not necessary to restrict to special noise. Let us however illustrate by a simple example with additive noise the definition, to put hands in a concrete example.

Proposition 42 *Let $b : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ continuous and satisfying*

$$|b(a,x) - b(a,y)| \leq L|x-y| \qquad \text{for all } a \in \mathbb{R}^m \text{ and } x, y \in \mathbb{R}^d.$$

Let $q \in C(\mathbb{R};\mathbb{R}^m)$. Consider the non-autonomous stochastic equation

$$dX_t = b(q(t), X(t))\,dt + dW_t$$

where W_t is a Brownian motion in $\mathbb{R}^d$. It generates a naRDS.

Proof. This equation can be easily solved by usual stochastic methods. For the purpose of introducing the associated naRDS we interpret this equation *pathwise*: for every $\omega \in C(\mathbb{R};\mathbb{R}^d)$ we consider the integral equation

$$x(t) = x_0 + \int_0^t b(q(s), x(s))\,ds + \omega(t).$$

By classical contraction principle one can easily prove it has a unique solution

$$x^{q,\omega,x_0} \in C\left(\mathbb{R};\mathbb{R}^d\right).$$

Then we set

$$\mathcal{Q} = C(\mathbb{R};\mathbb{R}^m) \qquad \vartheta_t q = q(t+\cdot)$$

$$\Omega = C\left(\mathbb{R};\mathbb{R}^d\right) \qquad \theta_t\omega = \omega(t+\cdot) - \omega(t)$$

$\mathcal{F}$ = Borel σ field of Ω, $\mathbb{P}$ = two-sided Wiener measure;

$$\varphi_t(q,\omega)(x_0) = x^{q,\omega,x_0}(t).$$

In other words, with the notation $\gamma = (q,\omega)$,

$$\varphi_t(\gamma)(x_0) = x_0 + \int_0^t b(q(s), \varphi_s(\gamma)(x_0))\,ds + \omega(t).$$

In order to say that this example satisfies the abstract properties of a naRDS we have to check a few conditions. Group properties of shifts are obvious; property $(\theta_t)_\sharp \mathbb{P} = \mathbb{P}$ was already discussed above. We have to prove the cocycle property

$$\varphi_{t+s}(\gamma) = \varphi_s(\Theta_t\gamma) \circ \varphi_t(\gamma).$$

We have

$$\begin{aligned}\varphi_{t+s}(\gamma)(x_0) &= x_0 + \int_0^{t+s} b(q(r), \varphi_r(\gamma)(x_0))\, dr + \omega(t+s) \\ &= x_0 + \int_0^{t} b(q(r), \varphi_r(\gamma)(x_0))\, dr + \omega(t) \\ &+ \int_t^{t+s} b(q(r), \varphi_r(\gamma)(x_0))\, dr + \omega(t+s) - \omega(t)\end{aligned}$$

$$\begin{aligned}&= \varphi_t(\gamma) + \int_t^{t+s} b(q(r), \varphi_r(\gamma)(x_0))\, dr + (\theta_t\omega)(s) \\ &\overset{r=t+u}{=} \varphi_t(\gamma) + \int_0^{s} b((\vartheta_t q)(u), \varphi_{t+u}(\gamma)(x_0))\, du + (\theta_t\omega)(s)\end{aligned}$$

hence we see that

$$z(s) := \varphi_{t+s}(\gamma)(x_0) \qquad s \geq 0$$

satisfies

$$z(s) = z_0 + \int_0^s b(q^*(u), z(u))\, du + \omega^*(s)$$

where $z_0 = \varphi_t(\gamma)$, $q^*(u) = (\vartheta_t q)(u)$, $\omega^*(s) = (\theta_t\omega)(s)$. By uniqueness of solutions to this equation,

$$z(s) = \varphi_s(q^*, \omega^*)(z_0).$$

Collecting all identities and definitions,

$$\begin{aligned}\varphi_{t+s}(\gamma)(x_0) &= \varphi_s(q^*, \omega^*)(z_0) \\ &= \varphi_s(\vartheta_t q, \theta_t\omega)(\varphi_t(\gamma))\end{aligned}$$

which means precisely $\varphi_{t+s}(\gamma) = \varphi_s(\Theta_t\gamma) \circ \varphi_t(\gamma)$. ■

Remark 43 *At the structural level (group and cocycle properties) there is nothing special in the additive noise. What is special is the possibility to solve, uniquely, the equation for all elements $\omega \in C(\mathbb{R}; \mathbb{R}^d)$ of the Wiener space, namely the pathwise solvability. Similar tricks hold only for few equations, like*

$$dX_t = b(q(t), X(t))\, dt + X_t dW_t.$$

However, in general there are two possibilities. One is the theory of rough paths, which at the price of some additional regularity of coefficients and a certain high level background in stochastic analysis, enables to solve "any" stochastic equation pathwise. One has to replace the Wiener space $C(\mathbb{R}; \mathbb{R}^d)$ by the more structured space of (geometric) rough paths. This *non trivial theory has the advantage to be fully pathwise and thus entirely analog to the previous additive noise example. The other possibility is to use the theory of stochastic flows. That theory claims that, given the stochastic-process solutions* $X_t^{x_0}(\omega)$*, which at time* t *are equivalence classes (hence not pointwise uniquely defined in* ω*), there is a version of the family such that we can talk of the space-time trajectories*

$$(t, x_0) \rightarrow X_t^{x_0}(\omega)$$

and, even more, we can do the same starting from an arbitrary time t_0*. Elaborating this concept with non trivial further elements, one can construct a naRDS (the technical difficulty is only in the stochastic part, not the non-autonomous one).*

Chapter 2

Generalities on RDS

2.1 Random dynamical systems

The theory of random dynamical systems develops with the introduction of probability instruments into classical deterministic models, to help representing uncertainties or summing up complexity. After the first work by Ulam and Neumann ([**?**]) in 1945 showing this need for intersection of probability theory and dynamical systems, the interest for the subject rises around the Eighties with the development of stochastic analysis and the discover of random and stochastic differential equations as sources for random transformations.

In this chapter we introduce the definition of random dynamical system (also abbreviated as RDS), and show how RDSs can be generated in continuous and discrete time. Then we present a finite dimensional model which exemplifies many properties of fluido-dynamical systems, and we outline the definition of the RDS associated to the stochastic Navier-Stokes equations. We end by giving the statement of the multiplicative ergodic theorem, which allows the definition of Lyapunov exponents for an RDS.

In all chapters we will write $\mathbb{T}$ for the set of *times*, referring to the groups $\mathbb{T} = \mathbb{R}$, $\mathbb{Z}$ (*two-sided time*), or to the semigroups $\mathbb{T} = \mathbb{R}^+$, $\mathbb{N}$ (*one-sided time*).

2.2 Definitions

The theory of random dynamical systems enlarges the classical theory of dynamical systems, investigating those situations in which a noise is supposed to perturb the system. The noise is formalized with a family of mappings $(\theta(t))_{t\in\mathbb{T}}$ of a measurable space $(\Omega, \mathcal{F})$ into itself, satisfying the following conditions:

(i) the map $(\omega, t) \mapsto \theta(t)\omega$ is measurable;

(ii) $\theta(0) = \mathrm{id}_\Omega$, if $0 \in \mathbb{T}$;

(iii) $\theta(s+t) = \theta(s) \circ \theta(t)$, for all s, $t \in \mathbb{T}$ (*semiflow property*).

In this case the triple $(\Omega, \mathcal{F}, (\theta(t))_{t\in\mathbb{T}})$ is called a *measurable dynamical system*. In addition, if P is a probability measure on $(\Omega, \mathcal{F})$ and if, for all t in $\mathbb{T}$, the map $\theta(t)$ preserves the measure P (i. e. $P(\theta(t)^{-1}(F)) = P(F)$ for all $F \in \mathcal{F}$), then $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$ is said to be a *metric dynamical system*.

Let us observe that condition (i) implies that, for each $t \in \mathbb{T}$, the map $\theta(t) : \Omega \mapsto \Omega$ is measurable. Moreover, if $\mathbb{T}$ is a group, from conditions (ii) and (iii) we gain that each $\theta(t)$ is invertible, and $\theta(t)^{-1} = \theta(-t)$.

In the sequel we will also write θ_t for $\theta(t)$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, i.e. a probability space such that if $F \in \mathcal{F}$, $P(F) = 0$ and $E \subset F$, then $E \in \mathcal{F}$.

Definition 44 *A* two-parameter filtration *is a two-parameter family* $\{\mathcal{F}_s^t\}_{\substack{s,t\in\mathbb{R},\\ s\leq t}}$ *of sub-σ-algebra of $\mathcal{F}$, such that*

(i) $\mathcal{F}_u^v \subset \mathcal{F}_s^t$, for $s \leq u \leq v \leq t$;

(ii) $\mathcal{F}_s^{t+} := \bigcap_{u>t} \mathcal{F}_s^u = \mathcal{F}_s^t$ and $\mathcal{F}_{s-}^t := \bigcap_{u<s} \mathcal{F}_u^t = \mathcal{F}_s^t$, for $s \leq t$;

(iii) $\mathcal{F}_s^t$ contains all P-null sets of $\mathcal{F}$, for $s \leq t$.

Define

$$\mathcal{F}_{-\infty}^t := \bigvee_{s\leq t} \mathcal{F}_s^t, \quad \mathcal{F}_s^\infty := \bigvee_{t\geq s} \mathcal{F}_s^t.$$

If $\{\mathcal{F}_s^t\}_{s\leq t\in\mathbb{R}}$ is a two-parameter filtration on $(\Omega, \mathcal{F}, P)$, a metric dynamical system θ_t on $(\Omega, \mathcal{F}, P)$ is a *filtered* dynamical system if it satisfies

$$\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}, \quad \text{for all} \quad u \in \mathbb{R}. \tag{2.1}$$

The definition of random dynamical system requires a modification of the semigroup property, to make it account for the noise perturbation:

Definition 45 *A* random dynamical system *on a measurable space $(X, \mathcal{B})$, over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$ is a measurable mapping*

$$\varphi : (\mathbb{T} \times \Omega \times X, \mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}) \to (X, \mathcal{B}),$$

satisfying the cocycle property*:*

$$\varphi(0, \omega, \cdot) = id_X \quad \textit{for all} \quad \omega \in \Omega \quad (\textit{if } 0 \in \mathbb{T}), \tag{2.2}$$

$$\varphi(t+s, \omega, \cdot) = \varphi(t, \theta(s)\omega, \varphi(s, \omega, \cdot)) \quad \textit{for all} \quad s, t \in \mathbb{T}, \quad \omega \in \Omega. \tag{2.3}$$

One can say that while a point $x \in X$ is moved by the map $\varphi(s, \omega, \cdot)$, ω is also shifted for a time s by θ, and $\theta(s)\omega$ is the new noise acting on $\varphi(s, \omega, x)$.

The map φ of the definition is also called a *perfect* cocycle, while, if condition (2.3) is weakened to hold for P-almost every ω, for fixed s and all t in $\mathbb{T}$, then φ is said to be a *crude* cocycle, and a *very crude* cocycle in case (2.3) holds for each fixed s and t.

We shall use the notation $\varphi(t, \omega)$ for the map $\varphi(t, \omega, \cdot) : X \to X$.

Remark 46 *Given a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, we will sometimes consider random dynamical systems which are defined for positive times only, being all theory extendible to this case.*

Remark 47 *If $\mathbb{T}$ is a group, by taking $t = -s$ in (2.3) and using (2.2), we find*

$$id_X = \varphi(-s, \theta(s)\omega) \circ \varphi(s, \omega) \quad \textit{for all } \ \omega, s,$$

while substituting $-t$ for s and $\theta(t)\omega$ for ω in (2.3) one gains

$$id_X = \varphi(t, \omega) \circ \varphi(-t, \theta(t)\omega) \quad \textit{for all } \ \omega, t.$$

This proves that the map $\varphi(t, \omega)$ is invertible and

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta(t)\omega) \quad \textit{for all } \ \omega, t.$$

If X is a topological space and, for each $\omega \in \Omega$, the mapping $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous, then the random dynamical system is called *continuous* or *topological*. If X is a finite dimensional manifold, then φ is a C^k *random dynamical system* if it is continuous and for each $(t, \omega) \in \mathbb{T} \times \Omega$ the map $x \mapsto \varphi(t, \omega)x$ is k-times differentiable with respect to x, and the derivatives are continuous in t and x.

Remark 48 *Given a probability space $(\Omega, \mathcal{F}, P)$, and a P-preserving bijective map $\theta : \Omega \to \Omega$, any family of continuous mappings $\psi(\omega) : X \to X$ such that $(\omega, x) \mapsto \psi(\omega)x$ is measurable generates a discrete random dynamical system $\varphi : \mathbb{N} \times \Omega \times X \to X$, by the formulas*

$$\varphi(0, \omega) = id_X, \quad \varphi(n, \omega) = \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega) \quad \textit{for } n \geq 1.$$

In this case, the cocycle property can be rewritten as

$$\varphi(n + 1, \omega)x = \varphi(n, \theta\omega)\psi(\omega)x.$$

If $\psi(\omega)x$ is invertible for all ω, and the map $(\omega, x) \mapsto \psi(\omega)^{-1}x$ is measurable, then the random dynamical system can be estended to negative times by

$$\varphi(n, \omega) = \psi(\theta^n\omega)^{-1} \circ \cdots \circ \psi(\theta^{-1}\omega)^{-1} \quad \textit{for } n \leq -1.$$

Conversely, each random dynamical system φ over $(\Omega, \mathcal{F}, P, \theta)$ with dicrete time $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{Z}$ is fully described by its time-one map $\psi(\omega) = \varphi(1, \omega) : X \to X$.

Suppose we are given a random dynamical system φ on a Polish space $(X, \mathcal{B})$, over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$. One can consider another measurable dynamical system, associated to the random dynamical system φ:

Definition 49 *The measurable map*

$$\begin{array}{rcl} \Theta : \mathbb{T} \times \Omega \times X & \rightarrow & \Omega \times X \\ (t, \omega, x) & \mapsto & (\theta(t)\omega, \varphi(t, \omega, x)), \end{array}$$

is called the skew product (flow) *of the metric dynamical system θ and the cocycle φ.*

The family of mappings $\Theta_t = \Theta(t, \cdot, \cdot)$, $t \in \mathbb{T}$, is a measurable dynamical system on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$. The semiflow property is an easy consequence of the flow and cocycle properties of θ and φ:

$$\begin{aligned} \Theta_0 &= \mathrm{id}_{\Omega\times X}, \\ \Theta_{t+s}(\omega, x) &= (\theta(t+s)\omega, \varphi(t+s, \omega)x) \\ &= (\theta(t) \circ \theta(s)\omega, \varphi(t, \theta(s)\omega)\varphi(s, \omega)x) = \Theta_t \circ \Theta_s(\omega, x), \end{aligned}$$

for all t, $s \in \mathbb{T}$, $\omega \in \omega$ and $x \in X$.

Deterministic dynamical systems are generated by ordinary differential equations; their role is played by *stochastic differential equations* in the case of random dynamical systems. An intermediate step is provided by considering *random differential equations*, i.e. families of ordinary differential equations with parameter ω, which can be solved "pathwise" for each fixed ω as a deterministic equation.

Example 50 *(RDS from Random Differential Equations)*

Let $\mathbb{T} = \mathbb{R}$, $X = \mathbb{R}^n$, and let $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$ be a metric dynamical system. Consider a measurable function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, and, for each fixed $\omega \in \Omega$, let the function $f_\omega : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given by $f_\omega(t, x) = f(\theta_t\omega, x)$. Suppose that f_ω is continuous, and locally Lipschitz in x. Then the solution of equation

$$\dot{x}_t = f(\theta_t\omega, x_t) \tag{2.4}$$

exists globally on $\mathbb{R}$ for each $\omega \in \Omega$ and is unique. The random dynamical system associated to the random differential equation (2.4) is given by

$$\varphi(t, \omega, x) = x + \int_0^t f(\theta_s\omega, \varphi(s, \omega, x))ds.$$

To prove the cocycle property, take $s, t \in \mathbb{R}$, and suppose that $s > 0$, $t > 0$ (the other cases are analogous). Write

$$\begin{aligned}\varphi(t, \theta(s)\omega, \varphi(s, \omega, x)) &= \varphi(s, \omega, x) + \int_0^t f(\theta_{u+s}\omega, \varphi(u, \theta(s)\omega, \varphi(s, \omega, x)))du \\ &= x + \int_0^s f(\theta_u\omega, \varphi(u, \omega, x))du \\ &+ \int_s^{t+s} f(\theta_z\omega, \varphi(z - s, \theta_s\omega, \varphi(s, \omega, x)))dz.\end{aligned}$$

This proves that the function

$$\psi(u, \omega, x) = \begin{cases} \varphi(u, \omega, x) & 0 \leq u \leq s, \\ \varphi(u - s, \theta_s\omega, \varphi(s, \omega, x)) & s \leq u \leq s + t, \end{cases}$$

satisfies

$$\psi(t + s, \omega, x) = x + \int_0^{t+s} f(\theta_u\omega, \psi(u, \omega, x))du,$$

and so by uniqueness

$$\varphi(t + s, \omega, x) = \psi(t + s, \omega, x) = \varphi(t, \theta(s)\omega, \varphi(s, \omega, x)).$$

Definition 51 *A mapping*

$$\begin{aligned}\phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \Omega &\longrightarrow \mathbb{R}^d \\ (s, t, x, \omega) &\mapsto \phi_{s,t}(x, \omega)\end{aligned}$$

is called a stochastic flow *if for P-almost every $\omega \in \Omega$ it satisfies:*

(i) $\phi_{s,s}(\omega) = Id_{\mathbb{R}^d}$ for all $s \in \mathbb{R}$;

(ii) $\phi_{s,t}(\omega) = \phi_{u,t}(\omega) \circ \phi_{s,u}(\omega)$ for all $s, t, u \in \mathbb{R}$.

If additionally $\phi_{s,t}(\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism (resp. is a k-times continuously differentiable homeomorphism) for all $s, t \in \mathbb{R}$ and P-almost every ω, then ϕ is called a *stochastic flow of homeomorphism* (resp. *of C^k-diffeomorphisms*).

2.3 RDS from Stochastic Differential Equations

An important class of random dynamical systems is generated by stochastic differential equations. A complex general result (see [3], chapter 2.3) asserts the existence of RDS associated to any stochastic differential equation involving stochastic integrals of Kunita-type, i.e. integrals which generalize classical Itô and Stratonovich integrals, introduced by Kunita in [**?**]. Here we state a particular case of this theorem involving classical Itô integrals only.

Consider the space $\Omega = \mathcal{C}_0(\mathbb{R}, \mathbb{R}^n)$ of continuous functions ω from $\mathbb{R}$ to $\mathbb{R}^n$, such that $\omega(0) = 0$. One can endow the space $\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$ with the topology given by the complete metric

$$d(\omega, \omega') = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{||\omega - \omega'||_k}{1 + ||\omega - \omega'||_k}, \quad ||\omega - \omega'||_k = \sup_{-k \le t \le k} |\omega(t) - \omega'(t)|,$$

and consider the induced topology on Ω. Denote by $\mathcal{F}^0$ the Borel σ-algebra on Ω.

Let $(\xi(t))_{t \in \mathbb{R}}$ be the coordinate process on Ω, i.e. the process given by $\xi(t, \omega) = \omega(t)$ for all $\omega \in \Omega$, $t \in \mathbb{R}$.

A continuous $\mathbb{R}^n$-valued process $(B_t)_{t \ge 0}$ is a Brownian motion on $\mathbb{R}^n$ if $B(0) = 0$, for all $0 \le s \le t$ the variable $B_t - B_s$ is normally distributed with mean 0 and covariance operator $(t - s)I_n$, and for all m-uples $0 \le t_1 \le \ldots \le t_m$ the random variables

$$B_{t_1}, \; B_{t_2} - B_{t_1}, \ldots, \; B_{t_m} - B_{t_{m-1}}$$

are independent ($(B_t)_{t \ge 0}$ has independent increments). A natural extension of Brownian motion to negative times is given by the following definition.

Definition 52 *A* two-sided Brownian motion $(W_t)_{t \in \mathbb{R}}$ *in* $\mathbb{R}^n$ *is a continuous process with independent increments, such that* $W_0 = 0$ *and, for each* s, $t \in \mathbb{R}$, *the increment* $W_t - W_s$ *is normally distributed with mean* 0 *and covariance matrix* $|t - s| I_n$.

If $(W_t)_{t \in \mathbb{R}}$ is a two-sided Brownian motion, we have in particular that the processes $(W_t)_{t \ge 0}$ and $(W_{-t})_{t \ge 0}$ are two independent Brownian motions on $\mathbb{R}^n$.

The measure P on $(\Omega, \mathcal{F})$ such that the coordinate process $(\xi(t))_{t \in \mathbb{R}}$ is a two-sided Brownian motion is called *(two-sided) Wiener measure*. The process $(\xi(t))_{t \in \mathbb{R}}$ is called *standard two-sided Brownian motion*. We denote by $\mathcal{F}$ the completion of the σ-algebra $\mathcal{F}^0$ with respect to P.

Define then the *shift operators* on Ω as

$$\theta_t(\omega)(s) = \omega(s + t) - \omega(t), \quad \text{for } s, t \in \mathbb{R}.$$

The family $(\theta_t)_{t \in \mathbb{R}}$ clearly satisfies the semiflow property. The map $(t, \omega) \mapsto \theta_t(\omega)$ is continuous, and then measurable.

The measure P is invariant with respect to the dynamical system $(\theta_t)_t$, as follows from the stationarity of the increments of the coordinate process: the law of the process $(\xi(t + s) - \xi(s))_{t \in \mathbb{R}}$ equals the law of $(\xi(t))_{t \in \mathbb{R}}$, and then

$$P(A) = P\{\omega \mid \omega. \in A\} = P\{\omega_{t+.} - \omega_t \in A\} = P\{\theta_t \in A\} \text{ for all } A \in \mathcal{F}^0.$$

Let $\mathcal{N}$ be the family of all subsets of P-null sets of $\mathcal{F}^0$:

$$\mathcal{N} = \{A \subset \Omega : \exists F \in \mathcal{F}^0 \text{ s.t. } A \subset F, P(F) = 0\}.$$

Define a two-parameter filtration $\{\mathcal{F}_s^t\}_{s\le t}$ as

$$\mathcal{F}_s^t = \sigma\{W_u - W_v : s \le v \le u \le t\} \bigvee \mathcal{N}, \quad \text{for all} \;\; s \le t.$$

Proposition 53 *For each $s \le t \in \mathbb{R}$ the following identities hold:*

$$\mathcal{F}_s^{t+} = \mathcal{F}_s^t = \mathcal{F}_s^{t-} := \sigma(\cup_{s\le u<t}\mathcal{F}_s^u),$$

$$\mathcal{F}_{s-}^t = \mathcal{F}_s^t = \mathcal{F}_{s+}^t := \sigma(\cup_{s<u\le t}\mathcal{F}_s^u).$$

Moreover, $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$, for all $u \in \mathbb{R}$.

Proof. Consider the processes $W_\cdot - W_s$, for fixed s, and $W_t - W_\cdot$, for fixed t. The equalities $\mathcal{F}_s^{t+} = \mathcal{F}_s^t$, $\mathcal{F}_s^t = \mathcal{F}_{s+}^t$ follow from the fact that for a d-dimensional strong Markov process with natural filtration $(\mathcal{F}_t)_t$, the augmented filtration $(\mathcal{F}_t^{\mathcal{N}})_t = (\mathcal{F}_t \vee \mathcal{N})_t$ is right-continuous ([32], proposition 2.7.7). The fact that the natural filtration of a d-dimensional adapted left-continuous process is left-continuous ([32], problem 2.7.1) implies that $\mathcal{F}_s^{t-} = \mathcal{F}_s^t$, and $\mathcal{F}_{s-}^t = \mathcal{F}_s^t$.

For the last sentence, take $u_1, ..., u_n, v_1, ..., v_n \in \mathbb{R}$ such that $s \le v_i \le u_i \le t$. Then for each $a \in \mathbb{R}$, $B_1, ..., B_n \in \mathcal{B}(\mathbb{R}^n)$, one has

$$\begin{aligned}
&\theta_a^{-1}\{\omega \in \Omega \mid \omega(u_1) - \omega(v_1) \in B_1, ..., \omega(u_n) - \omega(v_n) \in B_n\} \\
&= \{\theta_a^{-1}\omega \mid \omega(u_1) - \omega(v_1) \in B_1, ..., \omega(u_n) - \omega(v_n) \in B_n\} \\
&= \{\omega \mid \theta_a\omega(u_1) - \theta_a\omega(v_1) \in B_1, ..., \theta_a\omega(u_n) - \theta_a\omega(v_n) \in B_n\} \\
&= \{\omega \mid \omega(u_1 + a) - \omega(v_1 + a) \in B_1, ..., \omega(u_n + a) - \omega(v_n + a) \in B_n\} \in \mathcal{F}_{s+a}^{t+a}.
\end{aligned}$$

It is then clear that (2.1) holds. ∎

Hence $(\Omega, \mathcal{F}^0, \{\mathcal{F}_s^t\}_{s\le t}, (\theta(t))_{t\in\mathbb{R}}, P)$ is a filtered metric dynamical system. It is referred to as the *canonical metric dynamical system describing Brownian motion.*

Proposition 54 *The dynamical system $(\Omega, \mathcal{F}^0, (\theta(t))_{t\in\mathbb{R}}, P)$ is ergodic.*

Proof. It is a consequence of Kolmogorov's zero-one law. Consider the two-parameter filtration $\{\mathcal{F}_s^t\}_{s\le t}$ defined above, and define the σ-algebra

$$\mathcal{T}^\infty = \cap_{t\in\mathbb{R}}\mathcal{F}_t^\infty.$$

The independence of the σ-algebras $\mathcal{F}_s^u$ and $\mathcal{F}_t^z$ for all $s < u \le t < z$ allows to apply Kolmogorov's dichotomy and deduce that the σ-algebra $\mathcal{T}^\infty$ is degenerate, i.e. $P(A) \in \{0, 1\}$ for all $A \in \mathcal{T}^\infty$. The conclusion follows by observing that every $\theta(t)$-invariant set is contained in $\mathcal{T}^\infty$.

∎

To give the statement of the theorem, we have to introduce some spaces of functions on $\mathbb{R}^d$. For each $k \in \mathbb{Z}^+$, and $0 \leq \delta \leq 1$, we denote by $\mathcal{C}^{k,\delta}$ the space of functions $f : \mathbb{R}^d \to \mathbb{R}^d$ which are k times continuously differentiable and whose k-th derivative is locally δ-Hölder continuous. The space $\mathcal{C}^{k,\delta}$ is a Fréchet space with the following seminorms: for each compact convex subset K of $\mathbb{R}^d$

$$||f||_{k,0;K} = \sum_{0 \leq |\alpha| \leq k} \sup_{x \in K} |D^\alpha f(x)|,$$

$$||f||_{k,\delta;K} = ||f||_{k,0;K} + \sum_{|\alpha|=k} \sup_{x,y \in K, x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\delta}, \quad 0 < \delta \leq 1.$$

We then define the set $\mathcal{C}_b^{k,\delta}$ as the space of functions in $\mathcal{C}^{k,\delta}$ for which the norm

$$||f||_{k,0} = \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1+|x|} + \sum_{1 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)|,$$

$$||f||_{k,\delta} = ||f||_{k,0} + \sum_{|\alpha|=k} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\delta}, \quad 0 < \delta \leq 1$$

is finite.

We briefly recall the definition of Itô and backward Itô integral. Consider a continuous process $(X_u)_{u \geq s}$, which is adapted to $(\mathcal{F}_s^u)_{u \geq s}$ and such that $\sup_{u \in [s,t]} E[|X_u|^2] < \infty$ for a time $t > s$. For each partition $\sigma = \{s = t_0 \leq \ldots \leq t_n = t\}$ of $[s,t]$, consider the sum $I_\sigma = \sum_{k=0}^{n-1} X_{t_k}(W_{t_{k+1}} - W_{t_k})$. The limit in probability of I_σ, as $|\sigma| = \sup_k |t_k - t_{k-1}| \to 0$, exists and is called the Itô integral $\int_s^t X_u dW_u$ of $(X_u)_u$. For the definition of backward Itô integral, consider a continuous process $(X_u)_{u \geq s}$ which satisfies $\sup_{u \in [s,t]} E[|X_u|^2] < \infty$ for some $t > s$ and such that the variable X_u is $\mathcal{F}_u^t$-measurable (observe that the set $\mathcal{F}_u^t$ decreases as u increases). Then the limit of $\sum_{k=0}^{n-1} X_{t_{k+1}}(W_{t_{k+1}} - W_{t_k})$ exists in probability as $|\sigma| \to 0$, and is called backward Itô integral of $(X_u)_u$. It will be denoted by $\oint_s^t X_u dW_u$.

Recall that, if $f_0, ..., f_m$ are Borel measurable maps from $\mathbb{R}^d$ into itself, then a solution of the stochastic differential equation

$$dX_t = f_0(X_t)dt + \sum_{j=1}^m f_j(X_t) dW_t^j, \quad X_s = x \in \mathbb{R}^d, \tag{2.5}$$

is a continuous $\mathbb{R}^d$-valued adapted process $(X_t)_{t \geq s}$, such that $P\{X_s = x\} = 1$ and for each $t > s$ the equality

$$X_t = x + \int_s^t f_0(X_u) du + \sum_{j=1}^m \int_s^t f_j(X_u) dW_u^j$$

holds almost surely. One says that the solution to (2.5) is unique if, given any two solutions $(X_t)_{t\geq s}$, $(Y_t)_{t\geq s}$, then $P\{X_t = Y_t, t \geq s\} = 1$.

For the following result on existence and uniqueness of solutions of stochastic differential equations see for example [**?**] (theorems 3.4.1, 4.7.3 and 4.7.4):

Theorem 55 *Consider a standard Brownian motion $(W_t)_{t\in\mathbb{R}}$ in $\mathbb{R}^m$. Let $f_0 \in \mathcal{C}_b^{k,\delta}$, $f_j \in \mathcal{C}_b^{k+1,\delta}$, $j = 0, ..., m$, for some $k \geq 1$ and some $\delta > 0$. Suppose that the map*

$$f_0^* = f_0 - \sum_{j=1}^{m}\sum_{i=1}^{d}(f_j)_i \frac{\partial f_j}{\partial x_i}$$

is in $\mathcal{C}_b^{k,\delta}$. Then there exists a unique solution $(\phi_{s,t}(x))_{t\geq s}$ to the stochastic differential equation (2.5) for any $s \in \mathbb{R}$, $x \in \mathbb{R}^d$. Moreover, the solution can be extended to a stochastic flow of diffeomorphisms $(\phi_{s,t}(\omega, x))_{s,t\in\mathbb{R}}$, such that for each $s \in \mathbb{R}$ the equality

$$\phi_{s,t}(x) = x - \int_t^s f_0^*(\phi_{s,u}(x))du - \sum_{j=1}^{m} \oint_t^s f_j(\phi_{s,u}(x))dW_u^j$$

is satisfied for each $x \in \mathbb{R}^d$ and every $t \leq s$.

The theorem on generation of random dynamical systems from Itô stochastic differential equations is the following:

Theorem 56 *Consider a standard Brownian motion $(W_t)_{t\in\mathbb{R}}$ in $\mathbb{R}^m$, and let $f_0 \in \mathcal{C}_b^{k,\delta}$, $f_0^* = f_0 - \sum_{j=1}^m \sum_{i=1}^d (f_j)_i \frac{\partial f_j}{\partial x_i} \in \mathcal{C}_b^{k,\delta}$, $f_j \in \mathcal{C}_b^{k+1,\delta}$, $j = 0, ..., m$, for some $k \geq 1$ and some $\delta > 0$. Let $(\phi_{s,t}(\omega, x))_{s,t\in\mathbb{R}}$ be the stochastic flow of diffeomorphisms associated to equation (2.5) according to theorem (55). Then the map*

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R}^d \times \Omega &\longrightarrow \mathbb{R}^d \\ (t, x, \omega) &\mapsto \varphi(t, \omega, x) = \phi_{0,t}(x, \omega). \end{aligned}$$

is a C^k random dynamical system over the filtered dynamical system describing Brownian motion. Moreover, φ has stationary independent increments, i.e. for all $t_0 \leq t_1 \leq ... \leq t_n$ the random variables

$$\varphi(t_1) \circ \varphi(t_0)^{-1}, \ \varphi(t_2) \circ \varphi(t_1)^{-1}, \ldots, \ \varphi(t_n) \circ \varphi(t_{n-1})^{-1}$$

are independent, and the law of $\varphi(t + h) \circ \varphi(t)^{-1}$ is independent of t, for each t, h.

2.3.1 Additive noise

We shall prove theorem (56) in a particular case, when the functions $f_1, ..., f_m$ are constant. One speaks in this case of *additive noise*. The stochastic equation we consider is then of the form

$$X_t = x + \int_s^t b(X_u)du + W_t - W_s, \tag{2.6}$$

with b belonging to $\mathcal{C}_b^{k,\delta}$ for some $k \geq 1$, $\delta > 0$. We consider the stochastic flow of diffeomorphisms $(\phi_{s,t}(\omega, x))_{s,t\in\mathbb{R}}$ associated to this equation according to theorem (55), and define $\varphi(t,\omega)x = \phi_{0,t}(x,\omega)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$. Note that in this simple case, the existence and uniqueness of solutions of equation (2.6) can be proved also through a classical Cauchy theorem, for each fixed $\omega \in \Omega$.

We show that φ satisfies the crude cocycle property: for each $s \in \mathbb{R}$ there exists a P-null set N_s such that

$$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega) \;\text{ for all } t \in \mathbb{R},\; \omega \notin N_s. \tag{2.7}$$

Observe first that (2.7) holds if and only if ϕ satisfies

$$\phi_{s,s+t}(\omega) = \phi_{0,t}(\theta_s\omega) \;\text{ for all } t \in \mathbb{R},\; \omega \notin N_s, \tag{2.8}$$

thanks to flow property of the map ϕ. Indeed the cocycle property for φ rewrites as

$$\phi_{0,s+t}(\omega) = \phi_{0,t}(\theta_s\omega) \circ \phi_{0,s}(\omega).$$

Compose both sides with $\phi_{s,0}(\omega)$, and find $\phi_{0,s+t}(\omega) \circ \phi_{s,0}(\omega) = \phi_{s,s+t}(\omega) = \phi_{0,t}(\theta_s\omega)$, as wanted. Conversely, if $\phi_{s,s+t}(\omega) = \phi_{0,t}(\theta_s\omega)$, then

$$\varphi(t+s,\omega) = \phi_{0,s+t}(\omega) = \phi_{s,s+t}(\omega) \circ \phi_{0,s}(\omega) = \phi_{0,t}(\theta_s\omega) \circ \phi_{0,s}(\omega),$$

and (2.7) holds.

Let us prove (2.8). Consider the case $t > 0$. For each $x \in \mathbb{R}^d$, $s \in \mathbb{R}$, we have

$$\begin{aligned}\phi_{s,s+t}(x) &= x + \int_s^{s+t} b(\phi_{s,u}(x))du + W_{s+t} - W_s \\ &= x + \int_0^t b(\phi_{s,s+u}(x))du + W_{s+t} - W_s\end{aligned}$$

almost surely. Moreover, for almost every ω we have

$$\begin{aligned}\phi_{0,t}(\theta_s\omega, x) &= x + \int_0^t b(\phi_{0,u}(\theta_s\omega, x))du + (\theta_s\omega)(t) - (\theta_s\omega)(0) \\ &= x + \int_0^t b(\phi_{0,u}(\theta_s\omega, x))du + W_{s+t}(\omega) - W_s(\omega).\end{aligned}$$

Then the uniqueness of the solution for equation (2.6), and the continuity on x, imply the existence of a P-null set N_s depending on s such that (2.8) holds. The case $t < 0$ is analogous. Moreover, there exists a perfect cocycle which is indistinguishable from φ, according to a general result (theorem 1.3.2 of [3]).

The system $(\Omega, \mathcal{F}, (\theta_t)_{t\in\mathbb{R}}, P, \varphi)$ is called the random dynamical system associated to equation (2.6).

Observe that $\varphi(t)\circ\varphi(s)^{-1}$ equals $\phi_{0,t}\circ\phi_{0,s}^{-1} = \phi_{0,t}\circ\phi_{s,0} = \phi_{s,t}$, which is a $\mathcal{F}_s^t$-measurable random variable. Moreover, it follows by (2.8) that $\phi_{t,t+h} = \phi_{0,h}\circ\theta_t$ almost surely for each t, h, and then the law of $\varphi(t+h)\circ\varphi(t)^{-1}$ does not depend on t, and φ is an RDS with stationary independent increments.

Example 57 (One-dimensional Ornstein-Uhlenbeck process) *Consider the equation on* $\mathbb{R}$

$$dX_t = -\alpha X_t dt + \sigma dW_t, \quad X_{t_0} = x_0 \in \mathbb{R}. \tag{2.9}$$

Its solution, given by the Ornstein-Uhlenbeck process (see appendix **??***)*

$$X_t = e^{-\alpha(t-t_0)}x_0 + \int_{t_0}^t e^{-\alpha(t-s)}\sigma dW_s,$$

*can be rewritten, through an integration by parts formula (proposition 3.12 in [***?***]), for every* $\omega \in \Omega$, *as*

$$X_t(\omega) = e^{-\alpha(t-t_0)}x_0 + \sigma\omega(t) - \int_{t_0}^t \alpha e^{-\alpha(t-s)}\sigma\omega(s)ds.$$

Therefore the random dynamical system associated to (2.9) is

$$\varphi(t,\omega)x = e^{-\alpha t}x + \sigma\omega(t) - \int_0^t \alpha e^{-\alpha(t-s)}\sigma\omega(s)ds.$$

2.3.2 A finite dimensional model

We consider a finite dimensional model which covers many important examples, for instance the finite dimensional approximation of the Navier-Stokes equations (see paragraph 2.5). We consider the stochastic equation in $\mathbb{R}^n$

$$dX_t + (AX_t + B(X_t, X_t))dt = f dt + \sqrt{Q}dW_t, \quad t \geq t_0, \tag{2.10}$$

where A is an $n\times n$ symmetric matrix satisfyng $\langle Ax, x\rangle \geq \lambda|x|^2$ for each $x \in \mathbb{R}^n$, B is a bilinear continuous mapping with the property

$$\langle B(y,x), x\rangle = 0 \quad \forall x, y \in \mathbb{R}^n, \tag{2.11}$$

Q is a positive semidefinite symmetric matrix, and W_t is a two-sided Brownian motion over the canonical metric dynamical system $(\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{s\le t}, (\theta(t))_{t\in\mathbb{R}}, P)$. We shall consider $\mathcal{F}_{t_0}$-random variables as initial conditions. The equation in integral form is then

$$X_t = \eta - \int_{t_0}^t (AX_s + B(X_s, X_s) - f)ds + \sqrt{Q}W_t - \sqrt{Q}W_{t_0}. \tag{2.12}$$

To define the random dynamical system associated to (2.10), one could solve, for each fixed ω, a system of ordinary differential equation. We choose instead a probabilistic approach for the following existence and uniqueness result:

Proposition 58 *For every $\mathcal{F}_{t_0}$-measurable random variable η, equation (2.12) admits a unique solution $(X_t)_t$, which is continuous and adapted to $(\mathcal{F}_t)_t$.*

Proof. For uniqueness, suppose $(X_t^1)_t$, $(X_t^1)_t$ are two solutions of equation (2.10), and let C_B be a constant such that $B(x,y) \le C_B|x||y|$. Then the difference $V_t = X_t^1 - X_t^2$ satisfies

$$\frac{dV_t}{dt} = -AV_t - B(X_t^1, V_t) - B(V_t, X_t^2).$$

Taking the scalar product with V_t, and using property (2.11) we find

$$\langle \frac{dV_t}{dt}, V_t\rangle = -\langle AV_t, V_t\rangle - \langle B(V_t, X_t^2), V_t\rangle,$$

and consequently, by the inequality on A we have

$$\frac{d|V_t|^2}{dt} \le C_B|V_t|^2|X_t^2|.$$

The Gronwall lemma gives then

$$|V_t|^2 \le |V_{t_0}|^2 e^{C_B\int_{t_0}^t |X_s^2|ds}. \tag{2.13}$$

Uniqueness of the solution follows.

Existence of a solution can be proved by a truncation procedure as follows.

Observe first that the initial condition η can be supposed to be bounded. This can be seen by defining for each $m \in \mathbb{N}$ the set $A_m = \{\omega \in \Omega : |\eta(\omega)|^2 \le n\}$, and a random variable η_m as equal to η on A_m, and equal to 0 on A_m^{C}, and considering for each m the unique solution $(Y_t^m)_{t\ge 0}$ of equation (2.10) with initial condition η_m. The process Y^∞ defined as $Y_t^\infty = Y_t^m$ on A_m is then the unique solution of equation (2.12).

Fix a $T > t_0$, and a constant C such that $|\eta| \le C$. For each $m \ge C$, consider a Lipschitz continuous function B_m on such that $B_m(x) = B(x,x)$ for every $x \in \mathbb{R}^n$ with $|x| \le m$. Then the equation

$$dX_t = (-AX_t - B_m(X_t, X_t) + f)dt + \sqrt{Q}dW_t,$$

with initial condition η, has a unique adapted solution $(X_t^m)_t$, as a consequence of theorem (55). For each $m > C$, define the stopping time

$$\tau_m = \inf\{t_0 \le t \le T : |X_t^m| = m\} \wedge T.$$

On the stochastic interval $[t_0, \tau_m]$, X_t^m is a solution of the original equation, as

$$\begin{aligned} X^m_{t\wedge\tau_m} &= \eta + \int_0^{t\wedge\tau_m} (-AX_s^m - B_m(X_s^m) + f)ds + \sqrt{Q}W_{t\wedge\tau_m} - \sqrt{Q}W_{t_0} \qquad (2.14) \\ &= \eta + \int_0^{t\wedge\tau_m} (-AX_s^m - B(X_s^m, X_s^m) + f)ds + \sqrt{Q}W_{t\wedge\tau_m} - \sqrt{Q}W_{t_0}. \end{aligned}$$

Moreover, if k is greater than m, then the processes X_t^k and X_t^m are equal on $[t_0, \tau_m]$ with probability 1. A process X^∞ is then defined on $[t_0, \sup_{m>C} \tau_m]$ by $X^\infty = X_t^m$ on $[t_0, \tau_m]$, for every $m > C$. To estimate $\sup_{m>C} \tau_m$, apply Itô formula to (2.14), use the fact that $\langle B(y,x), x\rangle = 0$, and find

$$\begin{aligned} |X^m_{t\wedge\tau_m}|^2 + 2\int_0^{t\wedge\tau_m} \langle AX_s^m, X_s^m\rangle ds =& |\eta|^2 + 2\int_0^{t\wedge\tau_m} \langle X_s^m, f\rangle ds \\ &+ 2\int_0^{t\wedge\tau_m} \langle X_s^m, \sqrt{Q}dW_s\rangle + \mathrm{Tr}Q(t\wedge\tau_m). \end{aligned}$$

The coercivity of A and the inequality $\langle X_s^m, f\rangle \le |X_s^m||f|$ imply that

$$|X^m_{t\wedge\tau_m}|^2 \le |\eta|^2 + 2m(t\wedge\tau_m)|f| + 2\int_0^{t\wedge\tau_m} \langle X_s^m, \sqrt{Q}dW_s\rangle + \mathrm{Tr}Q(t\wedge\tau_m).$$

Then

$$E\left[\sup_{t\in[t_0,T]} |X^m_{t\wedge\tau_m}|^2\right] \le E[|\eta|^2] + 2mT|f| + T\mathrm{Tr}Q,$$

and in particular

$$E\left[I_{\{\tau_m<T\}}|X^m_{T\wedge\tau_m}|^2\right] \le Mm$$

for some constant $M > 0$. As $X^m_{T\wedge\tau_m} = m$ if $\tau_m < T$, we find

$$P\{\tau_m < T\} \le \frac{M}{m},$$

and as a concequence $P\{\sup_{m>C} \tau_m < T\} = 0$. Therefore $(X_t^\infty)_t$ is a solution on the interval $[t_0, T-\epsilon]$, for each $\epsilon > 0$, and the existence of a global solution follows by arbitrariness of T. ■

We write $X_t^{t_0,x}$ for the solution of (2.10) starting from x at time t_0. The random dynamical system associated to the stochastic equation is $\varphi(t,\omega)x = X_t^{0,x}(\omega)$. The cocycle property can be verified as in paragraph (2.3.1).

Example 59 A simplified GOY model (Gledzer, Ohkitani, Yamada model) *A system which is included in the described setting is the following. Consider, for each* $N \in \mathbb{N}$, *by the finite dimensional stochastic system*

$$u_{-1} = u_0 = u_N = u_{N+1} = 0,$$

$$\begin{aligned} du_n + \nu k_n^2 u_n dt =& k_n \left(\frac{1}{4} u_{n-1} u_{n+1} - u_{n+1} u_{n+2} + \frac{1}{8} u_{n-1} u_{n-2} \right) dt \\ &+ \sigma_n dW_t^n \quad \text{for } n = 1, ..., N, \end{aligned}$$

where $k_n = 2^n$, $\nu > 0$, *and* $(W_t^n)_{t \in \mathbb{R}}$ *are independent two-sided Brownian motions, for* $0 \leq n \leq N$.

It is a stochastic version of a simplification of the GOY model, a system belonging to the class of shell models, which are designed to capture many aspects of fluid dynamics, and are structured as simplified versions of the Navier-Stokes equations, by considering some variants of their Fourier approssimation.

Observe that the velocity on each single cell depends directly only by velocities in the nearest and next-nearest neighbour shells, and that the energy ($\sum_n |u_n|^2$) *is preserved when* $\nu = \sigma_n = 0$ *(namely without viscosity and external force).*

2.3.3 n-point motion

It is sometimes useful, for the study of the evolution of a dynamical system, to consider the analysis of n copies of the system, starting at different points, for example when focusing on sensitiveness of the system to initial conditions. In the stochastic settings described so far, we are interested in the study of the evolution of a random dynamical system for each fixed realization ω of the noise. Denote by φ the random dynamical system generated by the SDE (2.5) or (2.10). For each n-uple $x_1, ..., x_n \in \mathbb{R}^d$, we call the $(\mathbb{R}^d)^n$-valued process

$$(t, \omega) \mapsto (\varphi(t, \omega)x_1, ..., \varphi(t, \omega)x_n) \tag{2.15}$$

the *n-point motion* associated to the considered equation. For example, if φ is the random dynamical system associated to the stochastic equation (2.6), then the n-point motion rises from the system of stochastic equations

$$\begin{cases} X_t = x_1 + \int_s^t b(X_u)du + W_t - W_s \\ X_t = x_2 + \int_s^t b(X_u)du + W_t - W_s \\ \dots \\ X_t = x_n + \int_s^t b(X_u)du + W_t - W_s; \end{cases}$$

in particular, the same noise acts on all components.

The n-point motion (2.15) is a Markov process (for the definition see appendix **??**). For the proof we will need the following lemma:

Lemma 60 *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $\mathcal{F}$, $\mathcal{G}$ be two independent sub-σ-fields of $\mathcal{A}$. Let X be a $\mathcal{G}$-measurable random variable with values on a measurable space $(E, \mathcal{E})$, and let $\psi : E \times \Omega \to \mathbb{R}$ be measurable with respect to $\mathcal{E} \otimes \mathcal{F}$, and such that the map $\omega \mapsto \psi(X(\omega), \omega)$ is integrable. Define $\Psi(x) := E[\psi(x, \cdot)]$. Then*

$$E[\psi(X, \cdot)|\mathcal{G}] = \Psi \circ X.$$

Proof. Take ψ of the form $\psi(x, \omega) = h(x)f(\omega)$, where f is $\mathcal{F}$-measurable. Then

$$E[\psi(X, \cdot)|\mathcal{G}] = E[h(X)f(\cdot)|\mathcal{G}] = h(X)E[f] = \Psi \circ X.$$

Conclusion follows by a classical monotone class argument. ■

Proposition 61 *The n-point motion associated to equations (2.5) or (2.10) is a Markov process with respect to the filtration $(\mathcal{F}_0^t)_{t \geq 0}$, with transition probabilities given by*

$$P_t^{(n)}((x_1, ...x_n), B) = P\{\omega \in \Omega : (\varphi(t, \omega)x_1, ..., \varphi(t, \omega)x_n) \in B\},$$

for $t \in \mathbb{R}^+$, $B \in \mathcal{B}(\mathbb{R}^{nd})$.

Proof. Take $B \in \mathcal{B}(\mathbb{R}^{nd})$, and define $\psi : \mathbb{R}^{nd} \times \Omega \to \mathbb{R}$ by

$$\psi(x, \omega) = I_B((\varphi(t, \theta_s\omega)x_1, ..., \varphi(t, \theta_s\omega)x_n)).$$

Then

$$E[I_B(\varphi(s+t, \omega)x_1, ..., \varphi(s+t, \omega)x_n)|\mathcal{F}_0^s] = E[\psi((\varphi(s, \cdot)x_1, ..., \varphi(s, \cdot)x_n), \cdot)|\mathcal{F}_0^s];$$

moreover $\varphi(s, \cdot)x$ is $\mathcal{F}_0^s$-measurable, and ψ is $\mathcal{B}(\mathbb{R}^{nd}) \otimes \mathcal{F}_s^{s+t}$-measurable. As $\mathcal{F}_s^{s+t}$ and $\mathcal{F}_0^s$ are independent, we can apply lemma (60) and obtain

$$E[I_B(\varphi(s+t, \omega)x_1, ..., \varphi(s+t, \omega)x_n)|\mathcal{F}_0^s] = \Psi \circ (\varphi(s, \cdot)x_1, ..., \varphi(s, \cdot)x_n),$$

where $\Psi(x) = E[I_B(\varphi(t, \omega)x_1, ..., \varphi(t, \omega)x_n)] = P_t^{(n)}(x, B)$, i.e. (2.15) is a Markov process associated to the transition functions $(P_t^{(n)})_{t \in \mathbb{R}^+}$. ■

We still denote by $(P_t^{(n)})_{t \in \mathbb{R}^+}$ the Markov semigroup of operators on $B_b(\mathbb{R}^{nd})$ associated to the transition function $P_t^{(n)}$ defined above: for each $f \in B_b(\mathbb{R}^{nd})$, $x = (x_1, ..., x_n) \in \mathbb{R}^{nd}$, we have

$$P_t^{(n)}f(x) = \int_X f(y)P_t^{(n)}(x, dy) = \int_\Omega f((\varphi(t, \omega)x_1, ..., \varphi(t, \omega)x_n))P(d\omega).$$

As a consequence of continuity on initial data, the Markov semigroup $(P_t^{(n)})_{t \in \mathbb{R}^+}$ is Feller.

2.4 RDS as products of i.i.d. random mappings

We saw in remark 48 that a discrete random dynamical system can be generated by compositions of mappings of the form $\psi(\theta^n \cdot)$. The RDS for which these random variables are independent and identically distributed have been deeply studied, for example by Kifer [33]. They are also referred to as *RDS with independent increments*. Here we want to briefly describe how they are related to Markov chains, while in chapter **??** we will show why they are interesting, together with RDS generated by SDE, from the point of view of random invariant measures.

The proof of this result follows very closely that of proposition 61:

Proposition 62 *Let φ be a discrete dynamical system over a metric dynamical system $(\Omega, \mathcal{F}, P, \theta)$, given by the composition $\varphi(n,\omega) = \psi_n(\omega) \circ \psi_{n-1}(\omega) \circ \cdots \circ \psi_0(\omega)$ of i.i.d. random mappings $\psi_n = \psi \circ \theta^n$. Then for each n-uple $x_1, ..., x_n$ the n-point motion*

$$(k,\omega) \mapsto (\varphi(k,\omega)x_1, ..., \varphi(k,\omega)x_n)$$

is a homogeneous Markov chain with transition probability

$$P^{(n)}((x_1, ..., x_n), B) = P\{\omega \in \Omega : (\psi(\omega)x_1, ..., \psi(\omega)x_n) \in B\}. \tag{2.16}$$

Moreover, if the random dynamical system is continuous, the Markov chain is Feller.

2.5 An example in infinite dimensions: the stochastic Navier-Stokes equations

We consider the Navier-Stokes equations for a viscous incompressible fluid in two-space dimensions. We are given an open bounded subset Ω of $\mathbb{R}^2$, with boudary Γ. The equations for the velocity vector u and the pressure p, for a fluid with constant density equal to 1, are

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \\ \mathrm{div} u = 0, \end{cases} \tag{2.17}$$

where $\nu > 0$ represents the viscosity of the fluid, and f is the density of force applied to the fluid. The first equation expresses the momentum conservation, the second expresses the mass conservation.

Two kinds of boundary conditions, as well as an initial condition $u(x,0) = u_0(x)$, are usually considered. The first one

$$u|_{\Gamma=0}$$

corresponds to the boundary Γ being a solid at rest. The second has no physical interpretation, but presents some more affordable aspects. It can be considered in case the domain Ω_1 is a square, say $\Omega_1 = (0,l) \times (0,l)$, and consists in imposing

$$u(x + le_i, t) = u(x,t) \;\; \forall x \in \mathbb{R}^2, \; \forall t > 0.$$

In this case one imposes also that the integral of the velocity u over Ω equals zero. For this to be verified, it is sufficient that u_0 and f have zero-integral over Ω, as follows by integrating the equations on Ω and using the periodicity property.

Before giving the stochastic version of these equations, we describe how they can be formulated in a abstract form. We give a brief sketching of the whole problem, and we refer to [?], [?] for a complete description of the setting and details.

We denote by $H^m(\Omega)$ the Sobolev spaces of functions in $L^2(\Omega)$, whose derivatives of order less or equal to m are in $L^2(\Omega)$. $H^m(\Omega)$ is a Hilbert space with scalar product

$$(u,v)_m = \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v),$$

where $(D^\alpha u, D^\alpha v) = \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$.

We write $H^m_{per}(\Omega_1)$ for the space of restrictions on Ω_1 of Ω_1-periodic functions in $H^m(\mathbb{R}^n)$, which belong to $H^m(S)$ for each bounded open set S.

If X is a space of functions defined on Ω, we denote with $\mathbb{X}$ the space X^2 with the product structure, with $\dot{X}$ the set of the functions $u \in X$ such that $\int_\Omega u dx = 0$, and with X_0 the functions in X which are zero on Γ.

The following spaces are useful for establishing the mathematical setting of the equations:

$$V = \{u \in \mathbb{H}^1_0(\Omega), \mathrm{div} u = 0\}, \quad H = \{u \in \mathbb{H}^0_0(\Omega), \mathrm{div} u = 0\}$$

for the case of still boundary, and

$$V = \{u \in \dot{\mathbb{H}}^1_{per}(\Omega), \mathrm{div} u = 0\}, \quad H = \{u \in \dot{\mathbb{H}}^0_{per}(\Omega), \mathrm{div} u = 0\}$$

for the case of boundary periodic functions. On both cases V is a Hilbert space with scalar product $((u,v)) = \sum_{|\alpha|=1} (D^\alpha u, D^\alpha v)$ and norm $||u|| = ((u,u))^{1/2}$. Consider the bilinear form

$$a(u,v) = \sum_{i,j=1}^{2} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right),$$

and the trilinear form b given by

$$b(u,v,w) = \sum_{i,j}^{n} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

A weak form of the Navier-Stokes equations (2.17) can be given by taking the scalar product of the equations with a function $v \in V$. The pressure disappears and one finds

$$\frac{d}{dt}(u,v) + \nu a(u,v) + b(u,u,v) = (f,v), \tag{2.18}$$

The continuity of a and b guarantees the existence of a linear operator A and a bilinear operator B on $V \times V$ such that $(Au, v) = a(u, v)$, $(B(u, v), w) = b(u, v, w)$ for all $u, v, w \in V$. Equation (2.18) can then be written as an equation in V':

$$\frac{du}{dt} + \nu Au + B(u, u) = f. \tag{2.19}$$

This equation admits a unique solution u in $C([0,T]; H) \cap L^2(0,T;V)$ for each given f and u_0 in H, for each $T > 0$. Moreover, one can prove that the mapping $u_0 \mapsto u(t)$ is continuous from H into $D(A)$, and u belongs to $C([0,T]; V) \cap L^2(0,T; D(A))$ if u_0 is taken in V. This result on existence and continuity of solutions allows the definition of a semigroup of continuous operators

$$S(t) : u_0 \mapsto u(t), \quad t \geq 0$$

which define the dynamical system associated to Navier-Stokes equations.

A stochastic version of equations (2.19), with additive noise, is given by the equation

$$du + (\nu Au + B(u, u))dt = f dt + \sum_{j=1}^{m} \phi_j dW_j(t), \tag{2.20}$$

where $\phi_1, \ldots, \phi_m$ are functions in H and $W_1, ..., W_m$ are independent two-sided Brownian motions. To show how a random dynamical system can be associated to (2.20), consider the solution z of the Orstein-Uhlenbeck equation

$$dz = Azdt + \sum_{j=1}^{m} \phi_j dW_j(t).$$

Then, if u is a solution of (2.20), the process $v(t) = u(t) - z(t)$ satisfies formally

$$dv + (Av + B(v, v) + B(v, z) + B(z, v) + B(z, z))dt = f dt.$$

In this form, the equation can be solved pathwise for almost every ω. As can be done for equation (2.19), one can show that for each initial condition $v_0 \in H$ and for each $T > 0$ there exists a unique solution $v^{u_0}(\cdot, \omega) \in C([0,T]; H) \cap L^2(0,T;V)$, and belonging to $C([0,T]; V) \cap L^2(0,T; D(A))$ if v_0 is in V. Moreover the map $v_0 \mapsto v(t, \omega)$ is continuous for all $t > 0$. Then one can consider the map

$$\begin{array}{ccl} \mathbb{R}^+ \times \Omega \times H & \longrightarrow & H \\ (t, \omega, u_0) & \mapsto & \varphi(t, \omega)u_0 = v^{u_0}(t, \omega) + z(t, \omega). \end{array}$$

As in paragraph (2.3.1), one can show that the well-posedness of equation (2.20) implies the cocycle property, and then φ is a random dynamical system, defined fot $t \geq 0$, over the canonical two-sided metric dynamical system describing Brownian motion.

2.6 Multiplicative ergodic theorem and Lyapunov exponents

An important result of the theory of random dynamical systems is the so-called *Multiplicative Ergodic Theorem*, which allows the definition of Lyapunov exponents for RDS. Its full statements and the proof of all its variants can be found in [3], Part II. We give the statement first for the case of a linear cocycle Φ on $\mathbb{R}^d$, i.e. an RDS over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{T}})$, such that $\Phi(t,\omega)$ is linear for each $t \in \mathbb{T}$, $\omega \in \Omega$.

Theorem 63 *Suppose that Φ satisfies*

$$\sup_{0\le t\le 1} \log^+ ||\Phi(t,\cdot)|| \in L^1, \quad \sup_{0\le t\le 1} \log^+ ||\Phi(t,\cdot)^{-1}|| \in L^1.$$

Then there exists an invariant set $\tilde{\Omega}$ such that $P(\tilde{\Omega}) = 1$ and for each $\omega \in \tilde{\Omega}$ the limit $\Psi(\omega) = \lim_{t\to\infty} (\Phi(t,\omega)^*\Phi(t,\omega))^{1/2t}$ exists. Call $e^{\lambda_{p(\omega)}(\omega)} < \cdots < e^{\lambda_1(\omega)}$ the different eigenvalues of $\Psi(\omega)$, and $U_{p(\omega)}(\omega), \ldots, U_1(\omega)$ the corresponding eigenspaces, with $d_i(\omega) = \dim U_i(\omega)$. Then the functions $\omega \mapsto p(\omega)$, $\omega \mapsto \lambda_i(\omega)$, $\omega \mapsto U_i(\omega)$ and $\omega \mapsto d_i(\omega)$ are measurable, and they satisfy

$$p(\theta(t)\omega) = p(\omega) \text{ for all } t \in \mathbb{T},$$

$$\lambda_i(\theta(t)\omega) = \lambda_i(\omega), \ d_i(\theta(t)\omega) = d_i(\omega), \ \forall t, 1 \le i \le p(\omega).$$

For each $x \in \mathbb{R}^d \setminus \{0\}$, the limit

$$\lambda(\omega, x) = \lim_{t\to\infty} \frac{1}{t} \log ||\Phi(t,\omega)x||$$

exists and is called a Lyapunov exponent *for Φ. Moreover $\lambda(\omega, x) = \lambda_i(\omega)$ if and only if $x \in V_i(\omega) \setminus V_{i+1}(\omega)$, where*

$$V_i(\omega) = U_{p(\omega)} \oplus \cdots \oplus U_i(\omega) \ \text{ for } 1 \le i \le p(\omega), \quad V_{p(\omega)+1} = \{0\},$$

and $\lambda(\theta(t)\omega, \Phi(t,\omega)x) = \lambda(\omega, x)$, for all $x \in \mathbb{R}^d \setminus \{0\}$, $\Phi(t,\omega)V_i(\omega) = V_i(\theta(t)\omega)$ for all $1 \le i \le p(\omega)$.

For each $\omega \in \tilde{\Omega}$, $\mathbb{R}^d$ splits as

$$\mathbb{R}^d = \bigoplus_{k=1}^{p(\omega)} E_k(\omega),$$

where the E_k are random subspaces with dimension d_k, given by $E_k = V_i \cup V^-_{p(\omega)+k-1}$, where V_i^-, $1 \le i \le p^-(\omega) = p(\omega)$ is the family of subspaces associated to $\Phi(t,\omega)$.

In case $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{T}})$ is ergodic, then the map $\omega \mapsto p(\omega)$ is constant on $\tilde{\Omega}$, and λ_i, d_i are constant on $\{\omega \in \tilde{\Omega} : p(\omega) \ge 1\}$, for $i = 1, ..., d$.

We will now give the statement of the theorem for manifolds. Consider a C^1 RDS φ on a d-dimensional Riemannian manifold M, over the metric DS $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{T}})$, with associated skew product $(\Theta(t))_{t\in\mathbb{T}}$, and let ρ be an invariant measure for the one-point motion. Denote by TM the fiber boundle over M. By differentiating the identity $\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\circ\varphi(s,\omega)$, one finds that the differential

$$\begin{aligned} T\varphi(t,\omega) :& TM \to TM \\ & (x,v) \mapsto (\varphi(t,\omega)x, D\varphi(t,\omega,x)v) \end{aligned}$$

is a continuous cocycle over $(\theta_t)_{t\in\mathbb{T}}$.

Theorem 64 *Assume that $T\varphi$ satisfies*

$$\sup_{0\le t\le 1} \log^+ ||T\varphi(t,\omega,x)||_{x,\varphi(t,\omega)x} \in L^1(P), \tag{2.21}$$

$$\sup_{0\le t\le 1} \log^+ ||T\varphi(t,\omega,x)^{-1}||_{\varphi(t,\omega)x,x} \in L^1(P), \tag{2.22}$$

where $||\cdot||_{x,\varphi(t,\omega)x}$ is the norm of the differential as a linear mapping from T_xM to $T_{\varphi(t,\omega)x}M$. Then there exist Θ-invariant random variables p, d_i, λ_i which satisfy

$$1 \le p(\omega,x) \le d, \quad \lambda_1(\omega,x) > \cdots > \lambda_{p(\omega,x)}(\omega,x) > -\infty, \quad \sum_{i=1}^{p(\omega,x)} d_i(\omega,x) = d,$$

for (ω,x) in a Θ-invariant set $A\subset\Omega\times M$ of full measure, and such that the tangent space T_xM decomposes as

$$T_xM = \bigoplus_{k=i}^{p(\omega,x)} E_k(\omega,x)$$

for each $(\omega,x)\in A$, where E_k are random subspaces of dimension d_i satisfying

$$T\varphi(t,\omega,x)E_k(\omega,x) = E_k(\Theta(t)(\omega,x)),$$

and

$$\lim_{t\to\pm\infty} \frac{1}{t}\log ||T\varphi(t,\omega,x)v||_{\varphi(t,\omega)x} = \lambda_i(\omega,x) \quad \text{if and only if} \quad v\in E_i(\omega,x)\setminus\{0\}.$$

The λ_i are called Lyapunov exponents *of φ under $P\times\rho$. If ρ is ergodic, then p, d_i and λ_i are constant.*

Chapter 3

Random Attractors and Random Invariant Measures

3.1 Attractors for random dynamical systems

In Chapter 1 we have already investigated attractors for non-autonomous systems in a quite satisfactory manner. The results move directly to RDS, working ω-wise. The only issue which is really new from NADS to RDS is the concept of invariance but we have proved that the attractor constructed for NADS satistifes also the cocycle invariance property. However, we think it is pedagogically useful to warm-up with the typical computations of RDS re-stating and re-proving from scratch some elements of the theory of random attractors in the new language. This is the reason for some repetitions.

We are given a random dynamical system φ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{T}})$ on a Polish space, X, or more specifically on a complete separable metric space (X, d).

In the random setting, the analogous of the ω-limit set is defined for a random set K as the random set

$$\Omega_K(\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega)}.$$

For every $\omega \in \Omega$, $\Omega_K(\omega)$ is the set of points $y \in X$ such that there exist a sequence $t_n \to \infty$ and a sequence x_n in $K(\theta_{-t_n}\omega)$ such that $\varphi(t_n, \theta_{-t_n}\omega)x_n \to y$ as $n \to \infty$. With this observation, it is easy to verify that Ω_K is forward invariant for the RDS φ, in the sense of the definition given in paragraph 3.3.1. Let y be in $\Omega_K(\omega)$, and let $(t_n)_n$ and $(x_n)_n \subset K(\theta_{-t_n}\omega)$ be such that $\varphi(t_n, \theta_{-t_n}\omega)x_n \to y$. Then we find, by the cocycle property

$$\begin{aligned}
\varphi(t,\omega)y &= \lim_{n\to\infty} \varphi(t,\omega)\varphi(t_n, \theta_{-t_n}\omega)x_n = \lim_{n\to\infty} \varphi(t, \theta_{t_n}\theta_{-t_n}\omega)\varphi(t_n, \theta_{-t_n}\omega)x_n \\
&= \lim_{n\to\infty} \varphi(t+t_n, \theta_{-t_n}\omega)x_n = \lim_{n\to\infty} \varphi(t+t_n, \theta_{-t_n-t}\theta_t\omega)x_n \\
&= \lim_{n\to\infty} \varphi(s_n, \theta_{-s_n}\theta_t\omega)x_n,
\end{aligned}$$

with $s_n = t + t_n$. As x_n is in $K(\theta_{-t_n}) = K(\theta_{-s_n}\theta_t\omega)$, one gains $\varphi(t,\omega)y \in \Omega_K(\theta_t\omega)$.

There are several ways to generalize the notion of attraction when taking into account noise perturbations. The following one is the most classical.

Definition 65 *A random set A* attracts *another random set B if*

$$d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \to 0 \quad \text{as } t \to \infty \tag{3.1}$$

for P-almost every $\omega \in \Omega$.

The kind of attraction presented in the definition is sometimes referred to as *pullback attraction*. A weaker form of convergence is *attraction in probability*, according to which convergence (3.1) takes place in probability, i. e. for every $\epsilon > 0$

$$P\{d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) > \epsilon\} \to 0 \text{ as } t \to \infty.$$

In many situations, the pullback approach turns out to be more appropriate then the notion of *forward attraction*, according to which a random set A attracts another random set B if convergence

$$d(\varphi(t,\omega)B(\omega), A(\theta_t\omega)) \to 0$$

takes place for P-almost every ω as $t \to \infty$.

The following example shows that the two kinds of convergence are different.

Example 66 *Consider a metric dynamical system $(\Omega, \mathcal{F}, P, \theta)$, a random variable $f : \Omega \to \mathbb{R}^+$ and a constant α, with $0 \le \alpha \le 1$. The map $\psi : \Omega \times \mathbb{R} \to R$ given by*

$$\psi(\omega, x) = \alpha\frac{f(\theta\omega)}{f(\omega)}x$$

defines a discrete random dynamical system φ as described in remark 48. By an induction argument one sees that

$$\varphi(n,\omega)x = \alpha^n\frac{f(\theta^n\omega)}{f(\omega)}x.$$

Consider a random set B and take $A(\omega) = \{0\}$. We want to see under which condition the set A is forward or pullback attracting for B. For this purpose, we will use the result of the proposition below, which can be found in [3] (proposition 4.1.3): The proposition implies that the quantity

$$\frac{1}{n}\log\left(\alpha^n\frac{f(\theta^n\omega)}{f(\omega)}|B(\omega)|\right) = \log\alpha + \frac{\log f(\theta^n\omega)}{n} + \frac{1}{n}\log\frac{|B(\omega)|}{f(\omega)}$$

converges to $\log\alpha < 0$ or to ∞ as $n \to \infty$, and A is forward attracting if and only if $\frac{1}{n}\log f(\theta^n\omega)$ converges to zero. As for pullback attraction, one can consider

$$\frac{1}{n}\log\left(\alpha^n\frac{f(\omega)}{f(\theta^{-n}\omega)}|B(\theta^{-n}\omega)|\right) = \log\alpha + \frac{\log f(\omega)}{n} + \frac{1}{n}\log\frac{|B(\theta^{-n}\omega)|}{f(\theta^{-n}\omega)},$$

and A is pullback attracting if and only if $\frac{1}{n}\log\frac{|B(\theta^{-n}\omega)|}{f(\theta^{-n}\omega)}$ converges to zero.

Proposition 67 *Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{T}})$ be a metric dynamical system and let $g : \Omega \to R$ be a random variable. Then*

$$\limsup_{t\to\infty} \frac{1}{t} g(\theta_t \omega) \in \{0, \infty\} \quad \text{for } P\text{-almost every } \omega.$$

If one considers convergence in probability instead of almost sure convergence, then the two definitions coincide, thanks to the θ-invariance of P, which implies

$$P\{d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) > \epsilon\} = P\{d(\varphi(t, \omega)B(\omega), A(\theta_t\omega)) > \epsilon\}.$$

Definition 68 *A random compact set A is a* random attractor *for a random set B if*

(i) A is invariant;

(ii) A attracts B.

The definition of absorbing set in random context is given for the pullback point of view.

Definition 69 *A random set K* absorbs *a random set B, if for P-almost every $\omega \in \Omega$ there exists a time t_0 (which may depend on ω) such that*

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega) \quad \text{for all } t \geq t_0.$$

The following result is the analogous of theorem **??**.

Theorem 70 *Suppose that there exists a compact random set K which absorbs a random set B. Then the Ω-limit A of B is an attractor for B.*

Proof. Let t_0 be such that $\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega)$ for all $t \geq t_0$. Then

$$\Omega_B(\omega) \subset \bigcup_{s\geq t_0} \overline{\bigcap_{t\geq s} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)} \subset K(\omega),$$

which implies that $\Omega_B(\omega)$ is compact. To prove invariance, it is sufficient to show that $\Omega_B(\theta_t\omega) \subset \varphi(t, \omega)\Omega_B(\omega)$. Take $x \in \Omega_B(\theta_t\omega)$, then

$$x = \lim_{n\to\infty} \varphi(t_n, \theta_{-t_n}\theta_t\omega)x_n = \lim_{n\to\infty} \varphi(t, \omega)\varphi(t_n - t, \theta_{-t_n+t}\omega)x_n,$$

for some sequence $t_n \to \infty$ and $(x_n) \subset B(\theta_{-t_n+t}\omega)$. The sequence $\varphi(t_n - t, \theta_{-t_n+t}\omega)x_n$ is eventually contained in $K(\omega)$, and then it admits a subsequence converging to a point $y \in \Omega_B(\omega)$. By continuity of $\varphi(t, \omega)$ we find $x = \varphi(t, \omega)y \in \varphi(t, \omega)\Omega_B(\omega)$. Suppose now that A does not attract B. Then there exist a $\delta > 0$, a sequence $t_n \to \infty$ and a sequence $x_n \in B(\theta_{-t_n}\omega)$ such that $d(\varphi(t_n, \theta_{-t_n}\omega)x_n, \Omega_B(\omega)) \geq \delta$ for all n. For n

sufficiently big we have $(\varphi(t_n, \theta_{-t_n}\omega)x_n) \subset K(\omega)$, and then it possesses a convergent subsequence $(\varphi(t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k})_k$. The limit point belongs to $\Omega_B(\omega)$, thus contradicting $d(\varphi(t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k}, \Omega_B(\omega)) \geq \delta$. ■

The notion of global attractor for random dynamical systems is more delicate that the deterministic one. Following the paper by Crauel and Flandoli ([**?**]), we say that an invariant random compact set is a *global* or *universal attractor* if it attracts all bounded deterministic sets in X. The following result could be found in the same paper.

Theorem 71 *Suppose that there exists a compact random set K which absorbs every bounded deterministic set $B \subset X$. Then there exists a global attractor A for φ, given by the $\mathcal{F}^-$-measurable random set*

$$A(\omega) = \overline{\bigcup_{B \subset X} \Omega_B(\omega)}.$$

3.1.1 Attractor for the finite dimensional approximation of SNSE

We give the proof of existence of an attractor for equation (2.10), following the proof for the stochastic Navier-Stokes equations, due to Crauel and Flandoli ([**?**]). This finite dimensional version exemplifies all the techniques which are involved in the infinite-dimensional case, for the proof of dissipativity. The instruments combine those of the determinist case, such as the estimate of energy through continuity inequalities and Gronwall lemma, and some others typical of the stochastic case, for instance the pullback approach and the auxiliary Ornstein-Uhlenbeck process. The aspect of compactness needs a further effort for the full Navier-Stokes equations; nevertheless, the procedure strictly follows that of the deterministic case, for whom we refer to [**?**].

We want to show that there exists a bounded random set which attracts all bounded deterministic sets, i.e. for P-almost every ω there exists a random ball $B(0, r(\omega))$ such that for each bounded subset B of $\mathbb{R}^n$, there exists a time T such that

$$\varphi(t, \theta_{-t}\omega)B \subset B(0, r(\omega)), \quad \text{for all } t \geq T.$$

As $\varphi(t, \theta_{-t}\omega)x = X_0^{-t,x}(\omega)$, we have to estimate the modulus of $X_0^{-t,x}(\omega)$.

On this purpose, consider an auxiliary stochastic differential equation

$$dZ_t + (AZ_t + \alpha Z_t)dt = \sqrt{Q}dW_t,$$

where α is a positive constant, which will be specified later. As recalled in appendix **??**, a stationary solution of this equation is given by the Ornstein-Uhlenbeck process

$$Z_t = \int_{-\infty}^{t} e^{-(A+\alpha)(t-s)}\sqrt{Q}dW_s.$$

If we write shortly X_t for the solution of equation (2.10), the process given by the difference $Y_t = X_t - Z_t$ satisfies the random equation

$$\frac{dY_t}{dt} + AY_t + B(Y_t + Z_t, Y_t + Z_t) = -\alpha Z_t + f.$$

Taking the scalar product with Y_t, one finds

$$\frac{1}{2}\frac{d|Y_t|^2}{dt} = -\langle AY_t, Y_t\rangle - \langle B(Y_t + Z_t, Y_t + Z_t), Y_t\rangle - \langle \alpha Z_t, Y_t\rangle + \langle f, Y_t\rangle.$$

Using the property of A and B one gains the inequality

$$\begin{aligned}\frac{1}{2}\frac{d|Y_t|^2}{dt} + \lambda|Y_t|^2 &\leq -\langle B(Y_t + Z_t, Z_t), Y_t\rangle - \langle \alpha Z_t, Y_t\rangle + \langle f, Y_t\rangle \\ &\leq C|Y_t|^2|Z_t| + C|Y_t||Z_t|^2 + \alpha|Y_t||Z_t| + |f||Y_t|.\end{aligned}$$

Apply now the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, with $a = \sqrt{2\epsilon}|Y_t|$, $b = \frac{C}{\sqrt{2\epsilon}}(|Z_t|^2 + \alpha|Z_t| + |f|)$, where $0 < \epsilon < \lambda$, and find

$$\frac{1}{2}\frac{d|Y_t|^2}{dt} + \lambda|Y_t|^2 \leq C|Y_t|^2|Z_t| + \epsilon|Y_t|^2 + \frac{C}{\epsilon}(|Z_t|^2 + \alpha|Z_t| + |f|)^2,$$

which implies

$$\frac{d|Y_t|^2}{dt} + (\lambda_1 - C_1|Z_t|)|Y_t|^2 \leq C_1(|Z_t|^2 + \alpha|Z_t| + |f|)^2,$$

for some constants λ_1 and C_1. Using a differential version of the Gronwall lemma, proved by differentiating $|Y_t|^2 \exp\left(\int_{t_0}^t (\lambda_1 - C_1|Z_s|)ds\right)$, one finds

$$|Y_t^{t_0,x}|^2 \leq |x|^2 e^{-\int_{t_0}^t (\lambda_1 - C_1|Z_s|)ds} + \int_{t_0}^t e^{-\int_s^t (\lambda_1 - C_1|Z_u|)du} C_1(|Z_s|^2 + \alpha|Z_s| + |f|)^2 ds$$

and the inequality for $Y_0^{-t,x}$

$$\begin{aligned}|Y_0^{-t,x}|^2 &\leq |x|^2 e^{-\int_{-t}^0 (\lambda_1 - C_1|Z_s|)ds} \\ &\quad + \int_{-t}^0 e^{-\int_s^0 (\lambda_1 - C_1|Z_u|)du} C_1(|Z_s|^2 + \alpha|Z_s| + |f|)^2 ds.\end{aligned} \tag{3.2}$$

Recall now that the law of large numbers for stationary processes (theorem (**??**)) implies that

$$\lim_{t\to-\infty} \frac{1}{-t}\int_t^0 |Z_s|ds = E[Z_0].$$

Then if α is such that $C_1E[Z_0] \leq \frac{\lambda_1}{2}$, then we have $-\lambda_1 + C_1E[Z_0] \leq -\frac{\lambda_1}{2}$, and

$$\lim_{t\to-\infty} \frac{1}{t}\int_t^0 (\lambda_1 - C_1|Z_s|)ds = -\lambda_1 + C_1E[Z_0] \leq -\frac{\lambda_1}{2}.$$

This implies that

$$\lim_{t\to\infty} e^{-\int_{-t}^0 (\lambda_1 - C_1|Z_s|)ds} = 0,$$

which, together with the fact that

$$\lim_{t\to-\infty} \frac{(|Z_t|^2 + \alpha Z_t + |f|)^2}{t^\beta} = 0$$

for almost every ω, for β sufficiently big, gives

$$\int_{-\infty}^0 e^{-\int_s^0(\lambda_1 - C_1|Z_u|)du} C_1(|Z_s|^2 + \alpha|Z_s| + |f|)^2 ds < \infty.$$

The application of the above estimates to (3.2) gives the existence, for almost every ω, of a constant $C(\omega)$ such that, for each bounded set B,

$$\sup_{x\in B} |Y_0^{-t,x}(\omega)|^2 \leq C(\omega)$$

for t sufficiently big. Then, for almost every ω, there exists an $R(\omega)$ such that, for each bounded set B there exists a t_B such that

$$\sup_{x\in B} |X_0^{-t,x}(\omega)|^2 \leq \sup_{x\in B} C(|Y_0^{-t,x+Z_0(\omega)}(\omega)|^2 + |Z_0(\omega)|^2) \leq D(\omega)$$

for $t \geq t_B$, as wanted.

3.2 Random Measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and (X, d) be a Polish space, i.e. a complete separable metric space, with Borel σ-algebra $\mathcal{B}$. Denote by $B_b(X)$ (resp. $C_b(X)$) the space of real bounded measurable (resp. continuous) functions on X. In this setting a *random measure* is a map

$$\begin{array}{rcl} \mu : \mathcal{B} \times \Omega & \to & [0,1] \\ (B, \omega) & \mapsto & \mu_\omega(B), \end{array}$$

such that

(i) for P-almost every $\omega \in \Omega$, $B \mapsto \mu_\omega(B)$ is a probability measure;

(ii) for each $B \in \mathcal{B}$, the map $\omega \mapsto \mu_\omega(B)$ is measurable.

The random measure μ will also be indicated as $\mu = (\mu_\omega)_{\omega\in\Omega}$, and considered as a family of measures on $(X, \mathcal{B})$, indexed on Ω. Imposing condition (ii) to the family $(\mu_\omega)_{\omega\in\Omega}$ is equivalent to asking that, given any function $f \in C_b(X)$, the map

$$\omega \mapsto \mu_\omega(f) = \int_X f(x)\mu_\omega(x)$$

is a random variable (see appendix **??**). Such families of measures are also called *Markov kernels* or *transition probabilities*. One of the reasons for referring to them with a different name is the necessity, in RDS theory, of emphasizing the asymmetric roles of the spaces Ω and X, the first representing the possible noises and the second supporting the dynamic.

See [14] for a complete overview of random measures on Polish spaces.

3.3 Random measures and random dynamical systems

Let φ be a random dynamical system on a Polish space $(X, \mathcal{B})$, over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$, and let μ be a random measure on $X \times \Omega$.

Definition 72 *The random measure $\mu = (\mu_\omega)_{\omega\in\Omega}$ is said to be* invariant *for the random dynamical system φ if, for all $t \in \mathbb{T}$, one has*

$$E[\varphi(t,\cdot)\mu_\cdot(A)|\theta(t)^{-1}\mathcal{F}] = \mu_{\theta(t)\cdot}(A), \quad \textit{for all} \;\; A \in \mathcal{B}. \tag{3.3}$$

If θ is measurably invertible (for example if $\mathbb{T}$ is a group), then $\theta(t)^{-1}\mathcal{F} = \mathcal{F}$ and μ is invariant if and only if for all $t \in \mathbb{T}$

$$\varphi(t,\omega)\mu_\omega = \mu_{\theta(t)\omega} \quad \text{for } P\text{-almost every } \omega \in \Omega. \tag{3.4}$$

Example 73 *Suppose there exists a random variable $x_0 : \Omega \to X$ such that*

$$\varphi(t,\omega)x_0(\omega) = x_0(\theta_t\omega) \;\; \textit{for } P\textit{-almost every } \omega \in \Omega. \tag{3.5}$$

Then the random measure $\mu_\omega = \delta_{x_0(\omega)}$ is invariant. It is called a random Dirac measure. *See example 39.*

Consider the probability measure λ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$, given by

$$\lambda(A) = \int_\Omega P(d\omega) \int_X \mu_\omega(dx) I_A(\omega, x), \;\; \text{for each } A \in \mathcal{F} \otimes \mathcal{B}. \tag{3.6}$$

We have the following proposition.

Proposition 74 *The random measure $\mu = (\mu_\omega)_{\omega\in\Omega}$ is invariant for the RDS φ if and only if the measure λ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ given by (3.6) is invariant for the skew product associated to φ.*

Proof. Observe that, for each $F \in \mathcal{F}$, $B \in \mathcal{B}$,

$$\begin{aligned}
(\Theta_t\lambda)(F \times B) &= \lambda\{\Theta_t \in F \times B\} \\
&= \int_\Omega P(d\omega) \int_X \mu_\omega(dx) I_{\{\Theta_t \in F \times B\}}(\omega, x) \\
&= \int_\Omega P(d\omega) \int_X \mu_\omega(dx) I_{\{\theta_t \in F, \varphi(t,\cdot)\cdot \in B\}}(\omega, x) \\
&= \int_{\theta(t)^{-1}F} P(d\omega) \int_X \mu_\omega(dx) I_{\{\varphi(t,\cdot)\cdot \in B\}} \\
&= \int_{\theta(t)^{-1}F} P(d\omega) \int_X \mu_\omega\{x | \varphi(t,\omega)x \in B\} \\
&= \int_{\theta(t)^{-1}F} P(d\omega) \int_X (\varphi(t,\omega)\mu_\omega)(B), \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
\lambda(F \times B) &= \int_\Omega P(d\omega) I_F(\omega) \mu_\omega(B) = \int_\Omega P(d\omega) I_F(\theta(t)\omega) \mu_\omega(B) \\
&= \int_{\theta(t)^{-1}F} P(d\omega) \mu_{\theta(t)\omega}(B). \qquad (3.8)
\end{aligned}$$

Suppose then that (3.3) holds: then clearly (3.7) equals (3.8), and $\Theta_t\lambda = \lambda$. Conversely, suppose that λ is invariant under Θ_t, then for each $B \in \mathcal{B}$ and $t \in \mathbb{T}$, the function $\omega \mapsto \mu_{\theta(t)\omega}(B)$ is a version of the conditional expectation of the function $\omega \mapsto \varphi(t,\omega)\mu_\omega(B)$, with respect to $\theta(t)^{-1}\mathcal{F}$. Since $\mathcal{B}$ is countably generated, one can find an exceptional set which fits all $B \in \mathcal{B}$, and (3.3) holds. ■

We write $\mathcal{P}_P(\Omega \times X)$ for the set of probability measures λ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ of the form (3.6) for some random measure μ. Observe that Θ_t maps $\mathcal{P}_P(\Omega \times X)$ into itself. For each $\lambda \in \mathcal{P}_P(\Omega \times X)$, the random measure μ such that (3.6) holds is P-almost surely unique ([3], proposition 1.4.3). It is called the *factorization* of λ.

We call $\mathcal{I}_P(\varphi)$ the subset of $\mathcal{P}_P(\Omega \times X)$ of measures λ for which the associated random measure μ is invariant.

Define the space $L^1_P(\Omega, \mathcal{C}_b(X))$ of functions $f : \Omega \to \mathcal{C}_b(X)$ such that the map $(\omega, x) \mapsto f(\omega)(x)$ is measurable, and the integral $\int_\Omega \sup_{x\in X} |f(\omega)(x)| dP$ is finite. One can endow the space $\mathcal{P}_P(\Omega \times X)$ with the *topology of weak convergence*, i.e. the smallest topology such that the maps $\mu \mapsto \mu(f) = \int_{\Omega\times X} f d\mu$ are continuous for each $f \in L^1_P(\Omega, \mathcal{C}_b(X))$, $\mu \in \mathcal{P}_P(\Omega \times X)$.

Consider the action of the skew product $(\Theta_t)_t$ of φ on functions f belonging to $L^1_P(\Omega, \mathcal{C}_b(X))$ given by $\Theta_t f = f \circ \Theta_t$. Observe that $\Theta_t f$ belongs to $L^1_P(\Omega, \mathcal{C}_b(X))$ whenever f is in $L^1_P(\Omega, \mathcal{C}_b(X))$: for each t, the map $(\omega, x) \mapsto f(\theta_t\omega)(\varphi(t,\omega)x)$ is measurable, continuous in

x for each fixed ω and

$$\int_\Omega \sup_{x\in X} |(\Theta_t f)(\omega)(x)| dP = \int_\Omega \sup_{x\in X} |f(\theta_t\omega)(\varphi(t,\omega)x)| dP \tag{3.9}$$

$$\leq \int_\Omega \sup_{x\in X} |f(\theta_t\omega)(x)| dP < \infty. \tag{3.10}$$

Proposition 75 *If φ is a continuous random dynamical system on a Polish space X, the map $\mu \mapsto \Theta_t\mu$ on $\mathcal{P}_P(\Omega\times X)$ is affine and continuous, and the set $\mathcal{I}_P(\varphi)$ is convex and closed.*

Proof. The first property is obvious. For the second, suppose (μ^α) is a net converging to μ, i.e., for each $f \in L^1_P(\Omega, \mathcal{C}_b(X))$, $\mu^\alpha(f)$ converges to $\mu(f)$. Then $(\Theta_t\mu^\alpha)(f) = \mu^\alpha(\Theta_t f) \to \mu(\Theta_t f) = (\Theta_t\mu)(f)$, as $\Theta_t f$ belongs to $L^1_P(\Omega, \mathcal{C}_b(X))$.

The second part follows from proposition (74), as the set $\mathcal{I}_P(\varphi)$ concides with the set of fixed points of the map $\mu \mapsto \Theta_t\mu$ on $\mathcal{P}_P(\Omega\times X)$. ■

Existence of measures in $\mathcal{I}_P(\varphi)$ can sometimes be proved through a Krylov-Bogolyubov argument, as described in the following proposition:

Proposition 76 *Let φ be a continuous random dynamical system on a Polish space X, with continuous time $\mathbb{T}$, and let ν be in $\mathcal{P}_P(\Omega\times X)$. For each $T\in\mathbb{T}$, $T>0$, consider the measure μ_T defined by the means*

$$\mu_T(A) = \frac{1}{T}\int_0^T (\Theta_t\nu)(A), \quad \forall A \in \mathcal{F}\otimes\mathcal{B}. \tag{3.11}$$

Then every limit point of $(\mu_T)_T$ for $T\to\infty$, in the topology of weak convergence, is in $\mathcal{I}_P(\varphi)$.

Proof. Let $(T_k)_k$ be a sequence such that $T_k\to\infty$ and $\mu_{T_k}\to\mu$ weakly. For each $t>0$, we show that $(\Theta_t\mu)(f) = \mu(f)$ for all $f\in L^1_P(\Omega,\mathcal{C}_b(X))$. The equality $\Theta_t\mu=\mu$ then follows by the fact that all functions of the form $I_{A\times B}$, with $A\in\mathcal{F}$ and B closed belonging to $\mathcal{B}$, can be approximated by decreasing sequences in $L^1_P(\Omega,\mathcal{C}_b(X))$.

Observe that, if f is in $L^1_P(\Omega,\mathcal{C}_b(X))$, then $\mu_T f = \frac{1}{T}\int_0^T \nu(\Theta_t f)dt$. For each $t>0$, we have then that

$$\begin{aligned}
|(\Theta_t\mu_{T_k})(f) - \mu_{T_k}(f)| &= \frac{1}{T_k}\left|\int_0^{T_k} \nu(\Theta_{t+s}f)ds - \int_0^{T_k}\nu(\Theta_s f)ds\right| \\
&= \frac{1}{T_k}\left|\int_t^{T_k+t} \nu(\Theta_s f)ds - \int_0^{T_k}\nu(\Theta_s f)ds\right| \\
&= \frac{1}{T_k}\left|\int_{T_k}^{T_k+t} \nu(\Theta_s f)ds - \int_0^{t}\nu(\Theta_s f)ds\right| \\
&\leq \frac{2t}{T_k}||f||_{L^1_P(\Omega,\mathcal{C}_b(X))} \to 0 \text{ as } k\to\infty.
\end{aligned}$$

This convergence, together with the continuity of the map $\mu \mapsto \Theta_t\mu$, imply that $\Theta_t\mu = \mu$, for each $t > 0$. If t is in $\mathbb{T}$ and $t < 0$, then the invariance of Θ_t follows by the equality $\Theta_t\mu = \Theta_t\Theta_{-t}\mu$. ■

The existence of limit points for the sequence (3.11), and then of random invariant measures for φ, can be established through an anologous of Prohorov theorem for random measures.

Definition 77 *A set of measures $\Gamma \subset \mathcal{P}_P(\Omega \times X)$ is said to be* tight *if for every $\epsilon > 0$ there exists a compact set $C_\epsilon \subset X$ such that, for each $\lambda \in \Gamma$, $\lambda(\Omega \times C_\epsilon) \geq 1 - \epsilon$, or equivalently $\int_\Omega P(d\omega)\mu_\omega(C_\epsilon) \geq 1 - \epsilon$, if μ_ω is the factorization of λ.*

The following theorem is due to Crauel ([14], theorem 4.4):

Theorem 78 *If $\Gamma \subset \mathcal{P}_P(\Omega \times X)$ is tight, then every sequence $(\mu_n)_{n\in\mathbb{N}} \subset \Gamma$ admits a convergent subsequence.*

If φ is a continuous random dynamical system on a compact metric space X, random invariant measures always exist. Denote by $L_P^\infty(\Omega, \mathcal{M}(X))$ the set of P-essentially bounded measurable functions with values on the set of signed measures $\mathcal{M}(X)$ on $(X, \mathcal{B})$ of finite total variation. The set of random measures on X, which is a closed subset of $L_P^\infty(\Omega, \mathcal{M}(X))$, is compact by the Banach-Alaoglu's theorem. The set $\mathcal{P}_P(\Omega \times X)$ can be identified with the set of random measures, through the uniqueness of the factorization. We know by proposition (75) that the maps $(\Theta_t)_t$ are a family of commuting affine maps on $\mathcal{P}_P(\Omega \times X)$, and the existence of random invariant measures can be then proved by recurring to the

Theorem 79 *(Markov-Kakutani fixed point theorem) Let S be a nonempty compact and convex subset of a normed linear space X. If $\mathcal{H}$ is a commuting family of affine maps on S, then there exists an $x \in S$ such that $x = f(x)$ for all $f \in \mathcal{H}$.*

3.3.1 Support of a random measure

We denote by supp(σ) the support of a measure σ on $(X, \mathcal{B})$.

Definition 80 *Consider a random measure μ on $X \times \Omega$. Denote by N_μ the P-null set outside of which μ_ω is a probability measure. The* support *C of μ is the map on Ω with values on the subsets of X, given by*

$$\omega \mapsto C_\omega = \begin{cases} supp(\mu_\omega) & if\ \omega \in N_\mu^c, \\ X & if\ \omega \in N_\mu, \end{cases}$$

A map

$$\begin{aligned} A : \Omega &\to \mathcal{P}(X) \\ \omega &\mapsto A_\omega, \end{aligned}$$

is called a *random closed (open, compact) set* if A_ω is closed (open, compact) for each $\omega \in \Omega$, and the map $\omega \mapsto d(x, A_\omega)$ is measurable for each $x \in X$.

Proposition 81 *The support of a random measure is a closed random set.*

Proof. It is sufficient to prove that for each $\delta > 0$ the set $\{\omega \in \Omega \mid d(\omega, C_\omega) < \delta\}$ belongs to $\mathcal{F}$. One has

$$\begin{aligned} \{\omega \in \Omega \mid d(\omega, C_\omega) < \delta\} &= \{\omega \in \Omega \mid C_\omega \cap B(x,\delta) \neq \emptyset\} \\ &= N_\mu \cup \{\omega \in N_\mu^c : \operatorname{supp}(\mu_\omega) \cap B(x,\delta) \neq \emptyset\} \\ &= N_\mu \cup \{\omega \in N_\mu^c : \mu_\omega(B(x,\delta)) > 0\}, \end{aligned}$$

which belongs to $\mathcal{F}$ by definition of random measure. ■

Consider now a continuous RDS φ over a metric DS $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{T}})$. We say that a random set C is *forward (resp. backward) invariant* for the RDS φ if for each $t > 0$ (resp. $t < 0$)

$$\varphi(t,x) C_\omega \subset C_{\theta(t)\omega} \quad \text{for } P\text{-almost every } \omega \in \Omega.$$

C is called *invariant* if for each $t \in \mathbb{T}$

$$\varphi(t,\omega) C_\omega = C_{\theta(t)\omega} \quad \text{for } P\text{-almost every } \omega \in \Omega.$$

Proposition 82 *If μ is a random invariant measure for the continuous RDS φ, then the support C of μ is forward invariant if θ is invertible, and invariant if time is two-sided.*

Proof. We have that

$$\varphi(t,\omega) C_\omega \subset \operatorname{supp}(\varphi(t,\omega)\mu_\omega) = \operatorname{supp}(\mu_{\theta(t)\omega}) = C_{\theta(t)\omega},$$

with equality if time is two-sided. ■

Denote by $\mathcal{P}_P(C)$ the set of measures on $\Omega \times X$ supported on a random set C. Observe that C is (forward) invariant for φ if and only if $C \subset \Theta_t^{-1} C$ ($C = \Theta_t^{-1} C$) for $t > 0$, where Θ_t is the skew product associated to φ. This implies that if K is an invariant compact random set, the set $\mathcal{P}_P(K)$ is invariant for Θ_t. If X is compact, then $\mathcal{P}_P(K)$ is convex and compact and one can apply the Markov-Kakutani fixed point theorem to prove the existence of random invariant measures supported by K. If X is not compact, the existence of a random invariant measures supported on K still holds, with a more delicate proof, which can be found in [14] (corollary 6.13).

3.3.2 Markov measures

Let φ be a random dynamical system with times $\mathbb{R}^+$, over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(t))_{t\in\mathbb{R}})$, and we suppose that the one point motion associated to φ is a Markov process with transition probabilities $(P_t)_{t\in\mathbb{R}^+}$.

Markov measures are random measures with particular measurability properties, which are useful for establishing a corrispondence between random invariant measures for a random dynamical system φ and invariant measures for the associated Markov semigroup.

Define the σ-algebras

$$\mathcal{F}^+ = \sigma\{\omega \mapsto \varphi(t, \theta_s\omega, x) : x \in X, t, s \geq 0\},$$

$$\mathcal{F}^- = \sigma\{\omega \mapsto \varphi(t, \theta_{-s}\omega, x) : x \in X, 0 \leq t \leq s\}$$

representing respectively the future and the past of the RDS φ.

Random measures which are measurable with respect to $\mathcal{F}^-$, are called *Markov* random measures. Next theorem gives a motivation for this definition.

Proposition 83 *Let $\mu = (\mu_\omega)_{\omega\in\Omega}$ be a Markov random invariant measure for the random dynamical system φ, and suppose that $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent. Then $\rho = P\mu$ is an invariant measure for the Markov semigroup P_t associated to φ.*

Proof. Take $f \in B_b(X)$ and $t \in \mathbb{T}$. Then

$$\begin{aligned}(\rho P_t)[f] = \rho[P_t(f)] &= \int_X \rho(dx) \int_X P_t(x, dy) f(y)\\ &= \int_X P\mu(dx) \int_\Omega P(d\omega) f(\varphi(t, \omega, x)).\end{aligned}$$

Apply the last part of proposition (**??**) in appendix **??**, with $\mathcal{G} = \mathcal{F}^+$, and $h = f(\varphi(t, \cdot, \cdot))$, and find

$$(\rho P_t)[f] = \int_\Omega P(d\omega) \int_X \mu_\omega(dy) f(\varphi(t, \omega, y)).$$

Using the invariance of μ with respect to φ, and that of P with respect to θ, we can conclude

$$\begin{aligned}(\rho P_t)[f] &= \int_\Omega P(d\omega) \int_X \mu_{\theta(t)\omega}(dy) f(y)\\ &= \int_\Omega P(d\omega) \int_X \mu_\omega(dy) f(y) = \rho[f].\end{aligned}$$

■

Existence of invariant Markov measures supported by invariant random compact set is established by the following theorem, which can be found in [14]:

Theorem 84 *If K is a forward invariant random compact set which is measurable with respect to $\mathcal{F}^-$, then there exists an invariant Markov measure supported by K.*

Observe for istance that, in the setting of example 73, if x_0 is $\mathcal{F}^-$-measurable, then μ is a Markov random invariant measure, supported on $\{x_0(\omega)|\omega \in \Omega\}$.

3.4 From invariant measures to invariant random measures

In this paragraph we consider a continuous RDS φ generated by a stochastic differential equation as in Chapter 1, over the canonical metric dynamical system $(\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{s\leq t}, (\theta(t))_{t\in\mathbb{R}}, P)$. We want to present a result on convergence of random measures which leads to the construction of a random invariant measure for φ, starting from an invariant measure for the one-point motion P_t. The same theorem holds for a discrete random dynamical system on a Polish space $(X, \mathcal{B})$, given by the product of i.i.d. random mappings. The result has been studied in several versions for istance by Le Jan [35], Kunita [**?**] and Baxendale [5].

Suppose ρ is an invariant measure for the Markov semigroup associated to the one-point motion of φ, i.e.

$$\int_X \rho(dx) P_t(x, B) = \rho(B) \quad \forall x \in X, B \in \mathcal{B}, t \in \mathbb{R}$$

or more compactly

$$\rho P_t = \rho \quad \text{for each } t \in \mathbb{R},$$

with $P_t(x, B) = P\{\omega \in \Omega : \varphi(t, \omega)x \in B\}$ for each $x \in X$, $B \in \mathcal{B}$, $t \in \mathbb{R}$.

Define, for each $t \in \mathbb{R}^+$, a random measure $\mu^{(t)} = (\mu_\omega^{(t)})_{\omega\in\Omega}$ on X, by

$$\mu_\omega^{(t)} = \varphi(t, \theta(-t)\omega, \cdot)\rho.$$

This is a kind of pullback of the measure ρ. For each function f in $\mathcal{C}_b(X)$, the process $(\mu^{(t)}(f))_{t\in\mathbb{R}^+}$ is adapted to $(\mathcal{F}_{-t}^0)_{t\in\mathbb{R}^+}$. Moreover, each random variable $\mu^{(t)}(f)$ is in $L^1(\Omega)$.

Proposition 85 *For each function f in $\mathcal{C}_b(X)$, the process $(\mu^{(t)}(f))_{t\in\mathbb{R}^+}$ is a martingale with respect to the filtration $\{\mathcal{F}_{-t}^0\}_{0\leq t<\infty}$.*

Proof. Given s, $t \in \mathbb{R}^+$, recalling the definition of a RDS we can write

$$\begin{aligned}
\mu^{(t+s)} f(\omega) &= \int_X f(\varphi(t+s, \theta(-t-s)\omega, x))\rho(dx) \\
&= \int_X f(\varphi(t, \theta(-t)\omega, \cdot) \circ \varphi(s, \theta(-t-s)\omega, x))\rho(dx) \\
&= \int_X f(\varphi(t, \theta(-t)\omega, x))[\varphi(s, \theta(-t-s)\omega, \cdot)\rho](dx) \\
&= \int_X f(\varphi(t, \theta(-t)\omega, x))\lambda_\omega(dx),
\end{aligned}$$

where $\lambda = (\varphi(s, \theta(-t-s)\omega, \cdot)\rho)_{\omega\in\Omega}$. The map $\omega \mapsto f(\varphi(t, \theta(-t)\omega, x))$ is $\mathcal{F}^0_{-t}$-measurable, while $\omega \mapsto \varphi(s, \theta(-t-s)\omega, x)$ is $\mathcal{F}^0_{-t}$-independent. Then we can apply part (*ii*) of proposition (**??**) to find

$$E[\mu^{(t+s)} f|\mathcal{F}^0_{-t}] = \int_X P\lambda(dx) f(\varphi(t, \theta(-t)\cdot, x)).$$

One observes that, if $A \in \mathcal{B}$, then

$$\begin{aligned} P\lambda(A) &= \int_\Omega P(d\omega)[\varphi(s, \theta(-t-s)\omega, \cdot)\rho](A) \\ &= \int_\Omega P(d\omega) \int_X \rho(dx) I_A(\varphi(s, \theta(-t-s)\omega, x)) \\ &= \int_X \rho(dx) \int_\Omega P(d\omega) I_A(\varphi(s, \omega, x)) \\ &= \int_X \rho(dx) P_s(x, A) = (\rho P_s)(A), \end{aligned}$$

and proves the equality $P\lambda = \rho P_s$, which gives, by invariance of ρ

$$\begin{aligned} E[\mu^{(t+s)} f|\mathcal{F}^0_{-t}] &= \int_X f(\varphi(t, \theta(-t)\cdot, x))(\rho P_s)(dx) \\ &= \int_X f(\varphi(t, \theta(-t)\cdot, x))\rho(dx) = \mu^{(t)} f. \end{aligned}$$

■

Observe that, if M is the martingale of the theorem, for each $p \geq 1$, we have

$$E[|M_t|^p] = E[|\mu^{(t)}(f)|^p] = E\left[\left|\int_X f(\varphi(t, \theta(-t)\cdot, x))\rho(dx)\right|^p\right] \leq ||f||^p_\infty,$$

in particular M is uniformly integrable. We can then apply this theorem on convergence of right-continuous martingales (see [32], theorem 3.15, for the proof):

Theorem 86 *On a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t<\infty}, P)$, let $(X_t)_{0\leq t<\infty}$ be a right-continuous submartingale such that $\sup_{t\geq 0} E[X_t^+] < \infty$. Then the limit $X_\infty(\omega) = \lim_{t\to\infty} X_t(\omega)$ exists for almost every $\omega \in \Omega$, and X_∞ is in L^1.*

One can conclude that

$$\mu^{(t)}(f) \text{ converges a.s. for all } f \in C_b(X). \tag{3.12}$$

Next question is whether this kind of convergence implies the existence of a random measure $(\mu_\omega)_{\omega\in\Omega}$ such that

$$\mu_\omega^{(t)} \text{ converges to } \mu_\omega \text{ weakly for almost all } \omega \in \Omega. \tag{3.13}$$

Such a random measure μ is called the *statistical equilibrium* of the random dynamical system.

Condition (3.13), being clearly necessary for condition (3.12), has been shown to be sufficient if the space X is Radon, in a recent paper by Berti, Pratelli and Rigo [?]. A metric space is said to be *Radon* if every Borel probability measure on X is *tight*, i.e., taken any probability measure μ on $(X, \mathcal{B})$, for each $\epsilon > 0$ there exists a compact subset $C_{\mu,\epsilon} \subset X$ such that $\mu(C_{\mu,\epsilon}) \geq 1 - \epsilon$.

Since every Polish space is Radon (see [8], theorem 1.4), we can conclude that there exists a random measure μ such that (3.13) is satisfied.

Remark 87 *More generally, Berti, Pratelli and Rigo proved that for conditions (3.12) and (3.13) to be equivalent it is sufficient that the family of measures $(\lambda^{(t)})_{t\geq 0}$ defined by*

$$\lambda^{(t)}(A) = \int_\Omega P(d\omega) \int_X \mu_\omega^{(t)}(dx) I_A(\omega, x) = P[\mu^{(t)} I_A], \quad \textit{for each } A \in \mathcal{F} \otimes \mathcal{B}$$

is tight (definition (77)). This condition is also easy to verify in our case: observe that, for each $f \in C_b(X)$, the expected value of $\mu^{(t)} f$ is given by:

$$\begin{aligned} P[\mu^{(t)} f] &= \int_\Omega P(d\omega) \int_X \rho(dx) f(\varphi(t, \theta(-t)\omega, x)) \\ &= \int_X \rho(dx) \int_\Omega P(d\omega) f(\varphi(t, \theta(-t)\omega, x)) \\ &= \int_X \rho(dx) \int_\Omega P(d\omega) f(\varphi(t, \omega, x)) \\ &= \int_X \rho(dx) P_t f(x) = \rho[f]. \end{aligned} \tag{3.14}$$

Given any $\epsilon > 0$, take a compact set $C \subset X$ such that $\rho(C) \geq 1 - \epsilon$, and use (3.14) with $f = I_C$, finding $P\mu^{(t)}(C) = \rho(C) \geq 1 - \epsilon$.

For completeness, we give also the original proof of Kunita for the existence of a random measure μ such that (3.13) holds, in case $X = \mathbb{R}^n$. The martingale $\mu^{(t)} f$ converges almost surely for each f in $C_b(\mathbb{R}^n)$. In order to find an exceptional null set which fits every $f \in C_b(\mathbb{R}^n)$, consider a countable dense subset D of $C_b(\mathbb{R}^n)$. Then there exists a subset $\overline{\Omega}$ of Ω of full measure such that $\mu_\omega^{(t)} f$ converges for all $\omega \in \overline{\Omega}$ and $f \in D$. For each $\omega \in \overline{\Omega}$, define a positive linear functional μ^∞ on the linear subspace $\mathcal{D}$ of $C_b(\mathbb{R}^n)$ generated by D, as

$$\mu_\omega^\infty f := \lim_{t\to\infty} \mu_\omega^{(t)} f, \quad \text{for all } f \in \mathcal{D}.$$

We have $|\mu^\infty_\omega| \le ||f||_{C_b(\mathbb{R}^n)}$ for all $f \in \mathcal{D}$. Then we can apply the Hahn-Banach theorem and extend μ^∞_ω to a positive linear functional on $C_b(\mathbb{R}^n)$, such that $|\mu^\infty_\omega| \le ||f||_{C_b(\mathbb{R}^n)}$ for all $f \in C_b(\mathbb{R}^n)$. For each $\omega \in \overline{\Omega}$, the Riesz theorem now gives a measure μ_ω satisfying $\mu_\omega f = \mu^\infty_\omega f$. The convergence $\mu^{(t)}_\omega(\Omega) \to \mu_\omega(\Omega)$ implies that μ_ω is a probability measure. To see that $\mu = (\mu_\omega)_{\omega\in\Omega}$ is a random measure, observe that, for each $f \in C_b(X)$, $\mu(f)$ is pointwise limit of $\mathcal{F}^0_{-\infty}$-measurable functions; $\omega \mapsto \mu_\omega(f)$ is then measurable with respect to $\mathcal{F}^0_{-\infty}$, and μ is a Markovian random measure.

We can now prove convergence (3.13). Let f be in $C_b(\mathbb{R}^n)$, and let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $||f_m - f||_{C_b(\mathbb{R}^n)} \to 0$ as $m \to \infty$. Then we have:

$$\begin{aligned}\left|\int_X f d\mu^{(t)}_\omega - \int_X f d\mu_\omega\right| &\le \left|\int_X f d\mu^{(t)}_\omega - \int_X f_m d\mu^{(t)}_\omega\right| \\ &\quad + \left|\int_X f_m d\mu^{(t)}_\omega - \int_X f_m d\mu_\omega\right| + \left|\int_X f_m d\mu_\omega - \int_X f d\mu_\omega\right| \\ &\le \left|\int_X f_m d\mu^{(t)}_\omega - \int_X f_m d\mu_\omega\right| + 2||f_m - f||_{C_b(\mathbb{R}^n)}.\end{aligned}$$

For each $\epsilon > 0$, take m_0 such that $||f_{m_0} - f||_{C_b(\mathbb{R}^n)} < \epsilon/4$, and and t_0 such that for each $t \ge t_0$ $\left|\int_X f_{m_0} d\mu^{(t)}_\omega - \int_X f_{m_0} d\mu_\omega\right| < \epsilon/2$. Then for each $t \ge t_0$ we have $\left|\int_X f d\mu^{(t)}_\omega - \int_X f d\mu_\omega\right| < \epsilon$ as wanted.

To prove invariance of μ, observe that $\varphi(t,\omega)\mu_\omega$ is the limit of

$$\varphi(t,\omega)\varphi(s,\theta(-s)\omega)\rho = \varphi(s+t,\theta(-s)\omega)\rho$$

for $s \to \infty$; by taking $u = s + t$ one sees that this limit equals

$$\lim_{u\to\infty} \varphi(u,\theta(t-u)\omega)\rho = \lim_{u\to\infty} \varphi(u,\theta(-u)\theta(t)\omega)\rho = \mu_{\theta(t)\omega},$$

as desired.

From (3.14) it follows in particular that $P\mu = \rho$.

Example 88 *Consider again the random dynamical system φ of examples 57 and 39. Observe that, for each $t \ge 0$, $x \in \mathbb{R}$ and almost every $\omega \in \Omega$*

$$\begin{aligned}\varphi(t,\theta_{-t}\omega)x &= e^{-\alpha t}x - \sigma\omega(-t) - \int_0^t \alpha e^{-\alpha(t-s)}\sigma(\omega(s-t) - \omega(-t))ds \\ &= e^{-\alpha t}x - \sigma\omega(-t) - \int_0^t \alpha e^{-\alpha(t-s)}\sigma\omega(s-t)ds + \sigma\omega(-t)(1 - e^{-\alpha t}) \\ &= e^{-\alpha t}x - \sigma\omega(-t)e^{-\alpha t} - \int_{-t}^0 \alpha e^{\alpha s}\sigma\omega(s)ds,\end{aligned}$$

which converges to $-\int_{-\infty}^{0} \alpha e^{\alpha s}\sigma\omega(s)ds = x_0(\omega)$ *for almost every* ω*. Consequently, for each probability measure* λ *on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$*, the pullbacks* $\varphi(t, \theta_{-t}\omega)\lambda$ *converge to* $\delta_{x_0(\omega)}$ *as* $t \to \infty$*, for almost every* ω*. In particular this holds for the unique invariant measure of the Markov semigroup associated to* φ *(see appendix* **??***), and* $(\delta_{x_0(\omega)})_{\omega\in\Omega}$ *is the statistical equilibrium of the system.*

The results of Section 3.3.1 imply that the set of invariant random measures supported on the attractor is non-void.

Remark 89 *In the setting of this paragraph, if a global attractor* A *exists, then the statistical equilibrium* $(\mu_\omega)_{\omega\in\Omega}$ *is supported on* A*.*

Proof. Let ω be such that $\varphi(t, \theta_{-t}\omega)\rho$ converges weakly to μ_ω. For each $n \geq 1$, consider the closed set $U_n = \{x \in X : d(x, A(\omega)) \leq \frac{1}{n}\}$. Given any $\epsilon > 0$, let $B_\epsilon \subset X$ be a compact set such that $\rho(B_\epsilon) \geq 1 - \epsilon$. For every $n \geq 1$, if t is sufficiently big, we have $\varphi(t, \theta_{-t}\omega)B_\epsilon \subset U_n$, by the definition of attractor. Then, for every $n \geq 1$

$$\begin{aligned}\mu_\omega(U_n) &\geq \limsup_{t\to\infty}(\varphi(t, \theta_{-t}\omega)\rho)(U_n) = \limsup_{t\to\infty} \rho\{x \in X | \varphi(t, \theta_{-t}\omega)x \in U_n\} \\ &\geq \rho(B_\epsilon) \geq 1 - \epsilon,\end{aligned}$$

and consequently $\mu_\omega(U_n) = 1$ for all n. Since $U_n \downarrow A(\omega)$, we have $\mu_\omega(A(\omega)) = \lim_{n\to\infty} \mu_\omega(U_n) = 1$. ■

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