1 Notations

In this chapter we investigate infinite systems of interacting particles subject to Newtonian dynamics. Each particle is characterized by its position an velocity

$$(x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$$

at time t, in dimension d. The index i varies in a countable set I. We call *configuration* the family, denoted generically by Φ:

$$\Phi = (x_i, v_i)_{i \in I}.$$

Particles, as said above, are subject to the classical dynamics

$$x_i''(t) = -\sum_{j \in I \setminus \{i\}} \nabla U(x_i(t) - x_j(t))$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is the so called potential. Unless differently stated, we shall simplify some aspect of our investigation (to concentrate on the difficulties coming from the infinite number of particles) and assume

$$U \in C_c^2\left(\mathbb{R}^d\right)$$

namely twice continuously differentiable with compact support. Denote by $R_0 > 0$ the radius of a ball $B(0, R_0)$ such that

$$U = 0 \text{ outside } B(0, R_0).$$

As usual, we reformulate the system of second-order equations as a system of first-order ones

$$\begin{cases} x_i'(t) = v_i(t) \\ v_i'(t) = -\sum_{j \in I \setminus \{i\}} \nabla U(x_i(t) - x_j(t)) \end{cases}$$

and we consider the Cauchy problem given by these equations and the initial conditions

$$x_i|_{t=0} = x_i^0, \qquad v_i|_{t=0} = v_i^0, \qquad i \in I.$$

1.1 Intuitions about blow-up

The interaction potential U is smooth and compact support, hence we do not have troubles similar to those of point vortices. However, since the number of particles in infinite, we need that the sum $\sum_{j \in I \setminus \{i\}} \nabla U(x_i(t) - x_j(t))$ has only a finite number of terms, for every i and t (otherwise the sum of vectors could even have no meaning). At time $t = 0$ we impose this condition; are we sure that this local finiteness is maintained during the evolution? If not, we speak of blow-up.

In both the following intuitive examples consider the case when $I = \mathbb{Z}^d$ and $x_i^0 = i$, namely partices start uniformly distributed on the lattice.

1. Consider initial velocities which increase when $|i|$ is larger. Assume $U = 0$. It is not difficult to define values v_i such that all particles are in $B(0,1)$ at a given time $t_0 > 0$. Similarly, we may create infinite sectors $S_n \subset \mathbb{Z}^d$ that reach $B(0,1)$ at times t_n, with $\lim_{n\infty} t_n = 0$. We thus see not only that blow-up is possible, without any restriction on U and velocities, but it may also happen immediately, namely a solution may not exist.

2. By the lattice $\mathbb{Z}^d$, the space $\mathbb{R}^d$ is divided in hypercubes (call them cubes) of side one; color every second cube by red. Define the velocities v_i at the corners of each red cube in such a way that they are of intensity bounded, say $|v_i| \leq 1$ and point in the direction of the interior of the red cube, and lead to a collision such that one particle gather almost all the kinetic energy of all the others. This way, after a short time, we have much faster particles moving from the red cubes, one for every cube. Repeating this construction in a selfsimilar way, we may guess it is possible to produce faster and faster particles in shorter and shorter times and thus concentrate in finite time infinitely many particles in $B(0,1)$.

2 Locally finite and uniform configuration spaces

The object $\Phi = (x_i, v_i)_{i \in I}$ belongs to the product space $\left(\mathbb{R}^d \times \mathbb{R}^d\right)^I$. However, the sum $\sum_{j \in I \setminus \{i\}} \nabla U(x_i - x_j)$ should contain only a finite number of terms, otherwise the equations may have no meaning. Thus it is necessary to restrict configurations Φ to the smaller set of *locally finite configurations*

$$\mathcal{L}_f \subset \left(\mathbb{R}^d \times \mathbb{R}^d\right)^I$$

defined as

$$\mathcal{L}_f = \left\{ \Phi = (x_i, v_i)_{i \in I} : \sum_{i \in I} 1_{\{x_i \in B(0,R)\}} < \infty \text{ for every } R > 0 \right\}.$$

Remark 1 *With a suitable metric, $\mathcal{L}_f$ is a complete separable metric space and all functionals $F : \mathcal{L}_f \to \mathbb{R}$ of the form*

$$F(\Phi) = \sum_{i \in I} f(x_i, v_i) \qquad \Phi = (x_i, v_i)_{i \in I}$$

are continuous, where f is continuous bounded and $f(x, v) = 0$ for x outside a bounded set. We shall not use this remark below.

The space of configurations where we look for solutions will be a subspace of $\mathcal{L}_f$

$$X \subset \mathcal{L}_f \subset \left(\mathbb{R}^d \times \mathbb{R}^d\right)^I.$$

In this section we make the simplest choice, which has strong limitations for applications but will lead to a first set of simple results.

Given $\overline{\Phi} = (\overline{x}_j, 0)_{j\in I} \in \mathcal{L}_f$, set

$$X_{\overline{\Phi}} = \left\{ \Phi \in \mathcal{L}_f : \sup_{j\in I} \left(|x_j - \overline{x}_j| + |v_j| \right) < \infty \right\}.$$

Elements of $X_{\overline{\Phi}}$ are not too far from $\overline{\Phi}$ and have not too large velocities, uniformly in $j \in I$. Over this set we define the distance

$$d_\infty \left((x_j, v_j)_{j\in I}, (x'_j, v'_j)_{j\in I} \right) = \sup_{j\in I} \left(\left|x_j - x'_j\right|, \left|v_j - v'_j\right| \right)$$

which makes $\left(X_{\overline{\Phi}}, d_\infty\right)$ a complete separable metric space.

The set $X_{\overline{\Phi}}$ is not a vector space and $\left(X_{\overline{\Phi}}, d_\infty\right)$ is not a Banach space, in spite of the shape of d_∞ which looks like a norm. Following [1], we shall change variables and reduce the problem to a Banach space.

3 Change of variable

In order to have a Banach space, it is sufficient to change variable. We set

$$\begin{aligned} \xi_j(t) &= x_j(t) - \overline{x}_j \\ \xi_i^0 &= x_j^0 - \overline{x}_j \\ \overline{y}_{ij} &= \overline{x}_i - \overline{x}_j \end{aligned}$$

and consider the new system

$$\begin{cases} \xi'_i(t) = v_i(t) \\ v'_i(t) = - \sum_{j\in I\setminus\{i\}} \nabla U \left(\xi_i(t) - \xi_j(t) + \overline{y}_{ij} \right) \end{cases}$$

and we consider the Cauchy problem given by these equations and the initial conditions

$$\xi_i|_{t=0} = \xi_i^0, \qquad v_i|_{t=0} = v_i^0, \qquad i \in I.$$

Let us repeat the notations, adding the symbol Ψ for $\left(\xi_j, v_j\right)_{j\in I}$:

$$\underset{\left(\xi_j, v_j\right)}{\Psi} = \underset{(x_j, v_j)}{\Phi} - \underset{(\overline{x}_j, 0)}{\overline{\Phi}}.$$

We call Y_0 the set $\mathcal{L}_f$ read in the new variables:

$$Y_0 = \left\{ \Psi = (\xi_i, v_i)_{i\in I} : \sum_{i\in I} 1_{\{\xi_i + \overline{x}_i \in B(0,R)\}} < \infty \text{ for every } R > 0 \right\}$$

and, more important, we introduce the Banach space Y:

$$Y = \left\{ \Psi = (\xi_i, v_i)_{i \in I} : \|\Psi\|_\infty := \sup_{i \in I} (|\xi_i| \vee |v_i|) < \infty \right\}.$$

With the norm $\|\cdot\|_\infty$, Y is a separabe Banach space. We have

$$Y + \overline{\Phi} = X_{\overline{\Phi}}.$$

Then we introduce the operator

$$\begin{aligned} B &: Y_0 \to \left(\mathbb{R}^d \times \mathbb{R}^d\right)^I \\ B\left(\left(\xi_j, v_j\right)_{j \in I}\right)_i &= \left(v_i, -\sum_{j \in I \setminus \{i\}} \nabla U\left(\xi_i - \xi_j + \overline{y}_{ij}\right)\right) \end{aligned}$$

and we recognize that our original Cauchy problem is equivalent to

$$\left\{ \begin{array}{rcl} \Psi'(t) &=& B(\Psi(t)) \\ \Psi(0) &=& \Psi^0 \end{array} \right. .$$

4 About the number of particles

Counting the number of particles in a set is fundamental, in this topic. For the intuition, it is much better to use the orginal variables $\Phi = (x_j, v_j)$ for this purpose.

Definition 2 *For $a \in \mathbb{R}^d$, $\Phi = (x_j, v_j)_{j \in I} \in \mathcal{L}_f$ and $r \geq 0$, we set*

$$N(a, \Phi, r) = \sum_{j \in I} 1_{\{x_j \in B(a,r)\}}$$

$$N(\Phi, r) = \sup_{a \in \mathbb{R}^d} N(a, \Phi, r).$$

In the particular case of $\overline{\Phi} = (\overline{x}_j, 0)_{j \in I} \in \mathcal{L}_f$, we simply write

$$\overline{N}(r) = N\left(\overline{\Phi}, r\right).$$

Definition 3 *We say that Φ has bounded density if*

$$N(\Phi, r) < \infty$$

for every $r \geq 0$.

Example 4 *If $I=\mathbb{Z}^d$ and $\overline{x}_i = i$, then $\overline{\Phi} = (\overline{x}_j, 0)_{j\in I}$ has bounded density.*

The following lemma is very useful.

Lemma 5 *Given $\Phi, \Phi' \in X_{\overline{\Phi}} = Y + \overline{\Phi}$, and $r \geq \|\Phi - \Phi'\|_\infty$ for every $i \in I$ we have*

$$N(x_i, \Phi, r) \leq N\left(x_i', \Phi', r + 2\left\|\Phi - \Phi'\right\|_\infty\right).$$

If $\overline{\Phi}$ has bounded density and $\Psi \in Y$, then also $\Phi = \overline{\Phi} + \Psi$ has bounded density and

$$N(\Phi, r) \leq \overline{N}\left(r + 2\|\Psi\|_\infty\right).$$

Proof. One has

$$\begin{aligned} \left|x_i' - x_j'\right| &\leq |x_i - x_j| + \left|x_i - x_i'\right| + \left|x_j' - x_j\right| \\ &\leq |x_i - x_j| + 2\left\|\Phi - \Phi'\right\|_\infty. \end{aligned}$$

If $j \in I\backslash\{i\}$ satisfies $|x_i - x_j| \leq r$, then it also satisfies $\left|x_i' - x_j'\right| \leq r + 2\|\Phi - \Phi'\|_\infty$. This implies the inequality between the cardinality of points. The final claim comes simply from taking $\Phi' = \overline{\Phi}$ and the inequality

$$N\left(\overline{x}_i, \overline{\Phi}, s\right) \leq \overline{N}(s)$$

for every $s \geq 0$ and $i \in I$. ∎

5 Properties of B in Y

A priori, we have defined B as an operator from Y (or more generally from Y_0) to $\left(\mathbb{R}^d \times \mathbb{R}^d\right)^I$. In fact, it maps Y into Y and it is also locally Lipschitz continuous.

Lemma 6 *Assume that $\overline{\Phi}$ has bounded density. Then, for every $\Psi, \Psi' \in Y$ we have*

$$\|B(\Psi)\|_\infty \leq \|\Psi\|_\infty \vee \|\nabla U\|_\infty \overline{N}\left(R_0 + 2\|\Psi\|_\infty\right)$$

$$\left\|B(\Psi) - B\left(\Psi'\right)\right\|_\infty \leq \left\|\Psi - \Psi'\right\|_\infty \left(1 \vee 2\left\|D^2U\right\|_\infty \left(\overline{N}\left(R_0 + 2\|\Psi\|_\infty\right) + \overline{N}\left(R_0 + 2\left\|\Psi'\right\|_\infty\right)\right)\right).$$

Proof.

$$\begin{aligned} \|B(\Psi)\|_\infty &\leq \|\Psi\|_\infty \vee \sup_{i\in I}\left|\sum_{j\in I\backslash\{i\}} \nabla U\left(\xi_i - \xi_j + \overline{y}_{ij}\right)\right| \\ &\leq \|\Psi\|_\infty \vee \|\nabla U\|_\infty \sup_{i\in I} \sum_{j\in I\backslash\{i\}} 1_{\left\{\left|\xi_i - \xi_j + \overline{y}_{ij}\right| \leq R_0\right\}} \\ &= \|\Psi\|_\infty \vee \|\nabla U\|_\infty \sup_{i\in I} \sum_{j\in I\backslash\{i\}} 1_{\{|x_i - x_j| \leq R_0\}} \\ &\leq \|\Psi\|_\infty \vee \|\nabla U\|_\infty \sup_{i\in I} N(x_i, \Phi, R_0) \\ &\leq \|\Psi\|_\infty \vee \|\nabla U\|_\infty \overline{N}\left(R_0 + 2\|\Psi\|_\infty\right) \end{aligned}$$

$$
\begin{aligned}
\left\|B\left(\Psi\right)-B\left(\Psi'\right)\right\|_{\infty} &\leq \left\|\Psi-\Psi'\right\|_{\infty}\vee\sup_{i\in I}\left|\sum_{j\in I\setminus\{i\}}\left[\nabla U\left(\xi_i-\xi_j+\overline{y}_{ij}\right)-\nabla U\left(\xi_i'-\xi_j'+\overline{y}_{ij}\right)\right]\right| \\
&\leq \left\|\Psi-\Psi'\right\|_{\infty}\vee\left\|D^2U\right\|_{\infty}\sup_{i\in I}\sum_{j\in I\setminus\{i\}}\left(\left|\xi_i-\xi_i'\right|+\left|\xi_j-\xi_j'\right|\right)\Big(1_{\left\{\left|\xi_i-\xi_j+\overline{y}_{ij}\right|\leq R_0\right\}}+1_{\left\{\left|\xi_i'-\xi_j'\right.\right.} \\
&\leq \left\|\Psi-\Psi'\right\|_{\infty}\vee 2\left\|D^2U\right\|_{\infty}\left\|\Psi-\Psi'\right\|_{\infty}\sup_{i\in I}\sum_{j\in I\setminus\{i\}}\Big(1_{\left\{\left|\xi_i-\xi_j+\overline{y}_{ij}\right|\leq R_0\right\}}+1_{\left\{\left|\xi_i'-\xi_j'+\overline{y}_{ij}\right|\leq R_0\right.} \\
&= \left\|\Psi-\Psi'\right\|_{\infty}\vee 2\left\|D^2U\right\|_{\infty}\left\|\Psi-\Psi'\right\|_{\infty}\sup_{i\in I}\sum_{j\in I\setminus\{i\}}\left(1_{\left\{\left|x_i-x_j\right|\leq R_0\right\}}+1_{\left\{\left|x_i'-x_j'\right|\leq R_0\right\}}\right) \\
&\leq \left\|\Psi-\Psi'\right\|_{\infty}\vee 2\left\|D^2U\right\|_{\infty}\left\|\Psi-\Psi'\right\|_{\infty}\sup_{i\in I}\left(N\left(x_i,\Phi,R_0\right)+N\left(x_i',\Phi',R_0\right)\right) \\
&\leq \left\|\Psi-\Psi'\right\|_{\infty}\vee 2\left\|D^2U\right\|_{\infty}\left\|\Psi-\Psi'\right\|_{\infty}\left(\overline{N}\left(R_0+2\left\|\Psi\right\|_{\infty}\right)+\overline{N}\left(R_0+2\left\|\Psi'\right\|_{\infty}\right)\right).
\end{aligned}
$$

■

6 Local well posedness and global results in $X_{\overline{\Phi}}$

In the space of bounded displacements ξ_i and bounded velocities v_i we may prove local well posedness. The differential system, under the change of variables, in integral form is

$$\Psi\left(t\right)=\Psi^0+\int_0^t B\left(\Psi\left(s\right)\right)ds$$

that we investigate in the space $C\left(\left[0,T\right];Y\right)$.

Theorem 7 *Assume that $\overline{\Phi}$ has bounded density. Given $\Psi^0\in Y$ (or equivalently given $\Phi^0\in X_{\overline{\Phi}}$), there exists $T_0>0$ and a unique solution $\Psi\left(\cdot\right)\in C\left(\left[0,T\right];Y\right)$ (equivalently $\Phi\left(\cdot\right)\in C\left(\left[0,T\right];X_{\overline{\Phi}}\right)$). Given $r_0>0$, one can choose $T_0>0$ depending on r_0 such that the previous result is true for all $\Psi^0\in Y$ with $\left\|\Psi^0\right\|_{\infty}\leq r_0$.*

Proof. Step 1. For every $T_0>0$, denote by $\left\|\Psi\left(\cdot\right)\right\|_{\infty}$ the supremum norm $\sup_{t\in[0,T_0]}\left\|\Psi\left(t\right)\right\|_{\infty}$, whch makes $C\left(\left[0,T_0\right];Y\right)$ a Banach space; we hope there is no danger of confusion due to the use of the same symbol for different objects.

Consider the map $\Gamma:C\left(\left[0,T_0\right];Y\right)\to C\left(\left[0,T_0\right];Y\right)$ defined as

$$\Gamma\left(\Psi\left(\cdot\right)\right)\left(t\right)=\Psi^0+\int_0^t B\left(\Psi\left(s\right)\right)ds.$$

It is easy to check, based on Lemma 6 that Γ really maps $C\left(\left[0,T_0\right];Y\right)$ into itself, for every choice of $T_0>0$.

Given $r_0 > 0$, denote by $\mathcal{Y}_{T_0,2r_0}$ the closed ball or center zero and radius $2r_0$ in $C([0,T_0];Y)$: the set of all $\Psi(\cdot) \in C([0,T_0];Y)$ such that

$$\sup_{t\in[0,T_0]} \|\Psi(t)\|_\infty \leq 2r_0.$$

Let us check that, for T_0 small enough depending only on r_0, this ball is invariant by Γ. We have, from Lemma 6,

$$\begin{aligned} \|\Gamma(\Psi(\cdot))(t)\|_\infty &\leq \left\|\Psi^0\right\|_\infty + \int_0^t \|B(\Psi(s))\|_\infty ds \\ &\leq \left\|\Psi^0\right\|_\infty + \int_0^t \|\Psi(s)\|_\infty \vee \|\nabla U\|_\infty \overline{N}(R_0 + 2\|\Psi(s)\|_\infty) ds \\ &\leq r_0 + \left[2r_0 \vee \|\nabla U\|_\infty \overline{N}(R_0+4r_0)\right] T_0 \end{aligned}$$

hence $\leq 2r_0$ for every $T_0 \leq T_0^*$ with T_0^* satisfying

$$\left[2r_0 \vee \|\nabla U\|_\infty \overline{N}(R_0+4r_0)\right] T_0^* = r_0.$$

Step 2. For every $T_0 \leq T_0^*$, for every $\Psi(\cdot), \Psi'(\cdot) \in \mathcal{Y}_{T_0,2r_0}$, from Lemma 6 we have

$$\begin{aligned} &\left\|\Gamma(\Psi(\cdot))(t) - \Gamma\left(\Psi'(\cdot)\right)(t)\right\|_\infty \\ \leq\ & \int_0^t \left\|B(\Psi(s)) - B\left(\Psi'(s)\right)\right\|_\infty ds \\ \leq\ & \int_0^t \left\|\Psi(s) - \Psi'(s)\right\|_\infty \left(1 \vee 2\left\|D^2U\right\|_\infty \left(\overline{N}(R_0 + 2\|\Psi(s)\|_\infty) + \overline{N}\left(R_0 + 2\left\|\Psi'(s)\right\|_\infty\right)\right)\right) ds \\ \leq\ & \left(1 \vee 4\left\|D^2U\right\|_\infty \overline{N}(R_0+4r_0)\right) \int_0^t \left\|\Psi(s) - \Psi'(s)\right\|_\infty ds \\ \leq\ & \left(1 \vee 4\left\|D^2U\right\|_\infty \overline{N}(R_0+4r_0)\right) T_0 \left\|\Psi(\cdot) - \Psi'(\cdot)\right\|_\infty . \end{aligned}$$

Hence for any $T_0 \leq T_0^*$ such that

$$\left(1 \vee 4\left\|D^2U\right\|_\infty \overline{N}(R_0+4r_0)\right) T_0 < 1$$

the map Γ is a contraction, in the complete metric space $\mathcal{Y}_{T_0,2r_0}$. Hence it has a fixed point, that is the unique solution claimed by the theorem. ■

6.1 Criteria for global solutions

Proposition 8 *Assume that $\overline{\Phi}$ has bounded density. Given $\Phi^0 \in X_{\overline{\Phi}}$, assume there exists $C > 0$ such that, for every continuous-in-$X_{\overline{\Phi}}$ solution $\Phi(\cdot)$ on some interval $[0,T_0]$ we have $\|\Psi(\cdot)\|_\infty \leq C$, namely*

$$\sup_{s\in[0,T_0]} d_\infty\left(\Phi(s), \overline{\Phi}\right) \leq C.$$

Then the unique solution provided by Theorem 7 is global. Moreover, each one of the following conditions is sufficient:

$$\sup_{s\in[0,T_0]}\sup_{i\in I} N\left(x_i,\Phi\left(s\right),R_0\right)\leq C$$

$$\sup_{s\in[0,T_0]}\sup_{i\in I}\left|x_i\left(s\right)-\overline{x}_i\right|\leq C$$

$$\sup_{s\in[0,T_0]}\sup_{i\in I}\left|v_i\left(s\right)\right|\leq C.$$

Proof. A unique local solution exists on an interval T_0 satisfying (to preserve $\mathcal{Y}_{T_0,2C}$)

$$\left[2C\vee\left\|\nabla U\right\|_\infty\overline{N}\left(R_0+4C\right)\right]T_0\leq C$$

and (to be a contraction)

$$\left(1\vee 4\left\|D^2U\right\|_\infty\overline{N}\left(R_0+4C\right)\right)T_0<1.$$

The value $\Phi\left(T_0\right)$, however, satisfies the same condition $d_\infty\left(\Phi\left(T_0\right),\overline{\Phi}\right)\leq C$, by assumption, hence we may solve the equation on the interval $[T_0,2T_0]$ with the same value of T_0 found above. In a finite number of steps we cover any pre-defined interval of time. We leave the reader to check that the other conditions are sufficient. ■

6.2 Global solution in $d=1$

Lemma 9 *There exists $C_d>0$ with the following property. If Φ has bounded density, then*

$$N\left(\Phi,r\right)\leq C_dN\left(\Phi,1\right)\left(1+r^d\right).$$

Proof. It is sufficient to prove it for $r\geq 1$. Cover $B\left(a,r\right)$ by C_dr^d balls of the form $B\left(a',1\right)$, with suitable centers a'. Then

$$N\left(a,\Phi,r\right)\leq\sum_{a'}N\left(a',\Phi,1\right)\leq N\left(\Phi,1\right)C_dr^d.$$

Changing C_d if becessary, we get the result. ■

Theorem 10 *In $d=1$, assume $\overline{\Phi}$ has bounded density. The the local solutions of Theorem 7 are global.*

Proof. By the previous lemma we have

$$\overline{N}\left(r\right)\leq\overline{C}\left(1+r\right)$$

for a suitable constant $\overline{C} > 0$. If $\Phi(\cdot)$ is a solution on any time interval $[0, T_0]$, then, using the bound of Lemma 6, we have

$$\begin{aligned}
\|\Psi(t)\|_\infty &\leq \left\|\Psi^0\right\|_\infty + \int_0^t \|\Psi(s)\|_\infty \vee \|\nabla U\|_\infty \overline{N}(R_0 + 2\|\Psi(s)\|_\infty)\, ds \\
&\leq \left\|\Psi^0\right\|_\infty + \int_0^t \|\Psi(s)\|_\infty + \|\nabla U\|_\infty \overline{C}(1 + R_0 + 2\|\Psi(s)\|_\infty)\, ds \\
&\leq \left\|\Psi^0\right\|_\infty + \int_0^t \overline{C}\|\nabla U\|_\infty (1 + R_0)\, ds + \int_0^t \left(1 + 2\overline{C}\|\nabla U\|_\infty\right) \|\Psi(s)\|_\infty\, ds
\end{aligned}$$

and thus, by Gronwall lemma,

$$\|\Psi(t)\|_\infty \leq e^{\left(1+2\overline{C}\|\nabla U\|_\infty\right)t} \left(\left\|\Psi^0\right\|_\infty + \int_0^t \overline{C}\|\nabla U\|_\infty (1 + R_0)\, ds \right).$$

On any time interval $[0, T]$ the assumption of Proposition 8 is satisfied, hence the solution is global in $[0, T]$, hence on $[0, \infty)$. ■

Remark 11 *In generic dimension d, let us say that $\overline{\Phi}$ has strongly decaying density if*

$$\overline{N}(r) \leq \overline{C}(1 + r)$$

for a suitable constant $\overline{C} > 0$. In this case the proof of the previous theorem works and global existence holds. Of course the initial conditions Φ^0 allowd have also strongly decaying density, namely $N\left(\Phi^0, r\right)$ has at most linear growth, and this is very restrictive, in dimension $d > 1$.

7 Translation invariant measures

Except in dimension $d = 1$, we are not able to prove a global-in-time result in the class Y. However, similarly to the case of point vortices, it could be true that singularities are avoided for a.e. initial condition, with respect to a suitable measure. Let us start the investigation of this topic.

The most obvious idea would be to use a (potentially) invariant measure However, the natural ones for this purpose, the Gibbs measures, having a Maxwell distribution of velocities, is not supported on Y; more precisely Y has measure zero. This is a main reason to investigate larger spaces than Y; however, let us insist on Y becuase of its simplicity and replace the concept of invariant measure with the concept of *translation-invariant measure.* These new objects are not as powerful as the previous ones but may lead to interesting results.

Consider the Polish space $\left(X_{\overline{\Phi}}, d_\infty\right)$ defined above, with the Borel σ-algebra $\mathcal{B}$. A probability measure μ on $\left(X_{\overline{\Phi}}, \mathcal{B}\right)$ is called *translation invariant* if

$$\tau_a \mu = \mu$$

for every $a \in \mathbb{R}^d$, where $\tau_a \mu$ is the push-forward of μ under τ_a and τ_a will be now defined.

On $\mathbb{R}^d$, τ_a is the map defined as

$$\tau_a(x) = x + a.$$

It induces a map, still denoted by τ_a, on $\left(\mathbb{R}^d \times \mathbb{R}^d\right)^I$:

$$\tau_a (x_i, v_i)_{i \in I} = (x_i + a, v_i)_{i \in I} .$$

Restrictred to $X_{\overline{\Phi}}$, we see that $\tau_a \left(X_{\overline{\Phi}}\right) = X_{\overline{\Phi}}$. This map is measurable, hence the push-forward of a measure on $\left(X_{\overline{\Phi}}, d_\infty\right)$ is well defined.

Equivalent is to ask that

$$\int_{X_{\overline{\Phi}}} F(\tau_a \Phi) \mu(d\Phi) = \int_{X_{\overline{\Phi}}} F(\Phi) \mu(d\Phi)$$

for every point $a \in \mathbb{R}^d$ and bounded measurable functions $F : X_{\overline{\Phi}} \to \mathbb{R}$. As a particular case of F let us consider

$$F(\Phi) = \sum_{i \in I} f(x_i, v_i)$$

with f bounded measurable on $\mathbb{R}^d \times \mathbb{R}^d$, equal to zero for $|x| > R$ for some $R > 0$. We get

$$\int_{X_{\overline{\Phi}}} \sum_{i \in I} f(x_i + a, v_i) \mu(d\Phi) = \int_{X_{\overline{\Phi}}} \sum_{i \in I} f(x_i, v_i) \mu(d\Phi).$$

If μ is a Borel measure on $X_{\overline{\Phi}}$, a Borel measure on $Y = X_{\overline{\Phi}} - \overline{\Phi}$ is naturally defined as the push forward under the map $\Phi \mapsto \Psi := \Phi - \overline{\Phi}$. Let us call μ_Y such measure. We may reformulate the theory of translation invariant measures μ on $X_{\overline{\Phi}}$ by means of measures on Y with a suitable property:

Lemma 12 *The measure μ on $X_{\overline{\Phi}}$ is translation invariant if an only if the measure μ_Y on Y satisfies*

$$\int_Y G(\Psi) \mu_Y(d\Psi) = \int_Y G\left(\tau_a \Psi + \tau_a \overline{\Phi} - \overline{\Phi}\right) \mu_Y(d\Psi)$$

or equivalently

$$\int_Y G(\tau_a \Psi) \mu_Y(d\Psi) = \int_Y G\left(\Psi + \overline{\Psi}_a\right) \mu_Y(d\Psi)$$

where $\overline{\Psi}_a = \overline{\Phi} - \tau_a \overline{\Phi}$, for every bounded measurable functions $G : Y \to \mathbb{R}$.

Proof. Denoting the function $\Phi \mapsto G\left(\Phi - \overline{\Phi}\right)$ by $F(\Phi)$,

$$\begin{aligned}
\int_Y G(\Psi)\,\mu_Y(d\Psi) &= \int_{X_{\overline{\Phi}}} G\left(\Phi-\overline{\Phi}\right)\mu(d\Phi) = \int_{X_{\overline{\Phi}}} F(\Phi)\,\mu(d\Phi) \\
&= \int_{X_{\overline{\Phi}}} F(\tau_a\Phi)\,\mu(d\Phi) = \int_{X_{\overline{\Phi}}} G\left(\tau_a\Phi-\overline{\Phi}\right)\mu(d\Phi) \\
&= \int_{X_{\overline{\Phi}}} G\left(\tau_a\left(\Phi-\overline{\Phi}\right)+\tau_a\overline{\Phi}-\overline{\Phi}\right)\mu(d\Phi) \\
&= \int_Y G\left(\tau_a\Psi+\tau_a\overline{\Phi}-\overline{\Phi}\right)\mu_Y(d\Psi).
\end{aligned}$$

■

Remark 13 *However, the intuition is better on $X_{\overline{\Phi}}$, hence we leave the translation for investigations which profit from the Banach space property.*

Remark 14 *The v-component plays an auxiliary role in this framework. A general way to construct translation invariant measure on $\left(X_{\overline{\Phi}}, d_\infty\right)$ is to construct them with such a property on the purely spatial component, and then take product measure on the v-component, for instance independent identically distributed velocities to all particles; with bounded distribution, to be included in $X_{\overline{\Phi}}$. If we denote the spatial component of $X_{\overline{\Phi}}$ by $X^x_{\overline{\Phi}}$, that of Φ by Φ^x and the projection of μ on the spatial component by μ^x, we have*

$$\int_{X^x_{\overline{\Phi}}} \sum_{i\in I} f(x_i+a)\,\mu^x(d\Phi^x) = \int_{X^x_{\overline{\Phi}}} \sum_{i\in I} f(x_i)\,\mu^x(d\Phi^x)$$

for every bounded measurable compact support function f on $\mathbb{R}^d$.

Remark 15 *Let $I=\mathbb{Z}^d$, $\overline{\Phi}$ be given by $\overline{x}_i = i$, for every $i\in\mathbb{Z}^d$. An example of translation invariant measure on the spatial component of $\left(X_{\overline{\Phi}}, d_\infty\right)$ is given by the convex combination*

$$\mu(dx) = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \delta_{\overline{x}+q}(dx)\,dq$$

where $I=\mathbb{Z}^d$, $x=(x_i)_{i\in I}$, $\overline{x}=(\overline{x}_i)_{i\in I}$, $\overline{x}+q=(\overline{x}_i+q)_{i\in I}$, $q\in\mathbb{R}^d$. For this measure we have

$$\begin{aligned}
\int_{X^x_{\overline{\Phi}}} \sum_{i\in I} f(x_i+a)\,\mu^x(d\Phi^x) &= \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \int_{X^x_{\overline{\Phi}}} \sum_{i\in I} f(x_i+a)\,\delta_{\overline{x}+q}(dx)\,dq \\
&= \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \sum_{i\in I} f(i+q+a)\,dq = \sum_{i\in I}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} f(i+q+a)\,dq \\
&= \int_{\mathbb{R}^d} f(x+a)\,dx = \int_{\mathbb{R}^d} f(x)\,dx
\end{aligned}$$

independently of $a\in\mathbb{R}^d$.

8 Evolution of translation invariant measures

Recall from the proof of Theorem 7 that T_0 can be taken the same for all Φ^0 having the same distance from $\overline{\Phi}$. Thus, if a translation invariant measure μ_0 is supported on a ball $B_{X_{\overline{\Phi}}}\left(\overline{\Phi}, r_0\right)$ in $X_{\overline{\Phi}}$ of center $\overline{\Phi}$ and radius r_0, its time evolution is well defined for $t \in [0, T_0]$, where T_0 is a value defined by Theorem 7 with respect to r_0. If we denote by $\Phi\left(t; \Phi^0\right)$, for $t \in [0, T_0]$, the unique solution starting from $\Phi^0 \in B_{X_{\overline{\Phi}}}\left(\overline{\Phi}, r_0\right)$, call μ_t the push-forward of μ_0 under the map $\Phi^0 \mapsto \Phi\left(t; \Phi^0\right)$.

Lemma 16 *μ_t is translation invariant.*

Proof. Using the property $\tau_a \Phi\left(t; \Phi^0\right) = \Phi\left(t; \tau_a \Phi^0\right)$ (easy to check) and the invariance of μ_0:

$$\begin{aligned}
\int_{X_{\overline{\Phi}}} F\left(\tau_a \Phi\right) \mu_t\left(d\Phi\right) &= \int_{X_{\overline{\Phi}}} F\left(\tau_a \Phi\left(t; \Phi^0\right)\right) \mu_0\left(d\Phi^0\right) \\
&= \int_{X_{\overline{\Phi}}} F\left(\Phi\left(t; \tau_a \Phi^0\right)\right) \mu_0\left(d\Phi^0\right) \\
&= \int_{X_{\overline{\Phi}}} F\left(\Phi\left(t; \Phi^0\right)\right) \mu_0\left(d\Phi^0\right) \\
&= \int_{X_{\overline{\Phi}}} F\left(\Phi\right) \mu_t\left(d\Phi\right).
\end{aligned}$$

■

9 Specific energy

By *energy of particle i in configuration Φ* we mean

$$e\left(i, \Phi\right) = \frac{1}{2}\left(\left|v_i\right|^2 + \sum_{j \in I \backslash \{i\}} U\left(x_i - x_j\right)\right).$$

By *energy of configuration Φ in the Borel set B* of $\mathbb{R}^d$ we mean

$$e\left(\Phi, B\right) = \sum_{i \in I} e\left(i, \Phi\right) 1_{\{x_i \in B\}}.$$

Given a translation invariant measure μ, by *specific energy* we mean

$$\overline{e}\left(\mu\right) = \int_{X_{\overline{\Phi}}} e\left(\Phi, W_1\right) \mu\left(d\Phi\right)$$

where $W_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$. More generally we may define the average energy in $B \subset \mathbb{R}^d$ as

$$\overline{e}(\mu, B) = \int_{X_{\overline{\Phi}}} e(\Phi, B)\, \mu(d\Phi).$$

The set-function $B \mapsto \overline{e}(\mu, B)$ is additive.

Lemma 17 *$\overline{e}(\mu, B) = \overline{e}(\mu, B + a)$ for every $a \in \mathbb{R}^d$.*

Proof.

$$\begin{aligned}
\int_{X_{\overline{\Phi}}} e(\Phi, B + a)\, \mu(d\Phi) &= \frac{1}{2} \sum_{i \in I} \int_{X_{\overline{\Phi}}} \left(|v_i|^2 + \sum_{j \in I \setminus \{i\}} U(x_i - x_j) \right) 1_{\{x_i \in B + a\}} \mu(d\Phi) \\
&= \frac{1}{2} \sum_{i \in I} \int_{X_{\overline{\Phi}}} \left(|v_i|^2 + \sum_{j \in I \setminus \{i\}} U(x_i - a - (x_j - a)) \right) 1_{\{x_i - a \in B\}} \mu(d\Phi) \\
&= \frac{1}{2} \sum_{i \in I} \int_{X_{\overline{\Phi}}} \left(|v_i|^2 + \sum_{j \in I \setminus \{i\}} U(x_i - x_j) \right) 1_{\{x_i \in B\}} \mu(d\Phi) \\
&= \int_{X_{\overline{\Phi}}} e(\Phi, B)\, \mu(d\Phi).
\end{aligned}$$

■

The name specific energy can now be understood:

Lemma 18

$$\overline{e}(\mu) = \lim_{N \to \infty} \frac{1}{(2N + 1)^d} \overline{e}(\mu, W_N)$$

where $W_N = \left[-N - \frac{1}{2}, N + \frac{1}{2}\right]^d$.

Proof. Decompose W_N in $(2N + 1)^d$ disjoint hypercubes and apply additivity of $B \mapsto \overline{e}(\mu, B)$ along with the previous lemma. ■

Now, let μ_0 be a translation invariant measure supported on the ball $B_{X_{\overline{\Phi}}}\left(\overline{\Phi}, r_0\right)$ and let μ_t be its time-evolution, defined for $t \in [0, T_0]$, also translation invariant by the lemma above. Define the specific energy at time t as

$$\overline{e}_t := \overline{e}(\mu_t).$$

Lemma 19 *Assume that μ_0 is translation invariant and supported on the ball $B_{X_{\overline{\Phi}}}\left(\overline{\Phi}, r_0\right)$. Then the function $t \mapsto \overline{e}_t$ is constant.*

Proof. Step 1. By the previous lemma

$$\begin{aligned}
\overline{e}_t &= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \overline{e}\left(\mu_t, W_N\right) \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \int_{X_{\overline{\Phi}}} e\left(\Phi, W_N\right) \mu_t\left(d\Phi\right) \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \int_{X_{\overline{\Phi}}} e\left(\Phi\left(t;\Phi^0\right), W_N\right) \mu_0\left(d\Phi^0\right).
\end{aligned}$$

Heuristically, $e\left(\Phi\left(t;\Phi^0\right), \mathbb{R}^d\right) = e\left(\Phi^0, \mathbb{R}^d\right)$ (but both are infinite!), namely $e\left(\Phi\left(t;\Phi^0\right), W_N\right) \sim e\left(\Phi^0, W_N\right)$ for large N, hence

$$\begin{aligned}
\frac{1}{(2N+1)^d} \int_{X_{\overline{\Phi}}} e\left(\Phi\left(t;\Phi^0\right), W_N\right) \mu_0\left(d\Phi^0\right) &\sim \frac{1}{(2N+1)^d} \int_{X_{\overline{\Phi}}} e\left(\Phi^0, W_N\right) \mu_0\left(d\Phi^0\right) \\
&\sim \overline{e}_0.
\end{aligned}$$

The rigorous proof requires a control of the error. We have

$$e\left(\Phi\left(t;\Phi^0\right), W_N\right) = \frac{1}{2} \sum_{i\in I} \left(\left|v_i(t)\right|^2 + \sum_{j\in I\setminus\{i\}} U\left(x_i(t) - x_j(t)\right) \right) 1_{\{x_i(t)\in W_N\}}.$$

Since

$$\frac{1}{2}\frac{d}{dt} \left(\left|v_i(t)\right|^2 + \sum_{j\in I\setminus\{i\}} U\left(x_i(t) - x_j(t)\right) \right) = g_i(t)$$

where

$$\begin{aligned}
g_i(t) \quad &: = -v_i(t) \cdot \sum_{j\in I\setminus\{i\}} \nabla U\left(x_i(t) - x_j(t)\right) + \frac{1}{2} \sum_{j\in I\setminus\{i\}} \nabla U\left(x_i(t) - x_j(t)\right) \cdot \left(v_i(t) - v_j(t)\right) \\
&= -\frac{1}{2} \sum_{j\in I\setminus\{i\}} \nabla U\left(x_i(t) - x_j(t)\right) \cdot \left(v_i(t) + v_j(t)\right)
\end{aligned}$$

we have

$$\begin{aligned}
e\left(\Phi\left(t;\Phi^0\right), W_N\right) &= \frac{1}{2} \sum_{i\in I} \left(\left|v_i(t)\right|^2 + \sum_{j\in I\setminus\{i\}} U\left(x_i(t) - x_j(t)\right) \right) \left(1_{\{x_i(t)\in W_N\}} - 1_{\left\{x_i^0\in W_N\right\}} \right) \\
&\quad \frac{1}{2} \sum_{i\in I} \left(\left|v_i^0\right|^2 + \sum_{j\in I\setminus\{i\}} U\left(x_i^0 - x_j^0\right) \right) 1_{\left\{x_i^0\in W_N\right\}} + \sum_{i\in I} g_i(t)\, 1_{\left\{x_i^0\in W_N\right\}} \\
&= e\left(\Phi^0, W_N\right) + A_N\left(\Phi^0\right) + B_N\left(\Phi^0\right)
\end{aligned}$$

where

$$A_N\left(\Phi^0\right) = \frac{1}{2}\sum_{i\in I}\left(\left|v_i\left(t\right)\right|^2 + \sum_{j\in I\setminus\{i\}} U\left(x_i\left(t\right)-x_j\left(t\right)\right)\right)\left(1_{\{x_i(t)\in W_N\}} - 1_{\left\{x_i^0\in W_N\right\}}\right)$$

$$B_N\left(\Phi^0\right) = \sum_{i\in I} g_i\left(t\right)1_{\left\{x_i^0\in W_N\right\}}.$$

Thus

$$\overline{e}_t = \overline{e}_0 + \lim_{N\to\infty}\int_{X_{\overline{\Phi}}} \frac{A_N\left(\Phi^0\right)+B_N\left(\Phi^0\right)}{\left(2N+1\right)^d}\mu_0\left(d\Phi^0\right).$$

We have to prove that this limit is equal to zero.

Step 2. We give only the idea of the computation. We have

$$\left|A_N\left(\Phi^0\right)\right| \le \frac{1}{2}\sum_{i\in I}\left(r_0^2 + \left\|U\right\|_\infty N\left(i,\Phi\left(t;\Phi^0\right),R_0\right)\right)\left|1_{\{x_i(t)\in W_N\}} - 1_{\left\{x_i^0\in W_N\right\}}\right|.$$

Since $d_\infty\left(\Phi^0,\overline{\Phi}\right)\le r_0$, for $t\in[0,T_0]$ we have $d_\infty\left(\Phi\left(t;\Phi^0\right),\overline{\Phi}\right)\le R$ for a certain chosen $R>r_0$, hence $N\left(i,\Phi\left(t;\Phi^0\right),R_0\right)\le\overline{N}\left(R_0+2R\right)$, and thus, denoting by $C>0$ a constant independent of N and Φ^0, we get

$$\left|A_N\left(\Phi^0\right)\right| \le C\sum_{i\in I}\left|1_{\{x_i(t)\in W_N\}} - 1_{\left\{x_i^0\in W_N\right\}}\right|.$$

Since $\left|v_i\left(s\right)\right|\le R$, $\left|x_i\left(t\right)-x_i^0\right|\le RT_0$, hence only indexes i such that either $x_i^0\in W_N\setminus W_{N-RT_0}$ or $x_i\left(t\right)\in W_N\setminus W_{N-RT_0}$ may contribute to the sum; the volume of $W_N\setminus W_{N-RT_0}$ is of order $\left(2N+1\right)^{d-1}$ and thanks to the assumption on $\overline{N}\left(r\right)$ the numebr of i's with the previous properties is also of order equal or less than $\left(2N+1\right)^{d-1}$. It follows that

$$\frac{\left|A_N\left(\Phi^0\right)\right|}{\left(2N+1\right)^d} \le \frac{C}{2N+1}.$$

Step 3. One has

$$\begin{aligned} B_N\left(\Phi^0\right) &= -\frac{1}{2}\sum_{i\in I}\sum_{j\in I\setminus\{i\}}\nabla U\left(x_i\left(t\right)-x_j\left(t\right)\right)\cdot\left(v_i\left(t\right)+v_j\left(t\right)\right)1_{\left\{x_i^0\in W_N\right\}} \\ &= \frac{1}{2}\sum_{i,j\in I;i\neq j}\nabla U\left(x_i\left(t\right)-x_j\left(t\right)\right)\cdot v_i\left(t\right)\left(1_{\left\{x_j^0\in W_N\right\}} - 1_{\left\{x_i^0\in W_N\right\}}\right) \end{aligned}$$

hence

$$\left|B_N\left(\Phi^0\right)\right| \le \frac{1}{2}\left\|\nabla U\right\|_\infty r_0\sum_{i,j\in I;i\neq j}\left|1_{\left\{x_j^0\in W_N\right\}} - 1_{\left\{x_i^0\in W_N\right\}}\right|1_{\{|x_i(t)-x_j(t)|\le R_0\}}.$$

With arguments similar to those of Step 2 we deduce that $\left|B_N\left(\Phi^0\right)\right|$ is at most of order $(2N+1)^{d-1}$ and that

$$\frac{\left|B_N\left(\Phi^0\right)\right|}{(2N+1)^d} \leq \frac{C}{2N+1}.$$

The results of Steps 2 and 3 prove the final claim of Step 1. ■

References

[1] O. E. Lanford III, The classical mechanics of one-dimensional systems of infinitely many particles, *Comm. Math. Phys.* **9** (1968), 169-191.