$f : \mathbb{R} \to \mathbb{R}$, D_f dominio naturale, $x_0 \in Acc(D_f) \subseteq \mathbb{R}^*$

$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}^*$	$\lim_{x \to x_0} \frac{1}{f(x)} = ?$
$\mathbb{R} \setminus \{0\}$	$\frac{1}{\ell}$
$\pm\infty$	0
0	FORMA INDETERMINATA $\left(\frac{1}{0}\right)$
• 0^+ $\boxed{\exists \varepsilon > 0 \text{ per cui } f(x) > 0 \ \forall x \in (x_0-\varepsilon, x_0+\varepsilon) \setminus \{x_0\}}$	• $+\infty$ $\left(= \frac{1}{0^+}\right)$ ES $f(x) = x^2$, $\frac{1}{f(x)} = \frac{1}{x^2} \xrightarrow[x \to 0]{} +\infty$
• 0^- $\boxed{\exists \varepsilon > 0 \text{ per cui } f(x) < 0 \ \forall x \in (x_0-\varepsilon, x_0+\varepsilon) \setminus \{x_0\}}$	• $-\infty$ $\left(= \frac{1}{0^-}\right)$ ES $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$f, g : \mathbb{R} \to \mathbb{R}$, D_f = dom nat. di f, D_g = dom nat di g

$x_0 \in Acc(D_f) \cap Acc(D_g) \subseteq \mathbb{R}^*$

$\lim_{x \to x_0} f(x) = \ell_1 \in \mathbb{R}^*$	$\lim_{x \to x_0} g(x) = \ell_2 \in \mathbb{R}^*$	$\lim_{x \to x_0} (f(x) + g(x)) = ?$
$\mathbb{R}$	$\mathbb{R}$	$\ell_1 + \ell_2$
$+\infty$	$\mathbb{R}$	$+\infty$
$-\infty$	$\mathbb{R}$	$-\infty$
$+\infty$	$+\infty$	$+\infty$
$-\infty$	$-\infty$	$-\infty$

$+\infty$	$-\infty$	FORMA INDETERMINATA

$\lim_{x\to x_0} f(x) = l_1 \in \mathbb{R}^*$	$\lim_{x\to x_0} g(x) = l_2 \in \mathbb{R}^*$	$\lim_{x\to x_0} (f(x)\cdot g(x)) = ?$
$\mathbb{R}$	$\mathbb{R}$	$l_1 \cdot l_2$
$+\infty$	$(0, +\infty)$	$+\infty$
$+\infty$	$(-\infty, 0)$	$-\infty$
$+\infty$	0	FORMA INDETERMINATA
$-\infty$	$(0, +\infty)$	$-\infty$
$-\infty$	$(-\infty, 0)$	$+\infty$
$-\infty$	0	FORMA INDETERMINATA
$+\infty$	$+\infty$	$+\infty$
$+\infty$	$-\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$

OSS Per $\frac{f(x)}{g(x)}$ la forma indeterminata è $\frac{\pm\infty}{\pm\infty}$ $\left(= \pm\infty \cdot \underbrace{\frac{1}{\pm\infty}}_{=0}\right)$

$\frac{0}{0}$ $\left(= 0 \cdot \underbrace{\frac{1}{0}}_{=\pm\infty}\right)$

ESERCIZI • $\lim_{x\to 2} \frac{3}{2-x}$, $f(x) = 3$, $g(x) = 2-x$

$f(x) \xrightarrow[x\to 2]{} 3$, $g(x) \xrightarrow[x\to 2]{} 0$, $\frac{f(x)}{g(x)}$ è forma indet.

$\lim_{x\to 2^+} \frac{3}{2-x} = -\infty$, $f(x) \xrightarrow[x\to 2^+]{} 3$, $g(x) \xrightarrow[x\to 2^+]{} 0^-$

$\lim_{x\to 2^-} \frac{3}{2-x} = +\infty$, $f(x) \xrightarrow[x\to 2^-]{} 3$, $g(x) \xrightarrow[x\to 2^-]{} 0^+$

quindi $\lim\limits_{x\to 2} \frac{3}{2-x}$ non esiste

- $\lim\limits_{x\to+\infty} (x^4+3x^2+x+1) = +\infty$

  $+\infty + \infty + \infty + 1$

- $\lim\limits_{x\to+\infty} (x^4-3x^2) = \lim\limits_{x\to+\infty} x^2(x^2-3) = +\infty$

  $+\infty - \infty \qquad +\infty(+\infty-3)$

$\lim\limits_{x\to+\infty} (x^4-3x^2-x+1) = \lim\limits_{x\to+\infty} x^4\left(1-\frac{3}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}\right) = +\infty$

$+\infty(1-0-0+0)$

$\lim\limits_{x\to-\infty} (x^3-3x-2) = \lim\limits_{x\to-\infty} x^3\left(1-\frac{3}{x^2}-\frac{2}{x^3}\right) = -\infty$

$-\infty+\infty-2 \qquad -\infty(1-0-0)$

- $\lim\limits_{x\to+\infty} \frac{x^4+x^2}{3x^2+x} = \lim\limits_{x\to+\infty} \frac{x^4\left(1+\frac{1}{x^2}\right)}{x^2\left(3+\frac{1}{x}\right)} = \lim\limits_{x\to+\infty} x^2 \frac{\left(1+\frac{1}{x^2}\right)}{\left(3+\frac{1}{x}\right)} = +\infty$

  $+\infty \cdot \frac{1}{3}$

$\lim\limits_{x\to-\infty} \frac{x^3+2x^2-x}{x^4+x^3} = \lim\limits_{x\to-\infty} \frac{x^3\left(1+\frac{2}{x}-\frac{1}{x^2}\right)}{x^4\left(1+\frac{1}{x}\right)} = \lim\limits_{x\to-\infty} \frac{\left(1+\frac{2}{x}-\frac{1}{x^2}\right)}{x\left(1+\frac{1}{x}\right)} = 0^-$

$\frac{-\infty}{+\infty} \qquad \frac{1}{-\infty\cdot 1}$

- $\lim\limits_{x\to 0} \frac{x^4+x^2}{3x^2+x} = \lim\limits_{x\to 0} \frac{x^2(\overbrace{x^2+1}^{\to 1})}{x\,(\underbrace{3x+1}_{\to 1})} = \lim\limits_{x\to 0} \frac{x\cdot(x^2+1)}{(3x+1)} = 0$

  $\frac{0}{0} \qquad 0\cdot\frac{1}{1}$

$\lim\limits_{x\to 0} \frac{x^3+x^2+x}{5x^{100}+x^2} = \lim\limits_{x\to 0} \frac{x(\overbrace{x^2+x+1}^{\to 1})}{x^2(\underbrace{5x^{98}+1}_{\to 1})} = \lim\limits_{x\to 0} \frac{1}{x}\cdot \underbrace{\frac{x^2+x+1}{5x^{98}+1}}_{\to 1} \quad \nexists$

- $$\lim_{x \to +\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$$
$$+\infty - \infty$$

$f(x) = \sqrt{x+1}$, $D_f = \{x+1 \geq 0\} = [-1, +\infty)$

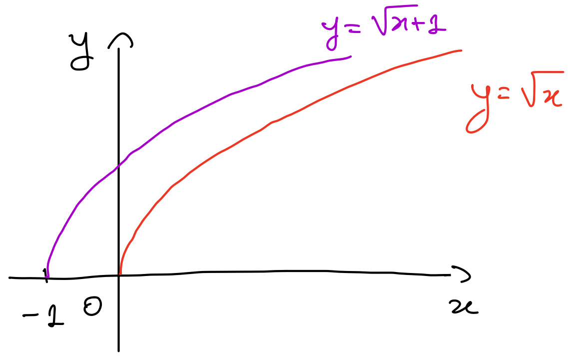
$g(x) = \sqrt{x}$, $D_g = \{x \geq 0\} = [0, +\infty)$

$$D_f \cap D_g = [0, +\infty) ,$$

$$Acc(D_f) \cap Acc(D_g) = [0, +\infty) \cup \{+\infty\}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x+1)^{\frac{1}{2}} = +\infty$$

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} x^{\frac{1}{2}} = +\infty$$



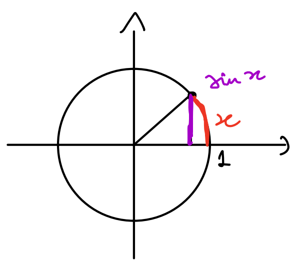
$$\sqrt{x+1} - \sqrt{x} = \left(\sqrt{x+1} - \sqrt{x}\right) \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} = \frac{(x+1) - x}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

$$\lim_{x \to +\infty} \left(\sqrt{x+1} - \sqrt{x}\right) = \lim_{x \to +\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0$$

$$\frac{1}{+\infty + \infty}$$

- $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$\frac{\sin(0)}{0} = \frac{0}{0}$$



$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$$

$$\frac{1 - \cos(0)}{0} = \frac{1-1}{0} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\operatorname{tg} x}{x} = \lim_{x \to 0} \frac{\sin(x)}{x \cdot \cos(x)} = \lim_{x \to 0} \underbrace{\frac{\sin(x)}{x}}_{\downarrow \, 1} \cdot \underbrace{\frac{1}{\cos(x)}}_{\downarrow \, 1} = 1$$

$$\frac{0}{0}$$

$$\lim_{x\to 0} \frac{2x \cdot \sin(x)}{1-\cos(x)} = \lim_{x\to 0} 2x \cdot \frac{\sin(x)}{x} \cdot x \cdot \frac{x^2}{1-\cos(x)} \cdot \frac{1}{x^2} =$$

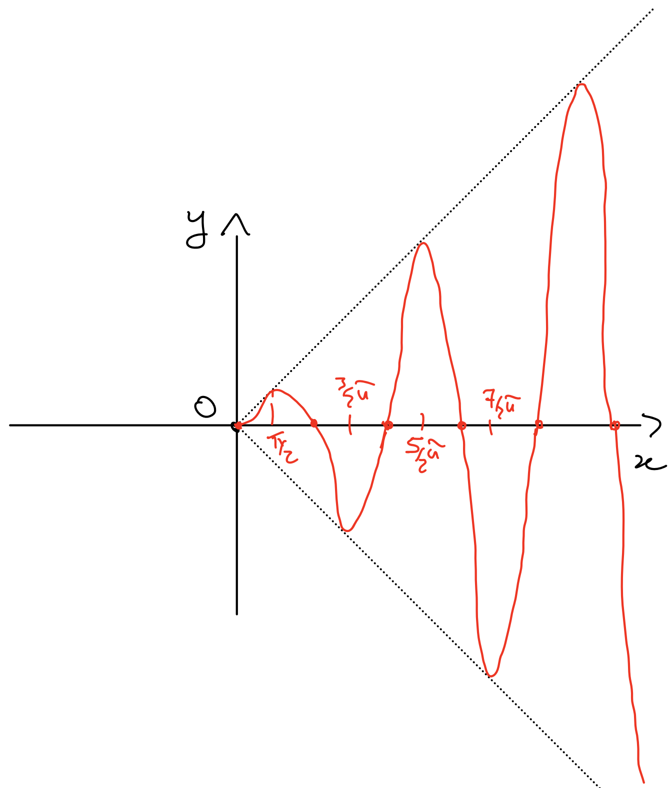
$$= \lim_{x\to 0} \underbrace{\frac{\sin(x)}{x}}_{\downarrow\, 1} \cdot \underbrace{\frac{x^2}{1-\cos(x)}}_{\downarrow\, 2} \cdot \underbrace{2}_{\downarrow\, 2} = 4$$

- $$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

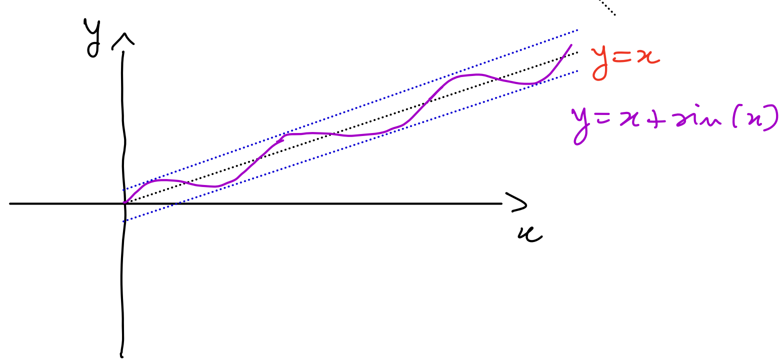
  $$\frac{\log(1)}{0} = \frac{0}{0}$$

- $$\lim_{x\to +\infty} x \cdot \sin(x)$$

  non esiste



- $$\lim_{x\to +\infty} \left(x + \sin(x) \right) = +\infty$$



$\forall x \in \mathbb{R}$ $\qquad x-1 \leq x+\sin(x) \leq x+1$

$$\lim_{x\to x_0} f(g(x)) = \lim_{t \to \lim_{x\to x_0} g(x)} f(t)$$

- $$\lim_{x\to +\infty} \log\left(1+\frac{1}{x}\right) = \lim_{x\to 1} \log(x) = 0$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)=1$$

$$\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right) \underset{\uparrow}{=} \lim_{x\to+\infty} \sin(x) \quad \nexists$$

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$

$$\lim_{x\to+\infty} \frac{\sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to+\infty} \frac{1}{x}\cdot \sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} t\cdot\sin(t)$$

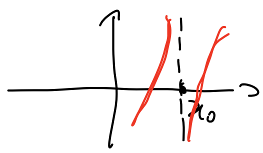
$$t=\frac{1}{x}, \quad \lim_{x\to+\infty} \frac{1}{x} = 0^+$$

$$\lim_{x\to+\infty} x\cdot\sin\left(\frac{1}{x}\right) = \ ?$$

$\longleftrightarrow$

<u>Def</u> $f: \mathbb{R}\to\mathbb{R}$, D dom naturale

- Se $+\infty$ $(-\infty)$ $\in Acc(D)$ si dice che f ha:
  - <u>asintoto orizzontale a $+\infty$ $(-\infty)$</u> se
  $$\lim_{\substack{x\to+\infty\\(-\infty)}} f(x) = c\in\mathbb{R}$$
  - <u>asintoto obliquo a $+\infty$ $(-\infty)$</u> se
  $$\lim_{\substack{x\to+\infty\\(-\infty)}} \frac{f(x)}{x} = a\in\mathbb{R} \quad e \quad \lim_{\substack{x\to+\infty\\(-\infty)}} \left(f(x)-ax\right) = b\in\mathbb{R}$$
  [$f(x)$ si comporta a $+\infty$ $(-\infty)$ come $y=ax+b$]
- Se $x_0\in Acc(D)\cap\mathbb{R}$ si dice che f ha <u>asintoto verticale a x_0</u> se $\lim_{x\to x_0^\pm} f(x) = \pm\infty$.



<u>ES</u> Studiare gli asintoti a $+\infty$ di:

$$f(x)=\frac{x^2-1}{x+2}, \quad f(x)=\sqrt{x^2+4x-1}$$