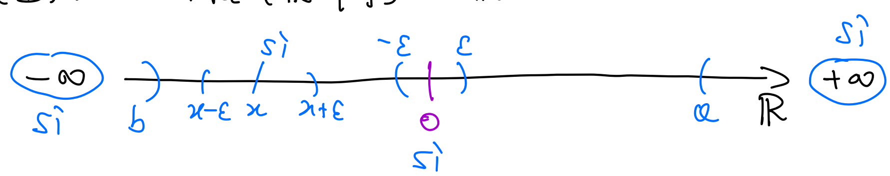
Limiti di funzioni $f:\mathbb{R}\to\mathbb{R}$, D dominio naturale

Def Sia $x_0 \in Acc(D) \subseteq \mathbb{R}^* = \mathbb{R}\cup\{\pm\infty\}$, si dice $\lim\limits_{x\to x_0} f(x) = \ell \in \mathbb{R}^*$
se $\forall$ intorno V di ℓ $\exists$ intorno U di x_0 tale che
se $x \in U \cap D \setminus \{x_0\}$ si ha che $f(x) \in V$.

Esercizi

- $\lim\limits_{x\to 0} \frac{1}{x^2} = +\infty$ ✓
- $\lim\limits_{x\to 0} \frac{1}{x} = +\infty$? $\quad f(x) = \frac{1}{x}$, $D = \mathbb{R}\setminus\{0\}$

$x_0 = 0 \in Acc(D)$? $\quad Acc(\mathbb{R}\setminus\{0\}) = \mathbb{R}^*$



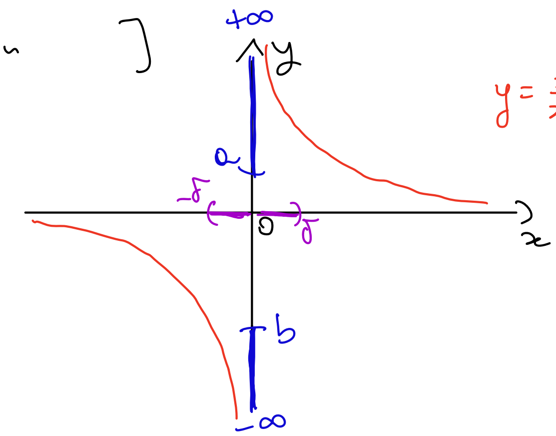
$\lim\limits_{x\to 0} \frac{1}{x} = +\infty \iff \forall a \in \mathbb{R}\cup\{-\infty\}\ \exists \delta>0$ tale che

se $x \in (-\delta,\delta)\setminus\{0\}$ si ha $\frac{1}{x} \in (a,+\infty)$ $\left[\frac{1}{x} > a\right]$

$[\forall x \quad " \quad\quad " \quad]$

$\Updownarrow$

$x < \frac{1}{a}$ $\quad x>0$ $\quad a>0$



<u>NO</u> perché se $x \in (-\delta, 0)$ si ha $\frac{1}{x} < 0$.

Def Sia $x_0 \in Acc(D)\cap\mathbb{R}$, si dice che $\lim\limits_{x\to x_0^+} f(x) = \ell \in \mathbb{R}^*$ (limite destro di f in x_0) se
$\forall$ intorno V di ℓ $\exists\, \delta>0$ tale che se $x \in (x_0, x_0+\delta)\cap D$
si ha $f(x) \in V$.

(<u>ES</u> $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$)

Def Sia $x_0 \in Acc(D) \cap \mathbb{R}$, si dice che $\lim_{x \to x_0^-} f(x) = \ell \in \mathbb{R}^*$ (limite sinistro di f in x_0) se

$\forall$ intorno V di ℓ $\exists\, \delta > 0$ tale che se $x \in (x_0 - \delta, x_0) \cap D$

si ha $f(x) \in V$.

(ES $\lim_{x \to 0^-} \frac{1}{x} = -\infty \iff$ Fissato $b \in \mathbb{R} \cup \{+\infty\}$ (ad es $b = -10^{10}$)

deve esistere $\delta > 0$ tale che $\forall\, x \in (-\delta, 0)$ deve valere $\frac{1}{x} < b$

$b < 0$, $x \in (-\delta, 0)$, $\frac{1}{x} < b \iff -\frac{1}{|x|} < b \iff -\frac{1}{|x|} < -|b|$

$\iff \frac{1}{|x|} > |b| \iff |x| < \frac{1}{|b|}$ ($b = -10^{10}$, $|x| < 10^{-10}$)

$\delta = \frac{1}{|b|}$.

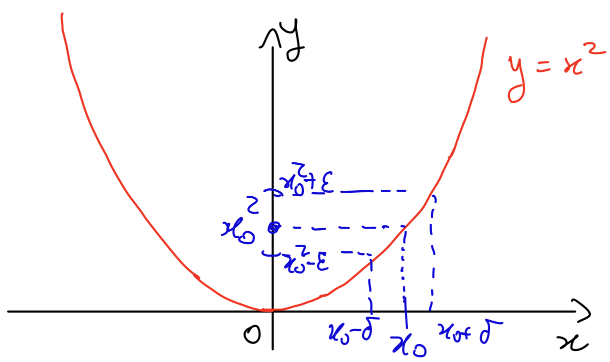
$x \in (-\delta, 0) \iff 0 > x > -\delta \iff 0 < |x| < \delta = \frac{1}{|b|}$

Teorema

- Il $\lim_{x \to x_0} f(x)$ se esiste è unico
- $\lim_{x \to x_0} f(x) = \ell$ è equivalente a $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \ell$.

ES $\lim_{x \to 0^+} \frac{1}{x} = +\infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. Quindi $\lim_{x \to 0} f(x)$ non esiste.

Esempi - $f(x) = x^2$, $D = \mathbb{R}$, $Acc(D) = \mathbb{R}^*$



$x_0 \in \mathbb{R}$, $\lim_{x \to x_0} x^2 = x_0^2$

$x_0 = +\infty$, $\lim_{x \to +\infty} x^2 = +\infty$

$x_0 = -\infty$, $\lim_{x \to -\infty} x^2 = +\infty$

$\lim_{x \to x_0} x^2 = x_0^2 \iff \forall \varepsilon > 0\ \exists\, \delta > 0$ tale che $\forall\, x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

si ha $f(x) \in (x_0^2 - \varepsilon, x_0^2 + \varepsilon)$ $[\, x_0^2 - \varepsilon < x^2 < x_0^2 + \varepsilon \,]$

Fisso $\varepsilon > 0$,

$$x^2 < x_0^2 + \varepsilon \iff x^2 - x_0^2 < \varepsilon \iff (x - x_0)(x + x_0) < \varepsilon$$

Trovare $\delta > 0$ per cui $x_0 - \delta < x < x_0 + \delta \iff -\delta < x - x_0 < \delta$

garantisce che $(x - x_0)(x + x_0) < \varepsilon$.

$(x_0 > 0)$

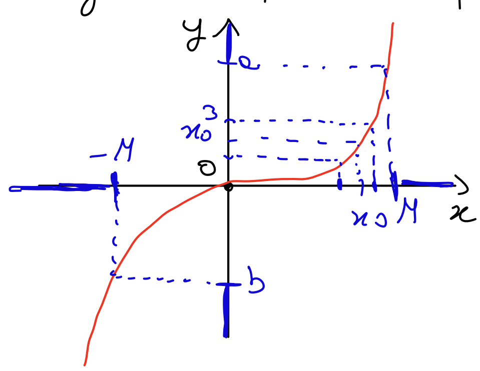
$$\underset{\substack{\updownarrow \\ \text{è vero se } \delta < x_0}}{(x - x_0)(x + x_0) <} \delta \cdot (3x_0) < \varepsilon \iff \delta < \frac{\varepsilon}{3x_0}$$

$$\lim_{x \to +\infty} x^2 = +\infty \iff \forall a \in \mathbb{R} \cup \{-\infty\} \ \exists M \in \mathbb{R} \cup \{-\infty\} \text{ tale che}$$

$\forall x \in (M, +\infty)$ si ha $f(x) \in (a, +\infty)$ $[x^2 > a]$
$[x > M]$

Fisso $a > 0$, esiste $M > 0$ tale che $x > M \Rightarrow x^2 > a$?
$M = \sqrt{a}$

- $f(x) = x^3$, $D = \mathbb{R}$, $Acc(D) = \mathbb{R}^*$



$x_0 \in \mathbb{R}$, $\lim_{x \to x_0} x^3 = x_0^3$

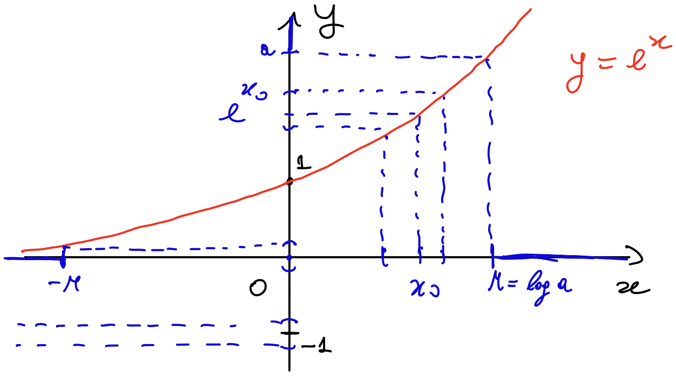
$x_0 = +\infty$, $\lim_{x \to +\infty} x^3 = +\infty$

$x_0 = -\infty$, $\lim_{x \to -\infty} x^3 = -\infty$

- $f(x) = x^k$, $k \in \mathbb{N}$

$x_0 \in \mathbb{R}$, $\lim_{x \to x_0} x^k = x_0^k$, $\lim_{x \to +\infty} x^k = +\infty$, $\lim_{x \to -\infty} x^k = \begin{cases} +\infty, & k \text{ pari} \\ -\infty, & k \text{ dispari} \end{cases}$

- $f(x) = e^x$, $D = \mathbb{R}$, $Acc(D) = \mathbb{R}^*$

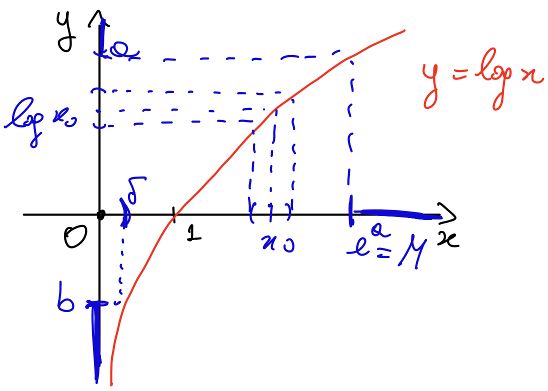


$x_0\in\mathbb{R}$, $\lim\limits_{x\to x_0} e^x = e^{x_0}$

$x_0=+\infty$, $\lim\limits_{x\to+\infty} e^x = +\infty$

$x_0=-\infty$, $\lim\limits_{x\to-\infty} e^x = 0$

- $f(x)=\log x$, $D=(0,+\infty)$, $Acc(D)=[0,+\infty)\cup\{+\infty\}$

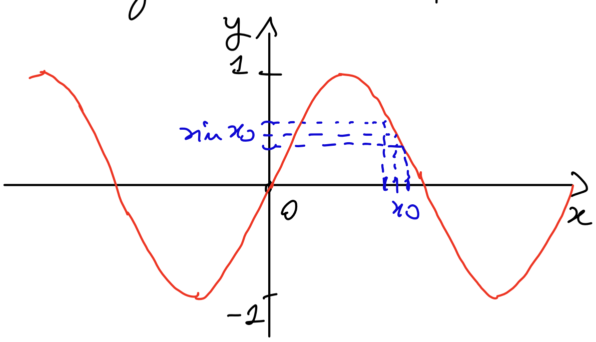


$x_0\in(0,+\infty)$, $\lim\limits_{x\to x_0}\log x = \log x_0$

$x_0=0$, $\lim\limits_{x\to 0}\log x = -\infty$

$x_0=+\infty$, $\lim\limits_{x\to+\infty}\log x = +\infty$

- $f(x)=\sin x$, $D=\mathbb{R}$, $Acc(D)=\mathbb{R}^*$



$x_0\in\mathbb{R}$, $\lim\limits_{x\to x_0}\sin x = \sin x_0$

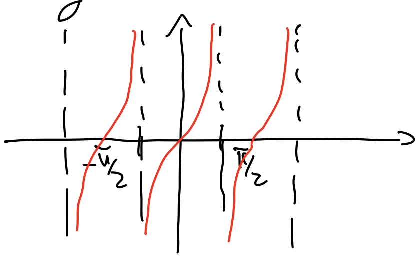
$x_0=+\infty$, $\lim\limits_{x\to+\infty}\sin x \nexists$ non esiste

$x_0=-\infty$, non esiste $\lim\limits_{x\to-\infty}\sin x$

- $f(x)=\cos x$, $D=\mathbb{R}$, $Acc(D)=\mathbb{R}^*$

$x_0\in\mathbb{R}$, $\lim\limits_{x\to x_0}\cos x = \cos x_0$, $\lim\limits_{x\to\pm\infty}\cos x$ non esiste.

- $f(x)=\operatorname{tg} x$, $D=\mathbb{R}\setminus\{\frac{\pi}{2}+k\pi \,/\, k\in\mathbb{Z}\}$, $Acc(D)=\mathbb{R}^*$



$x_0\in D$, $\lim\limits_{x\to x_0}\operatorname{tg} x = \operatorname{tg} x_0$

$x_0=\frac{\pi}{2}$, $\lim\limits_{x\to\frac{\pi}{2}^+}\operatorname{tg} x = -\infty$, $\lim\limits_{x\to\frac{\pi}{2}^-}\operatorname{tg} x = +\infty$

non esistono $\lim\limits_{x\to\pm\infty}\operatorname{tg} x$