$f: X \to X$, orbite di $x \in X = \{ f^n(x) \}_{n \in \mathbb{N}}$

ω-limite di $x = \{ y \in X \,/\, \exists\, n_k \nearrow +\infty$ t.c. $f^{n_k}(x) \xrightarrow[k \to +\infty]{} y \}$

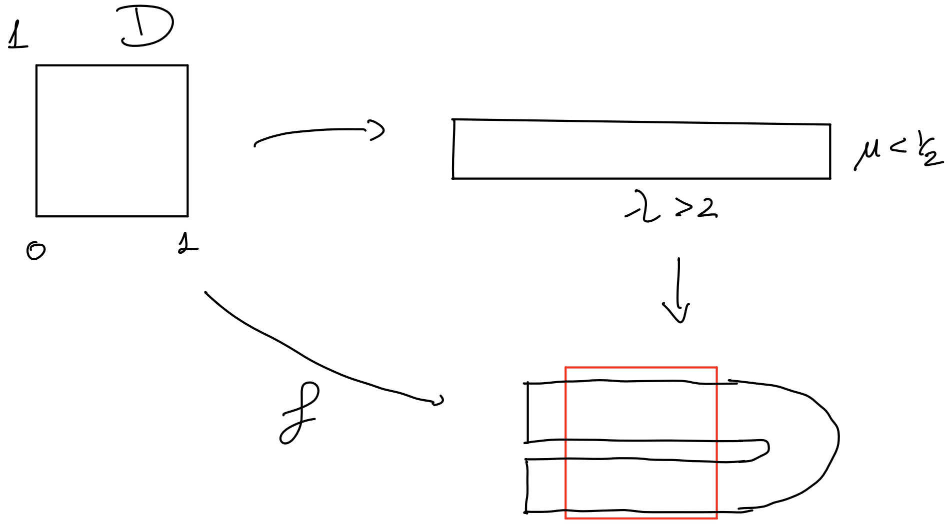
<u>Def</u> - Devaney

- $h_{top}(f) > 0$

$$\longleftrightarrow$$

<u>Esempi</u>

1. Ferro di cavallo di Smale , $D = [0,1] \times [0,1]$

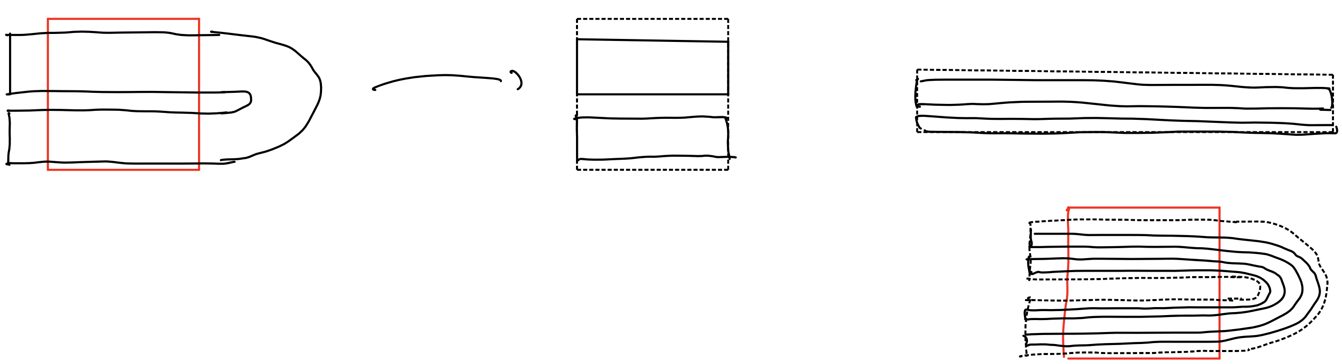


$$\Lambda := \bigcap_{n \in \mathbb{Z}} f^n(D)$$

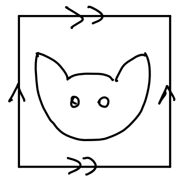
$$\varphi : \Lambda \longrightarrow \{0,1\}^{\mathbb{Z}} = \{ \omega = (\ldots \omega_{-2}\, \omega_{-1}\, \omega_0\, \omega_1\, \omega_2 \ldots) \,/\, \omega_i \in \{0,1\} \}$$

$$\begin{tikzcd}
\Lambda \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & \Lambda \arrow[d, "\varphi"] \\
\{0,1\}^{\mathbb{Z}} \arrow[r, "\sigma"'] & \{0,1\}^{\mathbb{Z}}
\end{tikzcd}$$

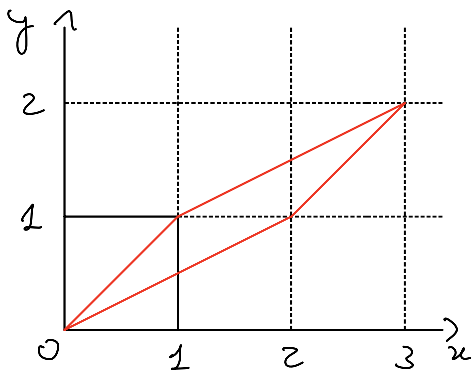
$$\varphi \circ f = \sigma \circ \varphi$$

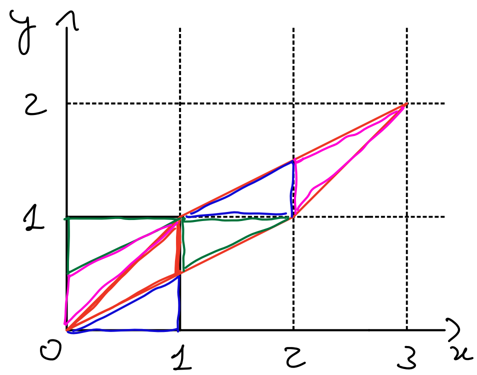


2. Gatto di Arnold, $\mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$



$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\det A = 1$., $\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}$





3. Mappa Standard, $X = S^1_x \times \mathbb{R}_y$

$$f(x,y) = \left(x + y - \frac{k}{2\pi} \sin 2\pi x,\ y - \frac{k}{2\pi} \sin 2\pi x \right)$$

<u>Dinamica simbolica</u> $\mathcal{A} = \{0, \dots, N-1\}$, N simboli

$$\Omega = \mathcal{A}^{\mathbb{N}_0} = \{ \omega = (\omega_0 \omega_1 \omega_2 \dots) \ / \ \omega_i \in \mathcal{A} \quad \forall i \in \mathbb{N}_0 \}$$

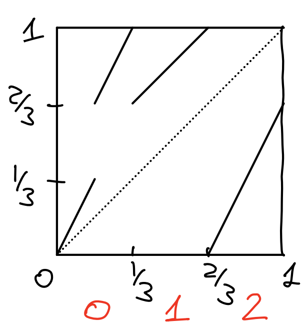
M matrice di transizione, $M \in \mathcal{M}(N \times N, \{0,1\})$

$M = (m_{ij})$ $\quad i,j \in \{0, \dots, N-1\}$, $\quad m_{ij} \in \{0,1\}$

$$\Omega_M = \{ \omega \in \Omega \ / \ m_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{N}_0 \}$$

<u>Esempio</u> $N = 3$ $\qquad M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$\omega = (0212 \dots) \in \Omega_M$



Subshift di tipo finito (Ω_M, σ)

Prop $h_{top}(\Omega_M, \sigma) = \log(\lambda_{Max})$, λ_{Max} è autovalore di modulo massimo di M.

OSS $M^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ $\quad \#(0\ldots0) = (M^2)_{00} = \{(000), (020)\}$

$\operatorname{tr} M^n = \#\{\omega \in \Omega_M \mid \sigma^n\omega = \omega\}$ $\quad h_{top}(\Omega_M, \sigma) = \lim_{n\to+\infty} \frac{1}{n} \log(\operatorname{tr} M^n)$

Teoria Ergodica

X spazio misurabile, σ-algebra di Borel $= \mathcal{B}$

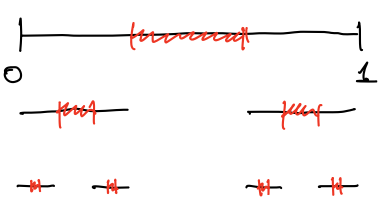
Es • $X = \mathbb{R}$, $A \in \mathcal{B}$, $\mu(A) = \int_A h(x)\,dx$, μ misura con densità $h \geq 0$

• A ha misura nulla se $\mu(A) = 0 = \int_A h(x)\,dx$

esempi sono punti, unione finita di punti, unione numerabile di punti

$\mu(\mathbb{Q}) = 0$. $\quad \mu(\{f^n(x)\}_{n\in\mathbb{N}}) = 0$.

esempio in $[0,1]$ = insieme di Cantor



C_1

C_2 $\quad C_\infty = \bigcap_{n\geq1} C_n$

C_3 $\quad m(C_n) = \frac{2^n}{3^n} \xrightarrow[n\to+\infty]{} 0$

$x = \sum_{i=1}^{+\infty} \frac{x_i}{3^i}$, $x_i \in \{0,1,2\}$ $\quad \forall x \in [0,1]$

Insieme di Cantor $= \{ x = \sum_{i=1}^{+\infty} \frac{x_i}{3^i} \mid x_i \in \{0,2\}\ \forall i \in \mathbb{N}\}$

OSS Se $\{C_n\} \subset \mathcal{B}$ e $C_n \subseteq C_{n-1}$ allora

$$\lim_{n\to+\infty} \mu(C_n) = \mu\left(\bigcap_n C_n\right)$$

Def $f : (X, \mathcal{B}) \to (X, \mathcal{B})$ misurabile

si dice che una misura μ è $\underline{f\text{-invariante}}$ se

$\mu(f^{-1}(A)) = \mu(A)$, $\forall A \in \mathcal{B}$

dove $f^{-1}(A) = \{ y \in X \,/\, f(y) \in A \}$.

Oss $\mu(A) = \int_A h(x)\,dx$

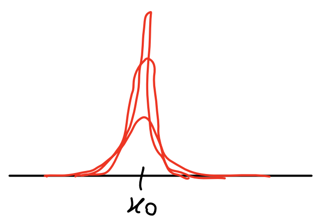
$$\mu(f^{-1}(A)) = \int_{f^{-1}(A)} h(x)\,dx \underset{\substack{y=f(x)\\ f \text{ invertibile},\ f \in C^1}}{=} \int_A \underline{h(f^{-1}(y))\, d(f^{-1}(y))} =$$

$$= \int_A h(f^{-1}(y)) \frac{1}{|f'(f^{-1}(y))|}\, dy$$

Es Se $x_0 \in X$ è fisso, $f(x_0) = x_0$, allora δ_{x_0} è f-invariante.

$$\delta_{x_0}(A) = \lim_{n \to +\infty} \int_A \rho_n(x)\,dx = \begin{cases} 1, & \text{se } x_0 \in A \\ 0, & \text{se } x_0 \notin A \end{cases}$$

$\{\rho_n\}$



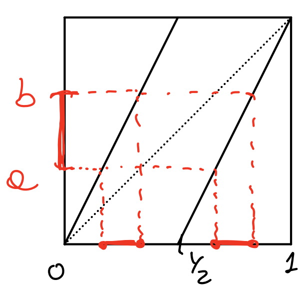
Se x_0 è periodico di periodo minimo p allora

$$\mu_{x_0} := \frac{1}{p} \sum_{i=0}^{p-1} \delta_{f^i(x_0)} \quad \text{è } f\text{-invariante}$$

Esempi

1. Endomorfismi lineari del cerchio $x \mapsto kx \pmod 1$, $k \in \mathbb{N}$, $k \geq 2$

la misura di Lebesgue è invariante.



$A = (a, b)$ $\quad m(A) = b - a$

$m(f^{-1}(A)) = m(\{ y \,/\, f(y) \in (a,b) \}) =$

$= k \cdot \frac{1}{k} m(A) = m(A).$

2. Full shift, $\Omega = \{0, \dots, N-1\}^{\mathbb{N}_0}$, $\sigma : \Omega \to \Omega$

$q = (q_0, \dots, q_{N-1})$, $q_i \geq 0$, $\sum_{i=0}^{N-1} q_i = 1$

Misura prodotto μ_q è σ-invariante

<u>Def</u> Fissato $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, $s_1, \dots, s_k \in \{0, \dots, N-1\}$,

$$C(n, k, s_1, \dots s_k) = \{\omega \in \Omega \,/\, \omega_n = s_1, \omega_{n+1} = s_2, \dots, \omega_{n+k-1} = s_k\}$$

$$\omega = (\dots\dots \underbrace{\omega_n \dots \omega_{n+k-1}}_{s_1 \dots s_k} \dots)$$

$$\mu_q(C(n, k, s_1, \dots s_k)) = \prod_{j=1}^{k} q_{s_j}$$

3. Subshift di tipo finito

$M \in \mathcal{M}(N \times N, \{0, 1\})$, (Ω_M, σ)

Π matrice stocastica $\Pi = (\pi_{ij}) \in \mathcal{M}(N \times N, [0, 1])$

$\sum_{j=0}^{N-1} \pi_{ij} = 1$, $\pi_{ij} > 0 \iff m_{ij} = 1$.

<u>Teo</u> Se $\exists\, \bar{n}$ t.c. $\Pi^n > 0$ $\forall n \geq \bar{n}$ allora $\exists!\, q = (q_0, \dots, q_{N-1})$, $q_i \geq 0$ $\sum_{i=0}^{N-1} q_i = 1$ t.c. $q\Pi = q$.

Misura μ_π è σ-invariante

$$\mu_\pi(C(n, k, s_1, \dots s_k)) = q_{s_1} \prod_{i=1}^{k-1} \pi_{s_i s_{i+1}}$$

$\longleftrightarrow$

<u>Def</u> Una misura μ è <u>ergodica</u> per $(X, \mathcal{B}, f)$ se $\forall A \in \mathcal{B}$ t.c. $f^{-1}(A) = A$ vale $\mu(A) = 0$ oppure $\mu(X \setminus A) = 0$.

<u>Teorema</u> (Ergodico di Birkhoff). Sia $(X, \mathcal{B}, f)$ e μ sia misura di probabilità f-invariante ed ergodica. Allora $\forall \varphi : X \to \mathbb{R}$, $\int_X |\varphi| \, d\mu < +\infty$, vale che

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int_X \varphi\, d\mu \quad \left(= \int_X \varphi(x)\, h(x)\, dx \right)$$

per ogni $x \in \tilde{X} \subset X$ con $\mu(\tilde{X}) = 1$.

<u>Applic.</u> $f: S^1 \to S^1$, $f(x) = 10x \pmod 1$, m misura di Lebesgue è di prob, f-invariante, ergodica.

Se $A = (a,b)$ allora per $\varphi = \chi_A$ vale che per m-q.o. $x \in [0,1)$

$$\lim_{n\to+\infty} \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} \chi_A(f^j(x))}_{\frac{1}{n}\#\{j\in\{0,\dots,n-1\}\,/\,f^j(x)\in A\}} = \int_X \chi_A\, dm = m(A) = b-a$$