Ancora sui Teoremi limite.

Legge debole dei grandi numeri

$\{X_k\}$ succ. di v.a. indipendenti ed equidistribuite.

Sia $\mu = E[X_k]$ $\forall k \geq 1$ e $E[X_k^2] < +\infty$ $\forall k \geq 1$

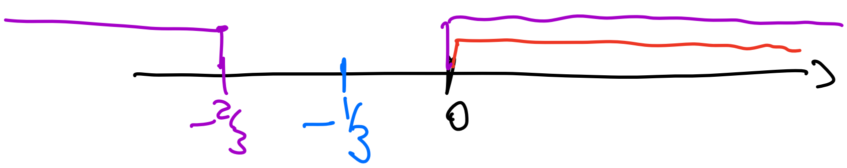
allora $\frac{X_1 + \dots + X_n}{n}$ conv. in probabilità μ, ossia

$$\forall \varepsilon > 0 \quad \lim_{n \to +\infty} \mathbb{P}\left(\left| \frac{X_1 + \dots + X_n}{n} - \mu \right| > \varepsilon \right) = 0$$

Applicazioni $\quad X_k$ sia v.a. discreta $E[X_k] = -\frac{1}{3}$

(es $p(-1) = \frac{2}{3}$, $p(1) = \frac{1}{3}$)

$$\mathbb{P}(X_1 + \dots + X_n > 0) = \mathbb{P}\left(\frac{X_1 + \dots + X_n}{n} > 0 \right) =$$



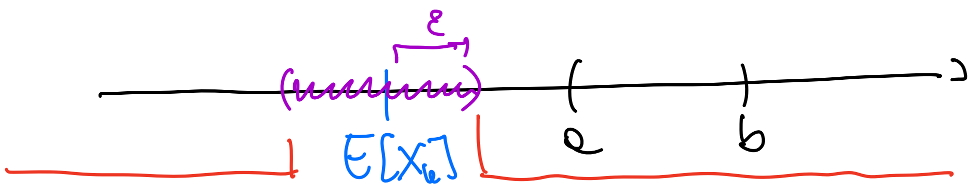
$$= \mathbb{P}\left(\frac{X_1 + \dots + X_n}{n} - \left(-\frac{1}{3}\right) > \frac{1}{3} \right) \leq$$

$$\leq \mathbb{P}\left(\left| \frac{X_1 + \dots + X_n}{n} - E[X_k] \right| > \overset{=\varepsilon}{\frac{1}{3}} \right) \xrightarrow[n \to +\infty]{} 0$$

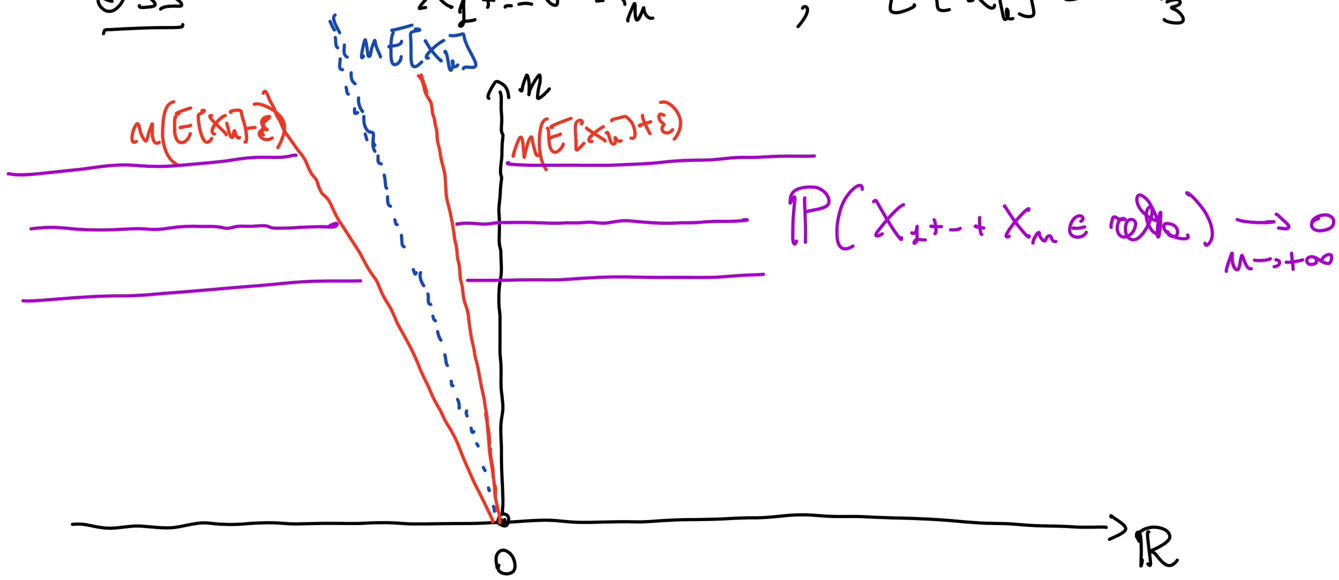
OSS sotto le stesse ipotesi, se $(a,b) \subset \mathbb{R}$ t.c.

$E[X_k] \notin (a,b)$ allora

$$\lim_{n\to+\infty} \mathbb{P}\left(\frac{X_1+\dots+X_n}{n} \in (a,b)\right) = 0$$



OSS $X_1+\dots+X_n$, $E[X_k] = -\frac{1}{3}$



$$\left|\frac{X_1+\dots+X_n}{n} - E[X_k]\right| > \varepsilon = \left|X_1+\dots+X_n - nE[X_k]\right| > \varepsilon n$$

$$\mathbb{P}\left(\left|\frac{X_1+\dots+X_n}{n} - E[X_k]\right| < \varepsilon\right) \xrightarrow[n\to+\infty]{} 1$$

Teorema del Limite Centrale

Sia $\{X_k\}$ succ. di v.a. indipendenti ed equidistribuite, siano $\mu = E[X_k]$ $\forall k \geq 1$ e $\sigma(X_k) \neq 0$ $\forall k \geq 1$.

Allora $\forall\, a, b \in \mathbb{R} \cup \{\pm\infty\}$, con $a < b$, si ha

$$\lim_{n\to+\infty} \mathbb{P}\left(a < \frac{X_1 + \dots + X_n - n\cdot\mu}{\sigma(X_n)\cdot\sqrt{n}} < b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt$$

$$= \Phi(b) - \Phi(a)$$

$\left(\dfrac{X_1 + \dots + X_n - n\mu}{\sigma(X_n)\sqrt{n}} \right.$ converge in distribuzione a $\left. N(0,1) \right)$

<u>OSS</u>

$$\frac{X_1 + \dots + X_n - n\mu}{\sigma(X_n)\sqrt{n}} = \frac{\left(\frac{X_1 + \dots + X_n}{n} - \mu\right)}{\sigma(X_n)/\sqrt{n}} =$$

$$= \frac{\sqrt{n}}{\sigma(X_n)} \left(\frac{X_1 + \dots + X_n}{n} - \mu \right)$$

<u>Applicazioni</u> $\{X_n\}$ v.a. indip. ed equidistribuite

Se $(a,b) \subset \mathbb{R}$ t.c. $E[X_n] \notin (a,b)$ allora

$$\mathbb{P}\left(a < \frac{X_1 + \dots + X_n}{n} < b\right) \xrightarrow[n\to+\infty]{} 0$$

Se $\sigma = \sigma(X_n) \neq 0$, allora ,

$$a < \frac{X_1 + \dots + X_n}{n} < b \iff an < X_1 + \dots + X_n < nb$$

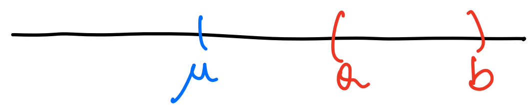
$$\iff an - n\cdot\mu < X_1 + \dots + X_n - n\mu < nb - n\cdot\mu$$

$$\iff \frac{n(a-\mu)}{\sigma\sqrt{n}} < \frac{X_1+\dots+X_n-n\mu}{\sigma\sqrt{n}} < \frac{n(b-\mu)}{\sigma\sqrt{n}}$$

$$\mathbb{P}\left(a < \frac{X_1+\dots+X_n}{n} < b\right) = \mathbb{P}\left(\frac{n(a-\mu)}{\sigma\sqrt{n}} < \frac{X_1+\dots+X_n-n\mu}{\sigma\sqrt{n}} < \frac{n(b-\mu)}{\sigma\sqrt{n}}\right)$$

$$\underset{n\geq 80}{\sim} \mathbb{P}\left(\frac{n(a-\mu)}{\sigma\sqrt{n}} < Z < \frac{n(b-\mu)}{\sigma\sqrt{n}}\right) \quad \text{con } Z \sim N(0,1)$$

$$= \int_{\sqrt{n}\frac{a-\mu}{\sigma}}^{\sqrt{n}\frac{b-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt \leq \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{n(a-\mu)^2}{2\sigma^2}} \cdot \frac{\sqrt{n}}{\sigma}(b-a)}_{\xrightarrow{n\to+\infty} 0}$$



ES $\quad \mu = -\frac{1}{3}$, $\quad \sigma = \sqrt{\mathrm{Var}(X_k)^2} = \frac{2}{3}\sqrt{2}$

$$\mathrm{Var}(X_k) = E[X_k^2] - (E[X_k])^2 = (-1)^2\frac{2}{3} + (1)^2\frac{1}{3} - \frac{1}{9} =$$
$$= 8/9$$

$$\mathbb{P}(X_1+\dots+X_n > 0) = \mathbb{P}\left(X_1+\dots+X_n - n\cdot\left(-\tfrac{1}{3}\right) > -n\left(-\tfrac{1}{3}\right)\right)$$

$$= \mathbb{P}\left(\frac{X_1+\dots+X_n + n\frac{1}{3}}{\frac{2}{3}\sqrt{2}\cdot\sqrt{n}} > \frac{n/3}{\frac{2}{3}\sqrt{2}\sqrt{n}} = \frac{\sqrt{n}}{2\sqrt{2}}\right) \underset{n \text{ grande}}{\sim}$$

$$\sim \mathbb{P}\left(Z > \frac{\sqrt{n}}{2\sqrt{2}}\right), \quad Z \sim N(0,1)$$

Se $n = 100$, $\quad \mathbb{P}\left(Z > \frac{5}{\sqrt{2}}\right) = 1 - \Phi\left(\frac{5}{\sqrt{2}}\right) \sim$

$$\sim 1 - \Phi(3.536) \sim 1 - 0.999795$$

$$\sim 0.0002$$

<u>ES</u> Gli aerei di una compagnia hanno 180 posti, ma la compagnia sa che chi ha prenotato un posto si presenta con probabilità 0.9.

Di conseguenza la compagnia vende 195 biglietti per ogni volo.

Qual è la probabilità che almeno un passeggero che ha prenotato non trovi posto?

Sia $\{X_k\}$ succ. di v.a di Bernoulli di parametro

$p = 0.9$, con $X_k = +1$ se il k-esimo passeggero si presenta.

$$\mathbb{P}(\text{"almeno un passeggero non trova posto"}) = \mathbb{P}(X_1 + X_2 + \dots + X_{195} \geq 181) =$$

$$E[X_k] = p = \frac{9}{10}, \quad Var(X_k) = p(1-p) = \frac{9}{10} \cdot \frac{1}{10} = \frac{9}{100}$$

$$\sigma = \sqrt{Var(X_k)} = \sqrt{\frac{9}{100}} = \frac{3}{10}$$

$$= \mathbb{P}\left(\frac{X_1 + \dots + X_{195} - 195 \cdot \frac{9}{10}}{\frac{3}{10}\sqrt{195}} \geq \frac{181 - 195 \cdot \frac{9}{10}}{\frac{3}{10}\sqrt{195}} \right) \sim$$

$$\sim \mathbb{P}\left(Z \geq \frac{181 - 195 \cdot \frac{9}{10}}{\frac{3}{10}\sqrt{195}} \right) \qquad Z \sim \mathcal{N}(0,1)$$

$$\sim 1 - \Phi(1.313) \sim 1 - 0.905 = 0.095$$

è un'approssimazione di $\mathbb{P}(X_1 + \dots + X_{195} \geq 181) =$

$= \mathbb{P}(Y \geq 181)$ con $Y \sim B(195, \frac{9}{10})$

$$= \sum_{k=181}^{195} \binom{195}{k} \left(\frac{9}{10}\right)^k \cdot \left(\frac{1}{10}\right)^{195-k}$$

<u>ES</u> Siano $X_1, \dots X_{100}$ v.a. che misurano le altezze di una classe di studenti. Supponiamo che $E[X_k] = 180$ cm con $Var(X_k) = 100$ cm.

(i) Stimare $\mathbb{P}\left(\frac{X_1 + \dots + X_{100}}{100} > 182 \right)$

(ii) Potendo modificare il numero di studenti, ossia avendo $X_1, \dots X_n$ v.a. con $E[X_k] = 180$, $Var(X_k) = 100$, per quali n si ha $\mathbb{P}\left(\frac{X_1 + \dots + X_n}{n} \leq 182 \right) \geq 0.98$?

(iii) Se le altezze dei professori hanno legge $N(170, 64)$, qual è la probabilità che un prof a caso sia più basso della media dei 100 studenti?

$$\longleftrightarrow$$

(i) $\mathbb{P}\left(\frac{X_1 + \dots + X_{100}}{100} > 182 \right)$

$$\frac{X_1 + \dots + X_{100}}{100} \sim E[X_k] = 180$$

$$\mathbb{P}\left(\frac{X_1 + \dots + X_{100}}{100} > 182 \right) = \mathbb{P}\left(\frac{X_1 + \dots + X_{100}}{100} - 180 > 182 - 180 \right)$$

$$= \mathbb{P}\Bigg(\underbrace{\frac{\sqrt{100}}{\sigma} \left(\frac{X_1 + \dots + X_{100}}{100} - 180 \right)}_{\frac{X_1 + \dots + X_{100} - 100 \cdot 180}{\sigma \sqrt{100}}} > \frac{\sqrt{100}}{\sigma} (182 - 180) \Bigg) \sim$$

$Z \sim N(0,1)$ $\qquad \sigma = \sqrt{Var(X_k)} = \sqrt{100} = 10$

$$\sim \mathbb{P}(Z > 2) = 1 - \Phi(2) \sim 1 - 0.97725$$

$$\sim 0.02275$$

(ii) $\mathbb{P}\left(\frac{X_1 + \dots + X_n}{n} < 182 \right) \geq 0.98$, n?

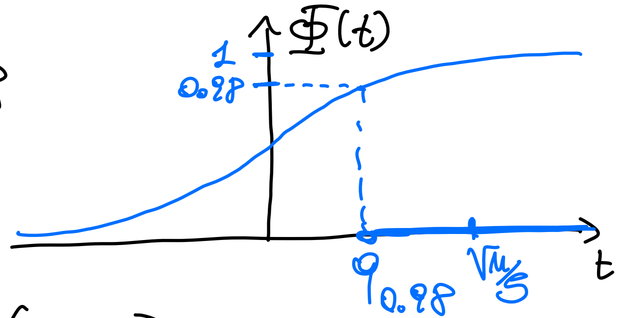
$$\mathbb{P}\left(\frac{\sqrt{n}}{\sigma}\left(\frac{X_1+\dots+X_n}{n}-\mu\right) < \frac{\sqrt{n}}{\sigma}(182-\mu)\right) \geq 0.98$$

$$\mu = 180, \quad \sigma = 10$$

$$Z \sim N(0,1)$$

$$\mathbb{P}\left(Z < \frac{\sqrt{n}}{10}\cdot(182-180)\right) = \mathbb{P}\left(Z < \frac{\sqrt{n}}{5}\right) \geq 0.98$$

$$\Phi\left(\frac{\sqrt{n}}{5}\right) \geq 0.98$$



$$\Phi\left(\frac{\sqrt{n}}{5}\right) \geq 0.98 = \Phi(q_{0.98})$$

$$\frac{\sqrt{n}}{5} \geq q_{0.98} \sim 2.055$$

$$n \geq (5\cdot 2.055)^2 \sim 105.576$$

$$\boxed{n \geq 106}$$

(iii) $\frac{X_1+\dots+X_{100}}{100}$, $\mu = 180$, $\sigma^2 = 100$

$$Y \sim N(170, 64)$$

$$\mathbb{P}\left(\frac{X_1+\dots+X_{100}}{100} > Y\right)$$

$$\frac{X_1+\dots+X_{100}-100\cdot\mu}{\sigma\cdot\sqrt{100}} \sim N(0,1)$$

$$X_1+\dots+X_{100} = \sigma\sqrt{100}\cdot Z + 100\cdot\mu, \quad Z\sim N(0,1)$$

$$\frac{X_1+\dots+X_{100}}{100} \sim N\left(\mu, \frac{\sigma^2}{100}\right)$$

$$Y\sim N(170,64), \quad -Y\sim N(-170,64)$$

$$\frac{X_1+\dots+X_{100}}{100} > Y \iff \frac{X_1+\dots+X_{100}}{100} - Y > 0$$

$\boxed{NO}$

$$\boxed{\frac{\sigma}{\sqrt{100}}\cancel{Z}_{Z_1} + \mu > 8\cancel{Z}_{Z_2} + 170 \iff \cancel{\left(\frac{\sigma}{\sqrt{100}}-8\right)Z+\mu}-170>0}$$

$$\frac{X_1+\dots+X_{100}}{100} - Y \sim N\left(\mu-170, \frac{\sigma^2}{100}+64\right)$$

$$P\left(\frac{X_1+\dots+X_{100}}{100} > Y\right) = P(W>0) =$$

$$W\sim N\left(\mu-170, \frac{\sigma^2}{100}+64\right)$$

$$Z\sim N(0,1), \quad W = \sqrt{\frac{\sigma^2}{100}+64}\; Z + (\mu-170)$$

$$= P\left(Z > \frac{170-\mu}{\sqrt{\frac{\sigma^2}{100}+64}}\right) \sim P(Z>-1.24)$$

$$= 1-\Phi(-1.24) = \Phi(1.24) \sim 0.89251$$