Chapter 1

Introduction and basic concepts

Definition 1.1. A *dynamical system* is a triple $(X, G, \mathcal{S})$ defined as the action $\mathcal{S}$ of a semigroup G with identity e on a set X, that is a function

$$\mathcal{S} : G \times X \to X$$

such that $\mathcal{S}(e, x) = x$ for all $x \in X$, and $\mathcal{S}(g_1, \mathcal{S}(g_2, x)) = \mathcal{S}(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

In the following, the set X is assumed to be a locally compact connected metric space.

Two main examples of dynamical system are given in the following definitions.

Definition 1.2. A *discrete-time dynamical system* is defined by the action of $\mathbb{N}_0$ on a set X defined through the iterations of a map $T : X \to X$ by

$$\mathcal{S}(n, x) = T^n(x),$$

where $T^n = T \circ \cdots \circ T$ is the composition of T with itself n times. A discrete-time dynamical system is denoted by the triple $(X, \mathbb{N}_0, T)$.

If the map T is invertible, the system can be extended to the action of the group $\mathbb{Z}$ on X. Examples of a discrete-time dynamical system are sequences defined by a recurrence relation. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_0 = a \in \mathbb{R}, \qquad x_n = f(x_{n-1}) \quad \forall\, n \geq 1,$$

for a real-valued function f. This corresponds to the dynamical system defined on $X = \mathbb{R}$ through the iterations of the map $f : \mathbb{R} \to \mathbb{R}$, that is $x_n = f^n(a)$.

Definition 1.3. A *continuous-time dynamical system* is defined by the action of $\mathbb{R}$ on a set $X \subset \mathbb{R}^n$ defined through the flow $\phi_t(\underline{x})$ of an autonomous ordinary differential equation $\dot{\underline{x}}(t) = F(\underline{x})$, that is

$$\mathcal{S}(t, \underline{x}) = \phi_t(\underline{x}),$$

where $\phi_t(\underline{x})$ is the solution of an ordinary differential equation[1] with initial condition $\underline{x}$, and $\phi_t : X \to X$ is a continuous function. A continuous-time dynamical system is denoted by the triple $(X, \mathbb{R}, \phi)$.

Definiton 1.3 includes the case of non-autonomous differential equations by using the standard procedure of "enlarging" the space of variables. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ define a time-dependent vector field $F(t, \underline{x})$ on $\mathbb{R}^n$ and consider the Cauchy problem

$$\begin{cases} \dot{\underline{x}}(t) = F(t, \underline{x}(t)) \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

If we let $\underline{y} = (\underline{x}, t) \in \mathbb{R}^{n+1}$ and $\tilde{F}(\underline{y}) = (F(t, \underline{x}), 1)$ be a vector field on $\mathbb{R}^{n+1}$, the previous non-autonomous Cauchy problem is equivalent to the autonomous problem

$$\begin{cases} \dot{\underline{y}}(t) = \tilde{F}(\underline{y}(t)) \\ \underline{y}(0) = (\underline{x}_0, 0) \end{cases}$$

A similar procedure can be applied to the case of sequences defined by a recurrence relation depending on n.

Analogously, it is known that ordinary differential equations of order greater than one can be reduced to systems of ordinary differential equations of order one, hence again included in Definition 1.3. The same is true for the discrete-time case. The following example shows how the procedure works.

Example 1.1. Let us consider the sequence $\{x_n\}$ defined as follows

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 1, \quad x_n = x_{n-1} + 2^{n-3}\, x_{n-2} + x_{n-3} \quad \forall\, n \geq 4.$$

[1] All ordinary differential equations we consider are assumed to have the property of local uniqueness of solutions and time-interval of existence of solutions given by $\mathbb{R}$ up to reparametrization.

We define the vector $\underline{y}_n = (x_n, x_{n-1}, x_{n-2}, n) \in \mathbb{R}^4$. Then using the previous recurrence we have

$$\underline{y}_{n+1} = \Big(x_n + 2^{n-2}\, x_{n-1} + x_{n-2}\,,\, x_n\,,\, x_{n-1}\,,\, n+1\Big) = T(\underline{y}_n) \quad \forall\, n \geq 3$$

with initial condition set to be $y_3 = (1, 1, 0, 3)$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(a, b, c, d) = \Big(a + 2^{d-2}\, b + c\,,\, a\,,\, b\,,\, d+1\Big).$$

The idea of an action of a semigroup on a set X can be used in more abstract contexts. Here we show only one example of algebraic nature that will be studied in more details in part IV of this book.

Example 1.2. Let X be a group, G be $\mathbb{R}$, and consider the action S on X given by multiplication for a one-parameter subgroup of X. For example, if $X = SL(2, \mathbb{R})$ the action of $\mathbb{R}$ defined by

$$S(t, x) = x \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in SL(2, \mathbb{R})$$

represents the geodesic flow on the hyperbolic Poincaré half-plane (see Chapter 10).

Definition 1.4. Given a dynamical system $(X, G, \mathcal{S})$, the *orbit* of a point $x \in X$ is the set $\mathcal{O}(x) := \{\mathcal{S}(g, x) \,:\, g \in G\}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the orbit of a point $x \in X$ is the set

$$\mathcal{O}(x) = \{T^n(x) \,:\, n \in \mathbb{N}_0\}\,. \tag{1.1}$$

If the map T is invertible, then we can consider the action of the group $\mathbb{Z}$ on X and define the *forward orbit* and *backward orbit* of a point $x \in X$ by

$$\mathcal{O}^+(x) := \{T^n(x) \,:\, n \geq 0\}\,, \quad \mathcal{O}^-(x) := \{T^n(x) \,:\, n \leq 0\}\,.$$

The orbit $\mathcal{O}(x)$ is then given by $\mathcal{O}^+(x) \cup \mathcal{O}^-(x)$.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *forward orbit* and *backward orbit* of a point $\underline{x} \in X$ are defined by

$$\mathcal{O}^+(\underline{x}) := \bigcup_{t \geq 0} \phi_t(\underline{x}), \qquad \mathcal{O}^-(\underline{x}) := \bigcup_{t \leq 0} \phi_t(\underline{x}), \tag{1.2}$$

and the orbit is $\mathcal{O}(\underline{x}) = \mathcal{O}^+(\underline{x}) \cup \mathcal{O}^-(\underline{x})$.

Definition 1.5. Given a dynamical system $(X, G, \mathcal{S})$, the *centralizer* of a point $x \in X$ is the sub-semigroup

$$\mathcal{C}(x) := \{g \in G \,:\, S(g, x) = x\}\,.$$

A point x is called *fixed* if $\mathcal{C}(x) = G$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, a point $x \in X$ is fixed if and only if $T(x) = x$. If x is not a fixed point but its centralizer is not G, x is called *periodic* and the minimum positive element in $\mathcal{C}(x)$ is the *minimal period* of x. For a fixed point $\mathcal{O}(x) = \{x\}$, and for a periodic point of minimal period p

$$\mathcal{O}(x) = \left\{x,\, T(x),\, T^2(x),\, \dots,\, T^{p-1}(x)\right\}.$$

For a non-invertible map there might be points which are not periodic but are pre-images of a periodic point. For such points x, the centralizer contains only the identity of G, but there exists $k \geq 1$ such that $\mathcal{C}(T^k(x))$ has a minimal positive element p. These points are called *pre-periodic with minimal period p*.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$ given by the solutions to $\dot{\underline{x}}(t) = F(\underline{x})$, a point $\underline{x} \in X$ is fixed if and only if $F(\underline{x}) = \underline{0}$. If $\underline{x}$ is not a fixed point but its centralizer is not trivial, $\underline{x}$ is called *periodic* and the minimum positive element in $\mathcal{C}(\underline{x})$ is the *minimal period* of T. A periodic point $\underline{x}$ of minimal period $T > 0$ satisfies

$$\phi_{t+T}(\underline{x}) = \phi_t(\underline{x}), \quad \forall\, t \in \mathbb{R},$$

and

$$\phi_{t+s}(\underline{x}) \neq \phi_t(\underline{x}), \quad \forall\, s \in (0, T),\, t \in \mathbb{R}.$$

For a fixed point $\mathcal{O}(\underline{x}) = \{\underline{x}\}$. For a periodic point of minimal period T

$$\mathcal{O}(\underline{x}) = \bigcup_{0 \leq t \leq T} \phi_t(\underline{x}),$$

and its orbits is called *a periodic orbit of period T*.

Definition 1.6. Given a dynamical system $(X, G, \mathcal{S})$, a set $A \subset X$ is called *invariant* if for each $x \in A$ it holds $\mathcal{S}(g, x) \in A$ for all $g \in G$.

For a continuous-time dynamical system one can introduce a weaker notion. We say that a subset A of X is *forward invariant* if for each $\underline{x} \in A$

it holds $\phi_t(\underline{x}) \in A$ for all $t \geq 0$. Analogously A is called *backward invariant* if the same relation holds for all $t \leq 0$. By definition, A is *invariant* if the previous relation holds for all $t \in \mathbb{R}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, we consider more situations. We say that a subset A of X is *forward invariant* if $T(A) \subseteq A$, A is called *fully invariant* if $T(A) = A$, A is called *completely invariant* if $T^{-1}(A) = A$. The different notions are useful in different approaches.

Finally, if the action of the group G on X can be interpreted in terms of time evolution, we can introduce notions about the forward and backward evolution of an orbit. In more general situations, one studies the set of all the possible limit points of an orbit as the sequence of the elements of the group acting varies.

Definition 1.7. For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the *ω-limit set* of a point $x \in X$ is the set

$$\omega(x) := \{y \in X \,:\, \exists\, n_k \to +\infty \text{ such that } T^{n_k}(x) \to y \text{ as } k \to \infty\}.$$

Definition 1.8. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *α-limit set* of a point $\underline{x} \in X$ is the set

$$\alpha(\underline{x}) := \left\{\underline{y} \in X \,:\, \exists\, t_k \to -\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Analogously the *ω-limit set* of a point $\underline{x} \in X$ is the set

$$\omega(\underline{x}) := \left\{\underline{y} \in X \,:\, \exists\, t_k \to +\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Proposition 1.1. *Given a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, let $\underline{x} \in X$ such that $\mathcal{O}^+(\underline{x})$ is bounded. Then the set $\omega(\underline{x})$ is non-empty, compact and invariant. If $\mathcal{O}^-(\underline{x})$ is bounded, the same holds for the set $\alpha(\underline{x})$.*

Proof (see [Gl94]). Given a point $\underline{x}$ with bounded forward orbit, let us consider a strictly increasing sequence $\{\tau_j\}_{j=0}^{\infty}$ of times in $\mathbb{R}^+$ with $\tau_0 = 0$ and $\tau_j \to +\infty$, and let $\underline{x}_j := \phi_{\tau_j}(\underline{x})$. We first show that

$$\omega(\underline{x}) = \bigcap_{j=0}^{\infty} \overline{\mathcal{O}^+(\underline{x}_j)}. \tag{1.3}$$

By the definition of the ω-limit set, it is immediate that $\omega(\underline{x}) \subset \overline{\mathcal{O}^+(\underline{x}_j)}$ for all $j \geq 0$. Hence it remains to show that if $\underline{y} \in \cap_{j=0}^{\infty} \overline{\mathcal{O}^+(\underline{x}_j)}$ then $\underline{y} \in \omega(\underline{x})$. By definition of closure of a set, for all $j \geq 0$ there exists a sequence $\{\underline{\xi}_n^j\}_n$ of points in $\mathcal{O}^+(\underline{x}_j)$ such that $\underline{\xi}^j_n \to \underline{y}$, hence there exists a sequence $\{t^j_n\}_n$ such that $\phi_{t^j_n}(\underline{x}_j) \to \underline{y}$. In particular we have proved that there exists a strictly increasing diverging sequence $\{\tau_j\}_{j=0}^{\infty}$ and sequences $\{t^j_n\}_n$ such that

$$\phi_{\tau_j + t^j_n}(\underline{x}) \xrightarrow[n\to\infty]{} \underline{y}, \quad \forall\, j \geq 0.$$

From $\{\tau_j + t^j_n\}_{j,n}$ we can then extract a diverging sequence $\{\tilde{t}_k\}_k$ such that $\phi_{\tilde{t}_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. Hence $\underline{y} \in \omega(\underline{x})$, and (1.3) is proved.

The first properties of $\omega(\underline{x})$ follow from (1.3). The sets $\{\overline{\mathcal{O}^+(\underline{x}_j)}\}_j$ define a decreasing sequence of non-empty closed sets, which are bounded because $\mathcal{O}^+(\underline{x})$ is bounded. Hence $\omega(\underline{x})$ is a non-empty compact set. It remains to prove that it is invariant.

Let $\underline{y} \in \omega(\underline{x})$, and let $\{t_k\}_k$ be a positively diverging sequence such that $\phi_{t_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. By the properties of a continuous-time dynamical system

$$\phi_{t+t_k}(\underline{x}) = \phi_t(\phi_{t_k}(\underline{x})) \xrightarrow[k\to\infty]{} \phi_t(\underline{y}), \quad \forall\, t \in \mathbb{R}.$$

Hence we have shown that $\phi_t(\underline{y}) \in \omega(\underline{x})$ for all $t \in \mathbb{R}$. This concludes the proof for the ω-limit set.

The proof for the α-limit set follows along the same lines. □

Proposition 1.2. *Given a discrete-time dynamical system* $(X, \mathbb{N}_0, T)$*, let* $x \in X$ *such that* $\mathcal{O}(x)$ *is bounded. Then the set* $\omega(x)$ *is non-empty and compact. If* T *is continuous then* $\omega(x)$ *is fully invariant.*

Proof. We can repeat the proof of Proposition 1.1 to show that the ω-limit set is non-empty and compact. In particular the proof follows from the analogue of (1.3).

Let $T : X \to X$ be a continuous map with respect to a topological structure on X. Then given $y \in \omega(x)$, and being $\{n_k\}_k$ the diverging sequence of naturals for which $T^{n_k} \to y$ as $k \to \infty$, we have

$$T^{n_k+1}(x) = T(T^{n_k}(x)) \xrightarrow[k\to\infty]{} T(y).$$

Hence $T(y) \in \omega(x)$, and $\omega(x)$ is a positively invariant set. On the other hand, since $\mathcal{O}(x)$ is bounded, the sequence $\{T^{n_k-1}(x)\}_k$ admits a convergent sub-sequence $\{T^{n_{k_j}-1}(x)\}_j$ with limit point z. Hence $z \in \omega(x)$. Again by continuity of T we find

$$T(z) = T\Big(\lim_{j\to\infty} T^{n_{k_j}-1}(x)\Big) = \lim_{j\to\infty} T^{n_{k_j}}(x) = y$$

since n_{k_j} is a subsequence of n_k. Hence $y \in T(\omega(x))$, and $\omega(x)$ is then fully invariant. □

We remark that the ω-limit set is not completely invariant in general. It is sufficient to think of the case in which the ω-limit set is a fixed point with more than one pre-image.

Definition 1.9. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the orbit of a point $\underline{y}$ is called *homoclinic* if there exists a fixed point $\underline{x}$ such that

$$\alpha(\underline{y}) = \omega(\underline{y}) = \{\underline{x}\}.$$

If there exist two distinct fixed points $\underline{x}_1$, $\underline{x}_2$ such that

$$\alpha(\underline{y}) = \{\underline{x}_1\} \quad \text{and} \quad \omega(\underline{y}) = \{\underline{x}_2\},$$

then the orbit of the point $\underline{y}$ is called *heteroclinic*.

Definition 1.9 can be adapted verbatim to the case of a discrete-time dynamical system $(X, \mathbb{N}_0, T)$ with invertible T.

Exercises

1.1. Let $T : [0, 1] \to [0, 1]$ be defined by

$$T(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \in (0, 1]; \\ 1, & \text{if } x = 0. \end{cases}$$

Show that for all $x \in [0, 1]$ the ω-limit set $\omega(x)$ is non-empty but not forward invariant.