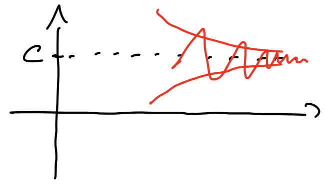
$f : D \subseteq \mathbb{R} \to \mathbb{R}$, D dominio naturale

$\lim_{x \to x_0} f(x)$, $x_0 \in \mathbb{R} \cup \{+\infty, -\infty\}$ punto di accumulazione per D

- Asintoti per f a $\pm\infty$

Asintoto orizzontale a $+\infty$ (o a $-\infty$) se

$$\lim_{\substack{x \to +\infty \\ (-\infty)}} f(x) = c \in \mathbb{R}.$$



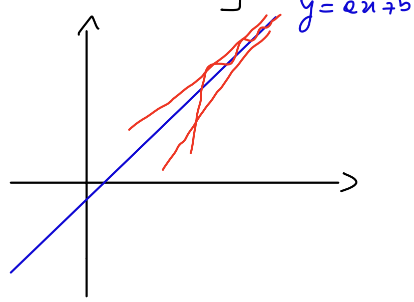
Asintoto obliquo a $+\infty$ (o a $-\infty$) se

$\exists\, a, b \in \mathbb{R}$ tali che

$$\lim_{\substack{x \to +\infty \\ (-\infty)}} \frac{f(x)}{x} = a \neq 0$$

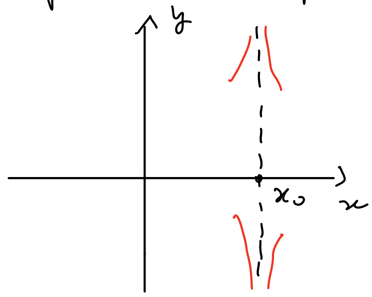
$$\lim_{\substack{x \to +\infty \\ (-\infty)}} [f(x) - ax] = b$$

$[f(x) \sim ax + b]$



- Asintoto verticale in $x_0 \in \mathbb{R}$ punto di acc per D

se $\lim_{x \to x_0^\pm} f(x) = \pm\infty$

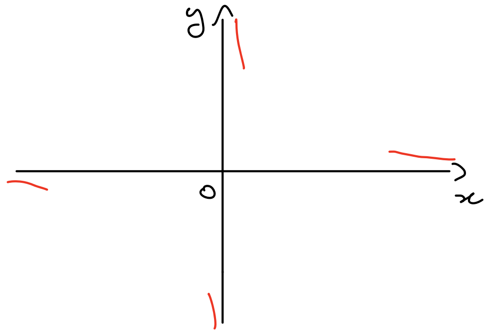


ES . $f(x) = \frac{1}{x^3}$, $D = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$

$\lim_{x \to +\infty} \frac{1}{x^3} = \frac{1}{+\infty} = 0^+$ asintoto orizzontale a $+\infty$

$\lim_{x \to -\infty} \frac{1}{x^3} = \frac{1}{-\infty} = 0^-$ " " a $-\infty$

$\lim_{x \to 0^+} \frac{1}{x^3} = \frac{1}{0^+} = +\infty$, $\lim_{x \to 0^-} \frac{1}{x^3} = \frac{1}{0^-} = -\infty$ asintoto verticale in 0



$$f(x) = \sqrt{\frac{x^3+1}{x-1}}$$

$$D = \left\{ x \in \mathbb{R} : \frac{x^3+1}{x-1} \geq 0 \ , \ x-1 \neq 0 \right\} = (-\infty, -1] \cup (1, +\infty)$$

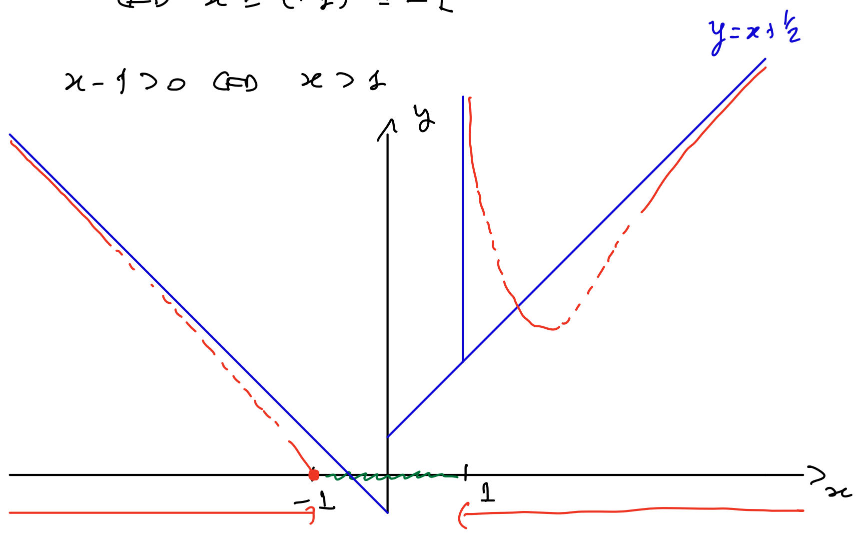
$$\frac{x^3+1}{x-1} \geq 0 \iff \begin{cases} x-1 \neq 0 \\ x^3+1 \geq 0 \\ x-1 > 0 \end{cases} \cup \begin{cases} x-1 \neq 0 \\ x^3+1 \leq 0 \\ x-1 < 0 \end{cases} \iff \begin{cases} x \neq 1 \\ x \geq -1 \\ x > 1 \end{cases} \cup \begin{cases} x \neq 1 \\ x \leq -1 \\ x < 1 \end{cases}$$

$$(1, +\infty) \cup (-\infty, -1]$$

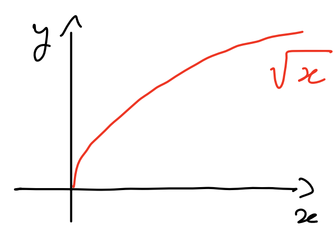
$$x^3+1 \geq 0 \iff x^3 \geq -1$$

$$\iff x \geq (-1)^{1/3} = -1$$

$$x-1 > 0 \iff x > 1$$



$$\lim_{x \to +\infty} \sqrt{\frac{x^3+1}{x-1}} = +\infty$$



$$\lim_{x \to +\infty} \frac{x^3+1}{x-1} = \lim_{x \to +\infty} \frac{x^3 \left(1 + \frac{1}{x^3}\right)}{x \left(1 - \frac{1}{x}\right)} =$$

$\frac{+\infty}{+\infty}$ f.i.

$$= \lim_{x \to +\infty} x^2 \cdot \frac{1 + \underbrace{\frac{1}{x^3}}_{\to 0^+}}{1 - \underbrace{\frac{1}{x}}_{\to 0^+}} = +\infty \cdot \frac{1}{1} = +\infty$$

$$0 \neq a = \lim_{x\to+\infty} \frac{\sqrt{\frac{x^3+1}{x-1}}}{x} = \lim_{x\to+\infty} \frac{1}{x}\sqrt{\frac{x^3+1}{x-1}} = \lim_{x\to+\infty} \sqrt{\frac{1}{x^2}\,\frac{x^3+1}{x-1}} =$$

$$= \lim_{x\to+\infty} \sqrt{\frac{x^3+1}{x^3-x^2}}$$

$$\lim_{x\to+\infty} \frac{x^3+1}{x^3-x^2} = \lim_{x\to+\infty} \frac{\cancel{x^3}\left(1+\frac{1}{x^3}\right)}{\cancel{x^3}\left(1-\frac{1}{x}\right)} = \lim_{x\to+\infty} \frac{1+\frac{1}{x^3}}{1-\frac{1}{x}} = 1.$$

$$b = \lim_{x\to+\infty}\left(\sqrt{\frac{x^3+1}{x-1}} - x\right) = \lim_{x\to+\infty}\left(\sqrt{\frac{x^3+1}{x-1}}-x\right)\frac{\left(\sqrt{\frac{x^3+1}{x-1}}+x\right)}{\left(\sqrt{\frac{x^3+1}{x-1}}+x\right)} =$$

$a=1$

$$= \lim_{x\to+\infty} \frac{\left(\sqrt{\frac{x^3+1}{x-1}}\right)^2-(x)^2}{\sqrt{\frac{x^3+1}{x-1}}+x} = \lim_{x\to+\infty}\frac{\frac{x^3+1}{x-1}-x^2}{\sqrt{\frac{x^3+1}{x-1}}+x} =$$

$$= \lim_{x\to+\infty} \frac{\frac{x^3+1-x^2(x-1)}{x-1}}{\sqrt{\frac{x^3+1}{x-1}}+x} = \lim_{x\to+\infty}\frac{\frac{x^3+1-x^3+x^2}{x-1}}{\sqrt{\frac{x^3+1}{x-1}}+x} = \lim_{x\to+\infty}\frac{\frac{x^2+1}{x-1}}{\sqrt{\frac{x^3+1}{x-1}}+x} =$$

$$\frac{x^2+1}{x-1} = \frac{x^{\cancel{2}}\left(1+\frac{1}{x^2}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)} = x \cdot \frac{1+\frac{1}{x^2}}{1-\frac{1}{x}}$$

$$\sqrt{\frac{x^3+1}{x-1}}+x = \sqrt{\frac{x^{\cancel{3}\,2}\left(1+\frac{1}{x^3}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)}} + x = \sqrt{x^2\cdot\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}+x \underset{x\to+\infty}{=} x\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}+x$$

$$= x\left[\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}+1\right]$$

$$= \lim_{x\to+\infty}\frac{\cancel{x}\cdot\frac{1+\frac{1}{x^2}}{1-\frac{1}{x}}}{\cancel{x}\left[\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}+1\right]} = \frac{1}{2}$$

$\sqrt{\frac{x^3+1}{x-1}}$ ha asintoto obliquo a $+\infty$ dato da $y = x + \frac{1}{2}$

$$\lim_{x\to1^+}\sqrt{\frac{x^3+1}{x-1}} = +\infty \qquad \text{asintoto verticale in } 1$$

$$\lim_{x\to 1^+} \frac{x^3+1}{x-1} = \frac{1^3+1}{1^+-1} = \frac{2}{0^+} = +\infty$$

$$\lim_{x\to -1^-} \sqrt{\frac{x^3+1}{x-1}} = \sqrt{\frac{(-1)^3+1}{-1-1}} = \sqrt{\frac{0^-}{-2}} = \sqrt{0^+} = 0$$

$$\lim_{x\to -\infty} \sqrt{\frac{x^3+1}{x-1}} = \lim_{x\to -\infty} \sqrt{\frac{x^{\cancel{3}\,2}(1+\frac{1}{x^3})}{\cancel{x}\,(1-\frac{1}{x})}} = \lim_{x\to -\infty} \sqrt{x^2 \cdot \frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}} = +\infty$$

$$0 \neq a = \lim_{x\to -\infty} \frac{\sqrt{\frac{x^3+1}{x-1}}}{x} = \lim_{x\to -\infty} \frac{\sqrt{x^2 \cdot \frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}}{x} = \lim_{x\to -\infty} \frac{|x|\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}}{x} =$$

$$= \lim_{x\to -\infty} \frac{(-x)\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}}}{x} = -1$$

$$b = \lim_{x\to -\infty} \left[\sqrt{\frac{x^3+1}{x-1}} - (-x)\right] = \lim_{x\to -\infty} \frac{\frac{x^3+1}{x-1} - x^2}{\underbrace{\sqrt{\frac{x^3+1}{x-1}} - x}_{+\infty \quad +\infty}} = \lim_{x\to -\infty} \frac{\frac{x^2+1}{x-1}}{\sqrt{\frac{x^3+1}{x-1}} - x} =$$

$$= \lim_{x\to -\infty} \frac{x \cdot \frac{1+\frac{1}{x^2}}{1-\frac{1}{x}}}{(-x)\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}} + (-x)} = \lim_{x\to -\infty} \frac{x}{-x} \frac{\frac{1+\frac{1}{x^2}}{1-\frac{1}{x}}}{\sqrt{\frac{1+\frac{1}{x^3}}{1-\frac{1}{x}}} + 1} = -\frac{1}{2}$$

$\sqrt{\frac{x^3+1}{x-1}}$ ha asintoto obliquo a $-\infty$ data da $y = -x - \frac{1}{2}$

$\longleftrightarrow$

$f : D \subseteq \mathbb{R} \to \mathbb{R}$, D dominio naturale

Def Sia $x_0 \in D$, se x_0 è anche punto di acc. per D, si dice che f è <u>continua in x_0</u> se $\lim_{x\to x_0} f(x) = f(x_0)$

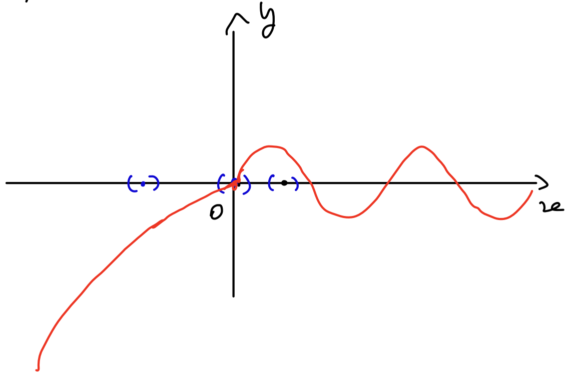
ES Non ha senso chiedersi se $f(x) = \frac{1}{x}$ sia continua o no in 0.

OSS Potenze, funzioni trigonometriche, esponenziali, logaritmo sono continue in tutti i punti del loro dominio naturale.

Prop Se f, g continue nei loro domini naturali, sono continue anche $f+g$, $f\cdot g$, $\frac{f}{g}$, $|f|$, $f\circ g$ nei loro domini naturali.

ES

$$f(x) = \begin{cases} \sin x, & x \geq 0 \\ -x^2+x, & x<0 \end{cases}, \quad D = [0,+\infty) \cup (-\infty,0) = \mathbb{R}$$



- f è continua in $(0,+\infty) \cup (-\infty,0) = \mathbb{R} \setminus \{0\}$
- f è continua in 0? $\lim_{x\to 0} f(x) = f(0) = \sin 0 = 0$?

$$\left.\begin{array}{l} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x = \sin 0^+ = 0 \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x^2+x) = 0 \end{array}\right] \Rightarrow \lim_{x\to 0} f(x) = 0$$

$\Rightarrow$ f è continua in 0

R. f è continua in $\mathbb{R}$

DISCONTINUITÀ Sia $x_0 \in D$, $f: D \subseteq \mathbb{R} \to \mathbb{R}$.

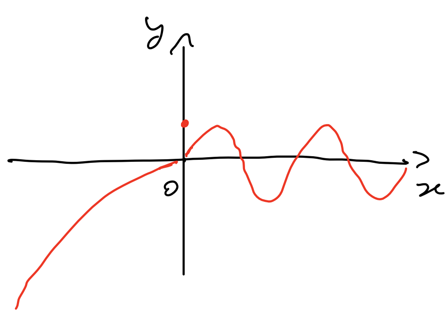
Se f non è continua in x_0, si dice che f ha in x_0:

1) Una discontinuità eliminabile se

$\lim_{x\to x_0} f(x)$ esiste ma è diverso da $f(x_0)$

ES

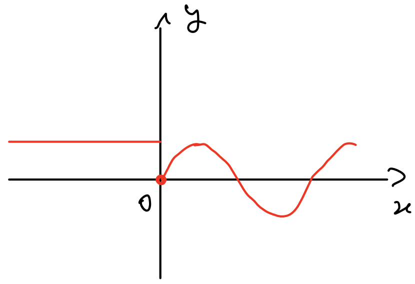
$$f(x) = \begin{cases} \sin x, & x>0 \\ 1, & x=0 \\ -x^2+x, & x<0 \end{cases}$$



$\lim_{x\to 0} f(x) = 0 \neq f(0) = 1$

2) Una <u>discontinuità di I specie</u> se

$\lim_{x \to x_0^+} f(x)$ e $\lim_{x \to x_0^-} f(x)$ esistono e sono finiti ma diversi tra loro



ES

$$f(x) = \begin{cases} \sin x, & x \geq 0 \\ 1, & x < 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^+} \sin x$$

$$\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^-} 1$$

3) Una <u>discontinuità di II specie</u> negli altri casi.

ES

$$f(x) = \begin{cases} \log(|x|), & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad D = \{x \neq 0 \text{ e } |x| > 0\} \cup \{0\} = \mathbb{R}$$

- f è continua in $\mathbb{R} \setminus \{0\}$
- f è continua in 0 ? $\quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \log(|x|) =$

$$= \lim_{x \to 0^+} \log(x) = -\infty$$

f ha in 0 una discontinuità di II specie.