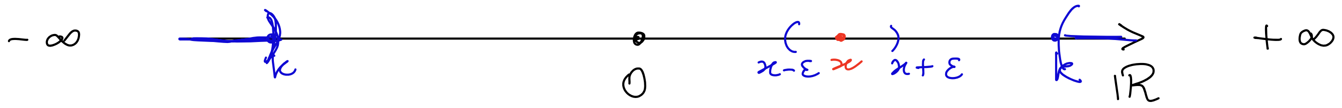


$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

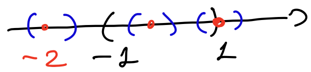
Def Intorni di $x \in \mathbb{R}$ e di $+\infty$ e $-\infty$.

- Fissato $\varepsilon > 0$, un intorno di $x \in \mathbb{R}$ di raggio ε è $(x-\varepsilon, x+\varepsilon)$.
  (oss $y \in (x-\varepsilon, x+\varepsilon) \iff d(x,y) < \varepsilon$)
- Fissato $\underset{\substack{\Uparrow \\ \mathbb{R}}}{k} >> 1$, un intorno di $+\infty$ è $(k, +\infty)$
- Fissato $k \in \mathbb{R}$, $k << -1$, un intorno di $-\infty$ è $(-\infty, k)$

Def Sia $A \subseteq \mathbb{R}$, un punto $x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ si dice di <u>accumulazione</u> per A se:

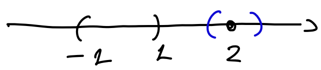
- $x \in \mathbb{R}$, $\forall\, \varepsilon > 0\ \exists\, y \in A$, $y \neq x$, tale che $y \in (x-\varepsilon, x+\varepsilon)$

  (ES $A = (-1, 1)$, ± 1 è di accumulazione per A

  ogni punto di A è di accumulazione per A

  ES $A = (-1, 1) \cup \{2\}$

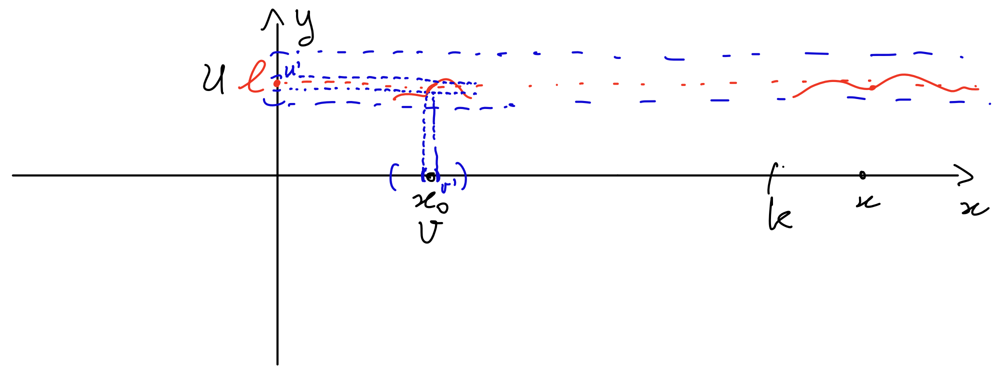
  

  2 non è di accumulazione, 2 è un punto isolato)

- $x = +\infty$, $\forall\, k >> 1$, $\exists\, y \in A$ tale che $y > k$ ($y \in (k, +\infty)$)
- $x = -\infty$, $\forall\, k << -1$, $\exists\, y \in A$ tale che $y < k$ ($y \in (-\infty, k)$).

Def Sia $f : D \subseteq \mathbb{R} \to \mathbb{R}$, D dominio naturale.
Dato $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ punto di accumulazione di D, si dice che $\lim_{x \to x_0} f(x) = \ell \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

se $\forall\, \mathcal{U}$ intorno di ℓ, $\exists\, V$ intorno di x_0, tale che $\forall x \in D \cap V \setminus \{x_0\}$ si ha $f(x) \in \mathcal{U}$.



- $\ell \in \mathbb{R}$, $x_0 \in \mathbb{R}$ $\quad (\forall x \in D \setminus \{x_0\}$ t.c. $|x - x_0| < \delta)$

  $\forall \varepsilon > 0\ \exists \delta > 0$ t.c. $\forall x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$

  si ha che $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$ $\quad (\,|f(x) - \ell| < \varepsilon)$

- $\ell \in \mathbb{R}$, $x_0 = +\infty$

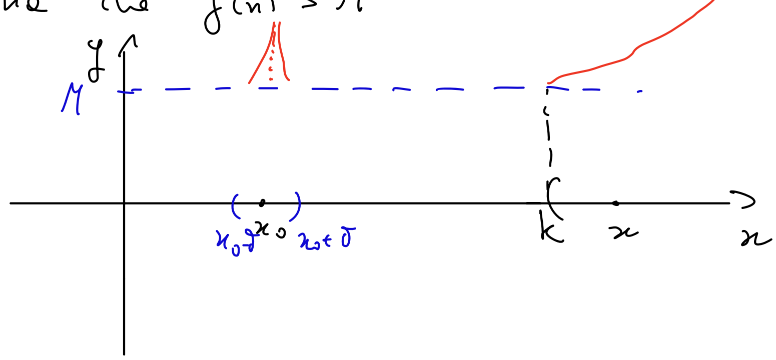
  $\forall \varepsilon > 0\ \exists K >> 1$ t.c. $\forall x \in D \cap (k, +\infty)$

  si ha che $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$

- $\ell = +\infty$, $x_0 = +\infty$

  $\forall M >> 1\ \exists K >> 1$ t.c. $\forall x \in D \cap (k, +\infty)$

  si ha che $f(x) > M$



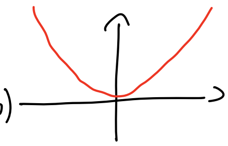
- $\ell = +\infty$, $x_0 \in \mathbb{R}$

  $\forall M >> 1\ \exists \delta > 0$ t.c. $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$

  si ha che $f(x) > M$

Esempi

- $f(x) = x^2$, $D = \mathbb{R} = (-\infty, +\infty)$



Punti di acc di $D = \mathbb{R} \cup \{+\infty, -\infty\}$

$\lim_{x \to +\infty} f(x) = +\infty \iff \forall M >> 1 \ \exists K >> 1$ tale che

$\forall x \in D \cap (K, +\infty)$ si ha che $f(x) > M$

Fissiamo $M >> 1$, $x^2 > M \iff x > \sqrt{M} = K$

$(M = 10^{100}, \ K = 10^{50}, \ \forall x > 10^{50}$ si ha $x^2 > 10^{100})$

$\lim_{x \to -\infty} f(x) = +\infty$

$\lim_{x \to x_0 \in \mathbb{R}} f(x) = x_0^2 \left[= f(x_0)\right]$

<u>OSS</u> $\lim_{\substack{x \to 2 \\ x_0}} x^2 = \overset{\ell}{4} \iff \forall \varepsilon > 0 \ \exists \delta > 0$ t.c.

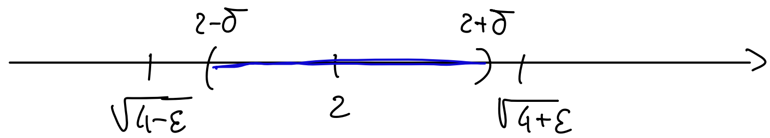
$\forall x \in D \cap (2-\delta, 2+\delta) \setminus \{2\}$ si ha che $|x^2 - 4| < \varepsilon$

Fissiamo $\varepsilon > 0$, $|x^2 - 4| < \varepsilon \iff x^2 \in (4-\varepsilon, 4+\varepsilon) \iff$

$4 - \varepsilon < x^2 < 4 + \varepsilon \iff \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$

$x \in (2-\delta, 2+\delta)$

$\sqrt{4-\varepsilon} \leq 2 - \delta$, $2 + \delta \leq \sqrt{4+\varepsilon}$



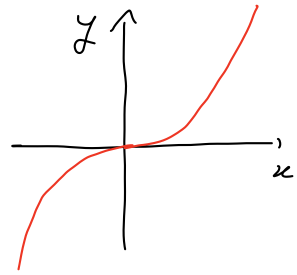
$\varepsilon = 10^{-2}$ $\sqrt{4-\varepsilon} \sim 1{,}9974$ $\sqrt{4+\varepsilon} \sim 2{,}0024$

$1{,}9975 \leq 2 - \delta$ $2 + \delta \leq 2{,}0023$

$1{,}999$ $\delta = 0{,}001$ $2{,}001$

- $f(x) = x^3$, $\quad D = \mathbb{R} = (-\infty, +\infty)$

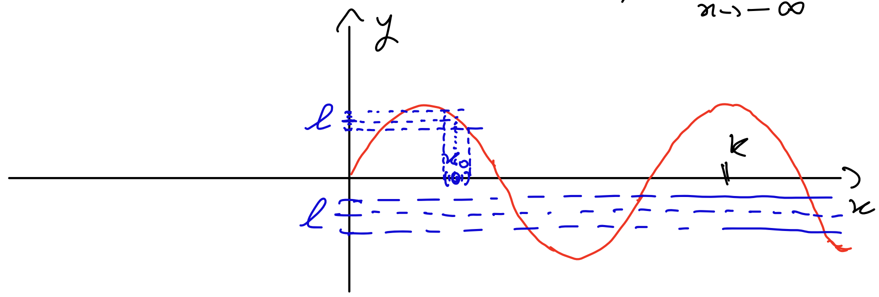
Punti di acc di $D = \overline{\mathbb{R}}$



$\lim_{x \to +\infty} x^3 = +\infty$, $\quad \lim_{x \to -\infty} x^3 = -\infty$, $\quad \lim_{x \to x_0 \in \mathbb{R}} x^3 = x_0^3$

- $f(x) = \sin x$, $\quad D = \mathbb{R} = (-\infty, +\infty)$, $\quad$ Punti di acc di $D = \overline{\mathbb{R}}$

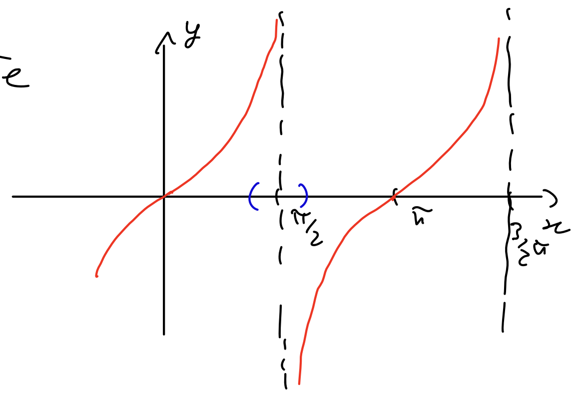
$\lim_{x \to +\infty} \sin x = ?$ non esiste, $\quad \lim_{x \to -\infty} \sin x =$ non esiste



$\lim_{x \to x_0 \in \mathbb{R}} \sin x = \sin x_0$.

- $f(x) = \tan x$, $\quad \frac{\pi}{2} \notin D$, ma è un punto di acc. di D

$\lim_{x \to \pi/2} \tan x$ non esiste



- $\lim_{x \to +\infty} e^x = +\infty$, $\quad \lim_{x \to -\infty} e^x = 0$, $\quad \lim_{x \to x_0} e^x = e^{x_0}$

- $\lim_{x \to +\infty} (\log x) = +\infty$, $\quad \lim_{x \to 0} (\log x) = -\infty$, $\quad \lim_{x \to x_0 \in (0, +\infty)} (\log x) = \log x_0$

$\longleftrightarrow$

Somme $\forall a \in \mathbb{R}$, $+\infty + a = +\infty$, $+\infty - a = +\infty$

$-\infty + a = -\infty$, $-\infty - a = -\infty$

$+\infty + \infty = +\infty$, $-\infty - \infty = -\infty$

$$\left(\lim_{x \to +\infty} (e^x + \log x) = \underbrace{\lim_{x\to+\infty} e^x}_{+\infty} + \underbrace{\lim_{x\to+\infty} \log x}_{+\infty} = +\infty \right)$$

$$\left(\lim_{x \to +\infty} (e^x + e^{-x}) = \underbrace{\lim_{x\to+\infty} e^x}_{+\infty} + \underbrace{\lim_{x\to+\infty} e^{-x}}_{0} = +\infty \right)$$

$+\infty - \infty$ = forme indeterminate $= -\infty + \infty$

$$\left[\begin{array}{l} \lim_{x\to+\infty} x^3 - x^2 = \underbrace{\left(\lim_{x\to+\infty} x^3\right)}_{+\infty} - \underbrace{\left(\lim_{x\to+\infty} x^2\right)}_{+\infty} = (+\infty) - (+\infty) = ? \\ \lim_{x\to+\infty} x^2 - x^3 = (+\infty) - (+\infty) = ? \end{array} \right]$$

Prodotto $a \in \mathbb{R}$, se $a > 0$, $a \cdot (+\infty) = +\infty$, $a \cdot (-\infty) = -\infty$

$a < 0$, $a \cdot (+\infty) = -\infty$, $a \cdot (-\infty) = +\infty$

$0 \cdot (+\infty)$, $0 \cdot (-\infty)$ forme indeterminate

$(+\infty) \cdot (+\infty) = +\infty$, $(+\infty) \cdot (-\infty) = -\infty$

$(-\infty) \cdot (-\infty) = +\infty$

Rapporto $\frac{a}{0}$ forme indeterminate, $\frac{+\infty}{0}$, $\frac{-\infty}{0}$

$\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{-\infty}{-\infty}$ forme indeterminate

Se $a \in \mathbb{R}$, $\frac{a}{+\infty} = \frac{a}{-\infty} = 0$

Potenze $a^b = e^{b \log a}$, $a > 0$, $e = +\infty$