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Quasistatic evolution of sessile drops and contact angle hysteresis

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Abstract. We consider the classical model of capillarity coupled with a rate-independent dissipation mechanism due to frictional forces acting on the contact line, and prove the existence of solutions with prescribed initial configuration for the corresponding quasistatic evolution. We also discuss in detail some explicit solutions to show that the model does account for contact angle hysteresis, and to compare its predictions with experimental observations.

Keywords: Capillary drops, Young's law, contact angle hysteresis, friction-induced dissipation, quasistatic evolution, rate-independent dissipative processes, minimal surfaces, finite perimeter sets.

MSC (2010): 76B45, 49J40, 49J45, 49Q05, 49Q20, 49S05, 74M10.

1. Introduction

More than 200 years ago, the laws of Young and Laplace governing the equilibrium shape of capillary drops were discovered [23, 37]. Gauss realized that these two laws arise from the condition that the capillary energy be stationary with respect to shape perturbations [21]. Since then, the topic has never lost appeal in the mathematics community (see, for example, [19] and the reference therein).

In recent years, renewed interest has been spurred by the physics literature on wetting phenomena (see [14, 15, 8]) and in particular by the research on super-hydrophobicity of rough hydrophobic surfaces. Understanding the impact of roughness on the shape of energy minimizing drops and on their adhesion properties has been the subject of a number of investigations, both in the physics literature (see [31, 32] and the many references cited therein), and in the mathematical literature (see [3, 17]).

The fact that liquid drops may adhere to solid substrates is a readily available observation: raindrops may stick to a vertical window pane. Interestingly, this phenomenon rests entirely on the fact that water drops may violate Young's law. If an interval of equilibrium contact angles is possible, rather than only one (as stipulated by Young's law), then an imbalance of the frictional forces at the contact line becomes available to equilibrate the gravitational force. The phenomenon that liquid drops on solid surfaces can exhibit more than one equilibrium contact angle is called contact angle hysteresis.

Accounting for contact angle hysteresis requires an amendment to classical capillarity theory. Here, we follow the phenomenological approach in [17], based on the introduction of a dissipation potential which states that the energy dissipated by a (slowly) moving contact line is proportional to the change in wetted area. This extends to the wetting problem the energy-based approach proposed by A. Mielke and coauthors for studing quasistatic, rate-independent, dissipative evolution processes (see [27, 28], and references therein). Similar ideas have been used in a variety of physical contexts ranging, for example, from the mathematical study of crack growth (see [20, 13]) and plastic flow (see [10, 12]), to the development of computational schemes for wetting problems (see [16, 36]).

In this paper we focus on a model problem, namely the quasistatic evolution of a water drop resting on a solid surface and driven by a prescribed, time-varying volume. A slowly evaporating drop provides a concrete example.[1]

Our achievements are twofold. First we prove a rigorous existence result (Theorem 3.9) and deduce some necessary and sufficient conditions that solutions must satisfy (Proposition 5.13). Second, we use these sufficient conditions to exhibit solutions in some concrete examples that can be easily compared with experimental evidence (Section 6). In this way, we can assess strengths and weaknesses of using the quasistatic evolution scheme to model contact angle hysteresis.

One of our conclusions is that while equilibrium configurations involving no motion of the contact line are reliably identified, our scheme may sometimes lead to unphysical contact line motion, with jumps occurring "earlier" than what should be expected based on physical intuition.

From the point of view of mathematical analysis, we have added a time-dependent constraint to the abstract approach of Mielke, yielding an additional term in the energy-dissipation balance; moreover we have treated a problem which is somewhat degenerate, in that the movement of the free surface of the drop (the liquid-vapour interface) carries no dissipation. As a consequence, our solutions do not have bounded variation with respect to time, and are not directly obtained as pointwise limits of time-discretized solutions (at least not in the usual way). Finally we were also able to show that under suitable assumptions on the physical parameters of the model and on the geometry of the container, the evolving drops can be represented as subgraphs of BV functions (Proposition 3.12).

The paper is organized as follows. Section 2 contains notation, background material on classical capillarity theory and on contact angle hysteresis, and a

[1] In this case the volume v of the drop is not a prescribed function of time t but is determined by some evaporation law expressing $\partial v/\partial t$ in terms of other quantities (typically the area of the free surface). However, in rate-independent dissipation processes time is just an order parameter, and could be replaced by any other parameter τ which is *increasing* in time; in this case we would take τ equal to minus the volume itself, find the desired solution as a function of τ, and then recover the physical time t as a function of τ using the fact that $\partial t/\partial\tau$ is now explicitly known via the evaporation law.

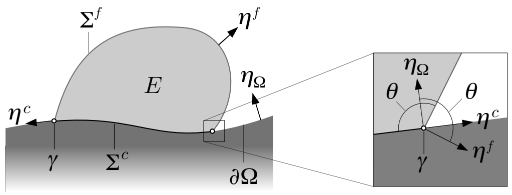

FIGURE 1. A drop E in the container Ω.

heuristic presentation of our model. In this section we make no attempt to provide precise mathematical statements and results, while all the material in the subsequent sections is intended to be mathematically rigorous. In Section 3 we state the main existence result (Theorem 3.9), which we then prove in Section 4. The technical lemmas needed for this goal are proved in Section 5. Finally, in Section 6 we present some concrete examples, some of which represent case studies of particular physical interest.

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2. QUASISTATIC EVOLUTION OF CAPILLARY DROPS

In this section we describe the quasistatic evolution of a capillary drop subject to time-dependent volume forces, a time-dependent volume constraint, and frictional forces acting on the contact line. After a short description of the setting of this problem, we define the quasistatic evolution in terms of the capillary energy and of the dissipation potential (§2.8), and then derive the corresponding flow rules (§2.10 and §2.11).

This section is mostly informal and heuristic. In particular, we will assume throughout that the objects we consider (sets, surfaces, and curves) are always as regular as necessary in order to define the quantities we use (curvature, velocity, and so on) and carry out the computations we need.

Let us begin with notation. Given a set A, we write $|A|$ for the measure of A, whether volume, area or length being usually clear from the context. Similarly, we write $\int_A f$ to denote the integral of the function f over the set A with respect to the appropriate measure.

For the rest of this section we fix a regular domain Ω in $\mathbb{R}^3$ (the container) and denote by η_Ω the inner (unit) normal to the boundary of Ω.

2.1. Drops. Given a set E contained in Ω (a drop) we consider the following objects (see Fig. 1):

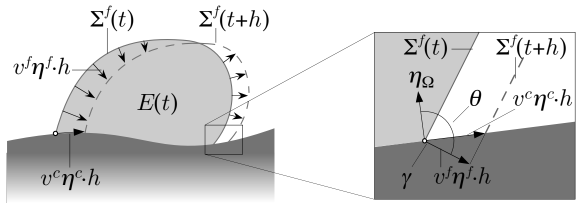

FIGURE 2. A moving drop at times t and $t+h$.

Σ^f : liquid-vapour interface, or free surface;
η^f : outer (unit) normal of the surface Σ^f;
H^f : mean curvature of the surface Σ^f (average of the principal curvatures);
Σ^c : liquid-solid interface, or contact surface;
γ : contact line, that is, the common boundary of Σ^c and Σ^f;
η^c : outer (unit) normal of the curve γ (tangent to the surface $\partial\Omega$);
θ : contact angle, that is, the angle between η^f and η_Ω, defined for every point of the contact line γ or, equivalently, the angle between the tangent planes to Σ^f and $\partial\Omega$.

Assume now that the drop E is moving in time;[2] we then define the following velocities (see Fig. 2):

v^f : outer normal velocity of the free surface Σ^f;
v^c : outer normal velocity of the contact line γ.

Thus v^c is a scalar function defined for every $x \in \gamma(t)$ and every $t \in [0,T]$ such that, for h sufficiently small, the curve $\gamma(t+h)$ can be represented as

$$\gamma(t+h) = \big\{x + (v^c h + o(h))\,\eta^c \;:\; x \in \gamma(t)\big\}\,.^{3} \tag{2.1}$$

A slightly more complicated formula allows us to represent $\Sigma^f(t+h)$ in terms of $\Sigma^f(t)$ and v^f.

Note that for every point of the contact line, v^f is the component of the vector $v^c\eta^c$ in the direction η^f (see Fig. 2), that is

$$v^f = v^c\,\eta^c\cdot\eta^f = v^c\sin\theta\,. \tag{2.2}$$

2.2. Capillary energy. The *capillary energy* associated with a drop E is given by

$$\mathscr{E} := \sigma_{\rm LV}|\Sigma^f| + \sigma_{\rm LS}|\Sigma^c| + \sigma_{\rm SV}|\partial\Omega\setminus\Sigma^c| + V\,. \tag{2.3}$$

[2] More precisely, E is a map that assigns to every $t \in [0,T]$ a set $E(t)$, representing the position of the drop at time t; when needed, we write $\Sigma^f(t)$ for the free surface at time t, $\gamma(t)$ for the contact line at time t, and so on.

[3] In this formula v^c and η^c are computed at t, x.

Here V stands for an additional, possibly time-dependent, volume energy of the form

$$V := \int_E \rho(t,x)\,dx\,, \tag{2.4}$$

where ρ is an assigned function on $[0,T]\times\Omega$. A typical example for the volume energy density is given by $\rho = -g\cdot x$, where the vector g is the gravitational force per unit volume acting on the drop. The vector g depends on t if we consider, for example, a container Ω tilted by an angle which varies in time, and use a reference frame moving with the container.

As usual, the *surface tensions* $\sigma_{\rm LV}, \sigma_{\rm LS}, \sigma_{\rm SV}$ which appear in (2.3) are positive parameters which satisfy the *wetting condition*

$$|\sigma_{\rm LS} - \sigma_{\rm SV}| \le \sigma_{\rm LV}\,. \tag{2.5}$$

The *Young angle* θ_Y is the angle in $[0,\pi]$ defined by

$$\cos\theta_Y := \frac{\sigma_{\rm SV} - \sigma_{\rm LS}}{\sigma_{\rm LV}}\,. \tag{2.6}$$

Using (2.6) we can write the capillary energy $\mathscr{E}$, up to addition of a constant, in a more convenient form:

$$\mathscr{E} := \sigma_{\rm LV}\big(|\Sigma^f| - \cos\theta_Y|\Sigma^c|\big) + V\,. \tag{2.7}$$

In the following we write $\mathscr{E}(E)$ or $\mathscr{E}(t,E)$ if we need to emphasize the dependence of the capillary energy $\mathscr{E}$ on the drop E and on the time t. Similarly, we write $V(E)$ or $V(t,E)$ for the volume energy V.

2.3. Equilibrium conditions. We usually consider drops of prescribed volume, which is possibly a function of time $w(t)$. When the drop E is at equilibrium with respect to the capillary energy $\mathscr{E}$—for example, when it is a (local) minimizer of $\mathscr{E}$ under the prescribed-volume constraint—then the contact angle θ agrees with the Young angle θ_Y at every point of the contact line (Young's law) and the mean curvature H^f verifies

$$-2\sigma_{\rm LV}H^f + \rho = \text{constant} = p \tag{2.8}$$

at every point of the free surface Σ^f (Laplace's law). The constant p in (2.8) is the Lagrange multiplier associated with the volume constraint, and agrees with the difference between the internal and the external pressure on the surface of the drop (in short, the pressure). If there is no volume contribution in the capillary energy $\mathscr{E}$ then the free surface Σ^f has constant mean curvature.

2.4. Derivation of the equilibrium conditions. We briefly sketch here the derivation of the equilibrium conditions given in the previous paragraph; the same calculations will be used later. Given a drop E, consider an arbitrary *variation* of E, that is, a map $h \mapsto E(h)$ such that $E(0) = E$ (and reasonably regular). Interpreting h as a time parameter, we denote by v^f and v^c the outer normal velocities of the free surface and of the contact line at time $h = 0$. The corresponding first variations of the volume, of the area of the contact surface, and so on, are given by the following well-known formulas:

$$\frac{d}{dh}|E(h)|\Big|_{h=0} = \int_{\Sigma^f} v^f \,, \tag{2.9}$$

$$\frac{d}{dh}|\Sigma^c(h)|\Big|_{h=0} = \int_{\gamma} v^c \,, \tag{2.10}$$

$$\frac{d}{dh}|\Sigma^f(h)|\Big|_{h=0} = \int_{\Sigma^f} -2H^f v^f + \int_{\gamma} \cos\theta\, v^c \,,) \tag{2.11}$$

$$\frac{d}{dh}V(E(h))\Big|_{h=0} = \int_{\Sigma^f} \rho v^f \,. \tag{2.12}$$

We then compute the first variation of the capillary energy $\mathscr{E}$ in (2.7):

$$\frac{d}{dh}\mathscr{E}(E(h))\Big|_{h=0} = \int_{\Sigma^f} (-2\sigma_{\rm LV} H^f + \rho) v^f + \int_{\gamma} \sigma_{\rm LV}(\cos\theta - \cos\theta_Y) v^c \,. \tag{2.13}$$

If E is a (local) minimizer of $\mathscr{E}$ at prescribed volume, then it must satisfy

$$\frac{d}{dh}\mathscr{E}(E(h))\Big|_{h=0} = 0 \tag{2.14}$$

for all variations $E(h)$ which are *volume-preserving*, that is, satisfy $|E(h)| = |E|$ for every h. Formula (2.9) implies that for such variations the integral of v^f over Σ^f vanishes.

The key remark now is that as $E(h)$ ranges among all admissible volume-preserving variations, the velocity-field v^f can be any (reasonably smooth) function on Σ^f with vanishing integral.

If the velocity-field v^f vanishes on the contact line γ, then v^c vanishes too, and replacing the left-hand side of (2.14) by the right-hand side of (2.13) we obtain

$$\int_{\Sigma^f} (-2\sigma_{\rm LV} H^f + \rho) v^f = 0 \,;$$

knowing that this equality holds for *every* function v^f vanishing at the boundary of Σ^f and with vanishing integral is sufficient to infer that $-2\sigma_{\rm LV} H^f + \rho$ must be constant on Σ^f, which is Laplace's law (2.8).

Consider now a velocity-field v^f that does not vanish on the contact line γ: replacing once again the left-hand side of (2.14) by the right-hand side of (2.13) and taking into account (2.8) we obtain

$$\int_{\gamma} (\cos\theta - \cos\theta_Y) v^c = 0 \,,$$

and since this holds for *every* (sufficiently smooth) function v^c on γ, we infer that $\cos\theta = \cos\theta_Y$ at every point of γ, which implies Young's law.

2.5. A variational formula for the pressure. Let E be a drop which minimizes the capillary energy $\mathscr{E}(E)$, or, equivalently,

$$\mathscr{E}'(E) := \sigma_{\rm LV}|\Sigma^f| + V(E) \,, \tag{2.15}$$

among all drops with the *same volume* and the *same contact surface*.

Then the argument in the previous paragraph shows that Laplace's law (2.8) is still satisfied (even though Young's law might not hold). In particular, we can still define the pressure p as the constant value of $-2\sigma_{\rm LV} H^f + \rho$.

Consider now an arbitrary variation $E(h)$ of E such that the velocity-field $v^c = 0$ vanishes everywhere on γ. Then the second integral at the right-hand side of (2.13) vanishes, and recalling (2.9) we obtain the following formula involving the pressure:[4]

$$\frac{d}{dh}\mathscr{E}(E(h))\Big|_{h=0} = p \cdot \frac{d}{dh}|E(h)|\Big|_{h=0} \,. \tag{2.16}$$

2.6. Dissipation potential and frictional forces. Given two drops $E, \tilde{E}$, we consider the following *dissipation potential* (or dissipation distance):

$$\mathscr{D}(E, \tilde{E}) := \mu|\Sigma^c \triangle \tilde{\Sigma}^c| \,, \tag{2.17}$$

where μ is a positive *friction coefficient* and $\Sigma^c \triangle \tilde{\Sigma}^c := (\Sigma^c \setminus \tilde{\Sigma}^c) \cup (\tilde{\Sigma}^c \setminus \Sigma^c)$ is the symmetric difference of Σ^c and $\tilde{\Sigma}^c$.

Consider now a drop E which is moving, and write E_0, γ_0 and v_0^c for the position of the drop, the contact line and its velocity at time 0: a simple computation shows that the *dissipation rate* $\mathscr{R}$ associated with $\mathscr{D}$ can be written in terms of γ_0 and v_0^c, and precisely[5]

$$\mathscr{R}(E_0, v_0^c) := \lim_{h\to 0^+} \frac{\mathscr{D}(E(0), E(h))}{h} = \mu \int_{\gamma_0} |v_0^c| \,. \tag{2.18}$$

This can be rephrased by saying that a small arc of the contact line which is moving with non-zero normal velocity v_0^c is subject to a frictional force per unit length equal to $-\mu\eta_0^c$ if $v_c > 0$ and $\mu\eta_0^c$ if $v_c < 0$.

2.7. Advancing and receding contact angles. The advancing and receding contact angles $\theta_{\rm adv}$ and $\theta_{\rm rec}$ are the angles in the interval $[0, \pi]$ defined by the following relations:

$$\begin{aligned} \cos\theta_{\rm adv} &= \cos\theta_Y - \frac{\mu}{\sigma_{\rm LV}} = \frac{\sigma_{\rm SV} - \sigma_{\rm LS} - \mu}{\sigma_{\rm LV}} \,, \\ \cos\theta_{\rm rec} &= \cos\theta_Y + \frac{\mu}{\sigma_{\rm LV}} = \frac{\sigma_{\rm SV} - \sigma_{\rm LS} + \mu}{\sigma_{\rm LV}} \,. \end{aligned} \tag{2.19}$$

To ensure that these definitions make sense, we assume the following stronger version of the wetting condition (2.5):

$$-\sigma_{\rm LV} < \sigma_{\rm SV} - \sigma_{\rm LS} - \mu < \sigma_{\rm SV} - \sigma_{\rm LS} + \mu < \sigma_{\rm LV} \,. \tag{2.20}$$

[4] We are simply re-affirming that the pressure p is the Lagrange multiplier associated to the minimization of $\mathscr{E}$ with prescribed volume (and prescribed contact surface).

[5] To obtain the second identity in (2.18), note that $\mathscr{D}(E(0), E(h)) = \mu(|\Sigma^+(h)| + |\Sigma^-(h)|)$ where $\Sigma^+(h) := \Sigma^c(h) \setminus \Sigma^c(0)$ and $\Sigma^-(h) := \Sigma^c(0) \setminus \Sigma^c(h)$; then an obvious variant of formula (2.10) yields that the *right* derivatives of $|\Sigma^+(h)|$ and $|\Sigma^-(h)|$ at $h = 0$ are given by integral over γ_0 of $(v_0^c)^+$ and $(v_0^c)^-$, respectively (for every real number a, we denote by a^+ and a^- its positive and negative part).

Remark. (i) In our setting, the dissipation rate depends only on the modulus of v_0^c, and therefore disregards whether the contact line is advancing or receding. By setting

$$\mathscr{D}(E,\tilde{E}) := \mu_{\rm adv}|\tilde{\Sigma}^c \setminus \Sigma^c| + \mu_{\rm rec}|\Sigma^c \setminus \tilde{\Sigma}^c| \,,$$

we could consider an *asymmetric* dissipation rate of the form

$$\mathscr{R}(E_0,v_0^c) = \int_{\gamma_0} \phi(v_0^c) \quad \text{with } \phi(v) := \begin{cases} \mu_{\rm adv}|v| & \text{for } v \geq 0 \\ \mu_{\rm rec}|v| & \text{for } v < 0. \end{cases}$$

We also remark that, while (2.18) seems physically plausible, (2.17) is not entirely justifiable when Σ^c and $\tilde{\Sigma}^c$ are not infinitesimally close. Indeed, in those cases where sudden jumps of the contact line occur, neglecting the contribution of viscous dissipative terms may be unphysical.

(ii) We emphasize that, within our approach, the existence of contact angle hysteresis at the macroscopic scale is postulated. In fact, it has long been known in the Physics literature, and proved rigorously in [7], that the existence of an interval of stable macroscopic contact angles can be deduced from a microscopic theory without hysteresis (so that the microscopic contact angle obeys the Young law), thanks to microscopic oscillations, either in the topography or in the chemical composition of the surface. In other words, the homogenization of the Young-Laplace law leads to an interval of stable contact angles, thus explaining the phenomenon of contact angle hysteresis.

(iii) When condition (2.20) is violated, it seems plausible that the system behaves as if the dissipation potential $\mathscr{D}$ given by (2.17) has been replaced by the one in point (i) above, with $\mu_{\rm adv}$ and $\mu_{\rm rec}$ chosen so that

$$\begin{aligned} \sigma_{\rm SV} - \sigma_{\rm LS} - \mu_{\rm adv} &:= \max\{-\sigma_{\rm LV}; \sigma_{\rm SV} - \sigma_{\rm LS} - \mu\} \,, \\ \sigma_{\rm SV} - \sigma_{\rm LS} + \mu_{\rm rec} &:= \min\{\sigma_{\rm LV}; \sigma_{\rm SV} - \sigma_{\rm LS} + \mu\} \,. \end{aligned}$$

2.8. Stability and solutions. We say that a drop E' is (globally) *stable* at time t if it minimizes $\mathscr{E}(t,E) + \mathscr{D}(E,E')$ among all E with the same volume as E'.

Let w be a strictly positive function defined on the interval $[0,T]$. We say that a map $t \mapsto E(t)$ defined for $t \in [0,T]$ is a solution (of the quasistatic evolution problem) satisfying the volume constraint $|E(t)| = w(t)$ if the following conditions are satisfied:

Global stability: $E(t)$ has volume $w(t)$ and is stable at time t for every $t \in [0,T]$.

Energy-dissipation balance: for every t_0, t_1 with $0 \leq t_0 < t_1 \leq T$ there holds

$$\begin{aligned} \mathscr{E}(t_1,E(t_1)) - \mathscr{E}(t_0,E(t_0)) = & \int_{t_0}^{t_1} \left[\int_{E(t)} \frac{\partial \rho}{\partial t}\, dx \right] dt \\ & + \int_{t_0}^{t_1} p\dot{w}\, dt - \int_{t_0}^{t_1} \mathscr{R}(E(t),v^c)\, dt \,, \end{aligned} \tag{2.21}$$

where ρ is the energy density that appears in (2.4), p is the pressure associated to the drop E at time t (see §2.5),[6] and $\mathscr{R}$ the dissipation rate (see §2.6). The quantities $\partial\rho/\partial t$, $\dot{w}$, p, and v^c at the right-hand side of (2.21) are all computed at time t.

2.9. Remark. (i) The definition of solution given above coincides with the one of *energetic solution* of a rate-independent process proposed by Mielke and coauthors, see for instance [27, 28].

(ii) Equation (2.21) represents an energy-dissipation balance because the sum of the first two integrals can be identified as the work done by the external forces in the time interval $[t_0,t_1]$, while the third integral represents the energy dissipated by friction. More precisely, since the derivative $\dot{w}$ of the volume of the drop agrees with the integral of the normal velocity v^f on the free surface Σ^f, the second integral in (2.21) represents the work done by the pressure p on Σ^f, that is, the amount of energy (from external sources) which is needed to keep the volume constraint $|E(t)| = w(t)$ at every t in $[t_0,t_1]$.

(iii) In an evolution governed by (2.21), the contact line moves at a speed v^c such that the dissipation rate $\mathscr{R}(E,v^c)$ exactly balances the energy release rate (namely, the negative of the sum of rate of change of capillary energy and power of the external forces). As will be clearer from the examples of Section 6, this condition may be violated when locally (but not globally) stable states are allowed in an evolution, and jumps may involve additional dissipation terms not included in $\mathscr{R}$. The additional dissipation terms are associated with dynamic processes occurring at time scales much faster than the (slow, quasistatic) time scale of the function $t \mapsto w(t)$ driving the evolution. Only at these slow time scales $\mathscr{R}$ describes dissipation correctly.

2.10. Flow rules. Consider a solution $t \mapsto E(t)$ as defined in §2.8. Then for every time t the free surface Σ^f satisfies Laplace's law (2.8) and the contact angle θ verifies the following conditions (with $\theta_{\rm rec}$ and $\theta_{\rm adv}$ defined in §2.7):

(i) $\theta = \theta_{\rm adv}$ at every point of γ where $v^c > 0$;

(ii) $\theta = \theta_{\rm rec}$ at every point of γ where $v^c < 0$;

(iii) $\theta_{\rm rec} \leq \theta \leq \theta_{\rm adv}$ at every point of γ where $v^c = 0$.

In other words, the contact angle is always between the angles $\theta_{\rm rec}$ and $\theta_{\rm adv}$, and must agree with the former one on the part of the contact line which is receding (moving inward), and with the latter one on the part which is advancing (moving outward).

2.11. Derivation of flow rules. Let t be fixed. We first prove that the global stability condition implies the following bounds for the contact angle, which yield statement (iii) in §2.10:

$$\theta_{\rm rec} \leq \theta \leq \theta_{\rm adv} \quad \text{at every point of } \gamma(t). \tag{2.22}$$

[6] By the global stability condition, $E(t)$ minimizes $\mathscr{E}'(t,E) := \sigma_{\rm LV}|\Sigma^f| + V(t,E)$ among all sets E with the same volume and the same contact surface, and therefore the pressure $p(t)$ is well-defined.

The derivation of these bounds takes the next three steps; statements (i) and (ii) are proved in the fourth and last step.

Step 1. A straightforward computation yields

$$\mathscr{E}(t,E) + \mathscr{D}(E,E(t)) = \mathscr{E}''(E) + C\,, \tag{2.23}$$

where C depend on $E(t)$ but not on E,[7] and

$$\mathscr{E}''(E) := \sigma_{\rm LV}\Big[|\Sigma^f| - \cos\theta_{\rm adv}|\Sigma^c\setminus\Sigma^c(t)| + \cos\theta_{\rm rec}|\Sigma^c(t)\setminus\Sigma^c|\Big] + V(E)\,.$$

Step 2. Let $\tilde E(h)$ be an arbitrary variation of $E(t)$, and denote by $\tilde v^f$ and $\tilde v^c$ the normal velocities at $h=0$ of the free surface $\tilde\Sigma^f(h)$ and of the contact line $\tilde\gamma(h)$.

We compute the first variation of $\mathscr{E}''$ as in §2.4: using (2.11), (2.12), the fact that the right derivatives of $\big|\tilde\Sigma^c(h)\setminus\Sigma^c(t)\big|$ and $\big|\Sigma^c(t)\setminus\tilde\Sigma^c(h)\big|$ at $h=0$ are given by the integrals over $\gamma(t)$ of $(\tilde v^c)^+$ and $(\tilde v^c)^-$, respectively (see footnote 2.6), and that $-2\sigma_{\rm LV}H^f+\rho$ is equal to $p(t)$ on $\Sigma^f(t)$ (see §2.8), we obtain

$$\begin{aligned}\frac{d}{dh^+}\mathscr{E}''\big(\tilde E(h)\big)\Big|_{h=0} = p(t)\int_{\Sigma^f(t)}\tilde v^f &\\ + \sigma_{\rm LV}\int_{\gamma(t)}(\cos\theta_{\rm rec}-\cos\theta)(\tilde v^c)^- &\\ + \sigma_{\rm LV}\int_{\gamma(t)}(\cos\theta-\cos\theta_{\rm adv})(\tilde v^c)^+\,. &\end{aligned} \tag{2.24}$$

Step 3. The global stability condition and identity (2.23) imply

$$\frac{d}{dh^+}\mathscr{E}''\big(\tilde E(h)\big)\Big|_{h=0} \geq 0 \tag{2.25}$$

for every *volume-preserving* variation $\tilde E(h)$. Recall that for such a variation the integral of the velocity-field $\tilde v^f$ over $\Sigma^f(t)$ is null and therefore the first integral at the right-hand side of (2.24) vanishes, while the velocity-field $\tilde v^c$ can be any reasonably smooth function on $\gamma(t)$.

Consider now volume-preserving variations $\tilde E(h)$ such that $\tilde v^c$ is positive: by replacing the left-hand side of (2.25) by the right-hand side of (2.24) we obtain

$$\int_{\gamma(t)}(\cos\theta-\cos\theta_{\rm adv})\tilde v^c \geq 0\ ;$$

since $\tilde v^c$ can be any positive function, we infer that $\cos\theta-\cos\theta_{\rm adv}\geq 0$ at every point of $\gamma(t)$, which implies the second inequality in (2.22). By considering volume-preserving variations such that $\tilde v^c$ is negative we derive the other inequality.

[7] If $\mathscr{E}$ is taken as in (2.7), then $C := -\sigma_{\rm LV}\cos\theta_Y|\Sigma^c(t)|$.

Step 4. If we differentiate the energy-dissipation balance (2.21) with respect to the variable t_1 and replace t_1 by t, we obtain

$$\frac{d}{dt}\mathscr{E}(t,E(t)) = \int_{E(t)}\frac{\partial\rho}{\partial t}\,dx + p(t)\,\dot w(t) - \mathscr{R}(E(t),v^c)\,. \tag{2.26}$$

On the other hand, a direct computation of the derivative of $\mathscr{E}(t,E(t))$, similar to the one used to get (2.24), yields

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(t,E(t)) = \int_{E(t)}\frac{\partial\rho}{\partial t} + p(t)\int_{\Sigma^f(t)}v^f &\\ + \sigma_{\rm LV}\int_{\gamma(t)}(\cos\theta-\cos\theta_Y)v^c\,. &\end{aligned} \tag{2.27}$$

Hence the difference between the right-hand sides of (2.26) and (2.27) must vanish, and using (2.9), (2.18), and (2.19) we get

$$\begin{aligned}0 &= \int_{\gamma(t)}(\cos\theta-\cos\theta_Y)v^c + \frac{\mu}{\sigma_{\rm LV}}|v^c| \\ &= \int_{\gamma(t)}\underbrace{(\cos\theta-\cos\theta_{\rm adv})(v^c)^+}_{f_+} + \underbrace{(\cos\theta_{\rm rec}-\cos\theta)(v^c)^-}_{f_-}\,.\end{aligned} \tag{2.28}$$

Finally, (2.22) implies that the functions f_+ and f_- are both non-negative, and therefore the integral in the second line of (2.28) vanishes *only if* $f_+=f_-=0$ at every point of $\gamma(t)$, that is, only if statements (i) and (ii) in §2.10 hold. □

2.12. About the existence of solutions. A fundamental issue regarding the solution defined in §2.8 is clearly its existence for a given initial configuration E_0. We briefly discuss here some aspects of this problem; a rigorous existence result will be given in the next section (Theorem 3.9).

(i) Existence cannot be expected to hold in general for obvious physical reasons. Consider, for instance, a drop subject to gravity on a plane which is horizontal at time 0 and gets more and more inclined as time passes; when the frictional force is no longer sufficient to balance the gravitational force, then the drop slides down. In the limit regime where inertia becomes negligible—the one of quasistatic evolution—the drop disappears instantly at infinity. Thus every existence result must contain assumptions which prevent this phenomenon.

(ii) We have encoded in our definition of solution the requirement that the drop is stable at the initial time. This requirement can be dropped, allowing for unstable initial states. In this case, however, the solution will jump at time zero to a stable state.

(iii) In Theorem 3.9 we prove the existence of solutions for a large class of initial states. However, these solutions are not as regular as required in §2.8, and satisfy the global stability condition and the energy-dissipation balance only in a suitable weak sense. While the regularity in space of these "weak" solutions can certainly be improved, the regularity in time cannot be expected to be much better than BV.

(iv) Due to a lack of convexity of the problem, the solution cannot be expected to be uniquely determined by the initial state. Cases of non-uniqueness and symmetry breaking are presented in Examples 6.4, 6.6, and 6.8.

(v) Even though the notion of quasistatic evolution we use is quite well-established (see [28, 29], and references therein) and it is of considerable mathematical interests to prove rigorous results in this context, it is not completely satisfactory in our setting. In particular, it is reasonable to think that the stability condition should be modified by requiring that $E(t)$ is a *local* minimizer of $\mathscr{E}(E) + \mathscr{D}(E, E(t))$ rather than an *absolute* minimizer. This might affect the way the solution jumps, making it more "realistic": jumps will not occur as soon as a more favorable competitor becomes available, but only when the current state ceases to be a local minimizer. Also, the form of the energy-dissipation balance (2.21) is affected: at a jump time it becomes an inequality because extra energy is lost in a transition from a less stable to a more stable configuration (see Example 6.6 and Remark 6.7(ii)). This "gap" is filled by additional dissipation mechanisms associated with fast time scales, as suggested in Remark 2.9(iii).

This idea is also at the basis of recent proposals that consider alternative notions of solutions, based on vanishing viscosity approximation schemes, in which the extra dissipation at the jump time is the one due to viscous forces (see, for example, [11, 29, 30]).

3. Rigorous existence results

In Theorem 3.9 we state a rigorous existence result for solutions with prescribed volume and prescribed initial configuration. To this end, however, we replace the "strong" notion of solution given in §2.8 with a "weak" one described in §3.7.

The need for a weak formulation is partly due to the intrinsic lack of smoothness of solutions and partly due to the approach we use to prove Theorem 3.9: we first construct time-discretized solutions by iterated minimizations (§4.1), and then take the limit as the discretization parameter tends to 0 (Theorem 4.3). In order to perform both operations we must enlarge the class of admissible drops E to include all finite perimeter sets (see §3.2) and consider maps $E(t)$ with almost no regularity in t. Consequently the geometric and mechanical quantities involved in the definition of the capillary energy $\mathscr{E}$, the dissipation $\mathscr{D}$, and the energy-dissipation balance must be carefully re-defined for this class of objects. This will be done in §3.6 and §3.7.

Let us begin by clarifying some notation used in this and the following sections. We denote points in $\mathbb{R}^3$ as $x = (x_1, x_2, x_3)$. Sets and functions are always assumed to be at least Borel measurable. When it is not clear from the context, we explicitly denote the d-dimensional volume of a set A by $\mathscr{H}^d(A)$ instead of $|A|$ (more precisely, $\mathscr{H}^d(A)$ is the d-dimensional Hausdorff measure of A).

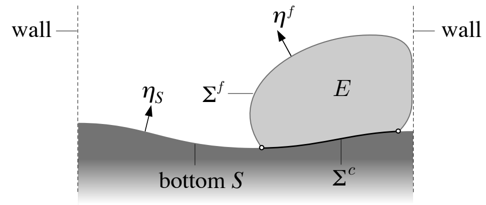

FIGURE 3. Container with completely hydrophobic walls.

3.1. Assumptions on the container and the energy. The container Ω is an open set in $\mathbb{R}^3$ of the form

$$\Omega := \{x \in U \times (0, +\infty) \; : \; x_3 > g(x_1, x_2)\} \tag{3.1}$$

where U is a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary and $g : U \to [0, +\infty)$ is function of class C^1 with bounded gradient; we denote by $\mathrm{Lip}(g)$ the Lipschitz constant of g.

The "bottom" of the container is the part of $\partial\Omega$ corresponding to the graph of g, and is denoted by S. The "wall" is the part of $\partial\Omega$ contained in $\partial U \times \mathbb{R}$ (see Fig. 3).

Unlike the previous section, we assume that the wall of the container is made of a material different from the bottom and completely hydrophobic. To realize this assumption, we modify the definitions of the contact and the free surface of a drop E as follows:

$$\Sigma^c := \partial E \cap S \quad \text{and} \quad \Sigma^f := \partial E \setminus S \, . \tag{3.2}$$

In other words, the contact surface is now the interface between the drop and the bottom of the container (but not the wall). The capillary energy $\mathscr{E} = \mathscr{E}(t, E)$ is then defined by formula (2.7).

We assume furthermore that the energy density ρ associated to the volume energy V (see formula (2.4)) is a positive function of class C^1 on $[0, T] \times \Omega$ with bounded derivative (in particular ρ is Lipschitz) and linear growth at infinity, that is, there exists a positive constant c_0 such that

$$\rho(t, x) \geq c_0 x_3 \, . \tag{3.3}$$

Remark. The only purpose of the specific assumptions on Ω and ρ made above is to simplify some of the statements and proofs contained in this and the next sections. In particular, the growth assumption on ρ has been added to prevent the drop from disappearing at infinity during the evolution, and is used only in the proof of Proposition 5.4. Clearly, the same result could be obtained under different or less restrictive assumptions.

3.2. Drops as finite perimeter sets. In this and the next sections, a drop is a set E with *finite perimeter* in $\mathbb{R}^3$ which is contained in Ω.

Accordingly, ∂E denotes the *essential boundary* of E in $\mathbb{R}^3$ rather than the topological boundary; this modification should be taken into account in the definition of the free surface Σ^f and the contact surface Σ^c (formula (3.2)). Along the same line, we now define the normal vector-field η^f as the *approximate outer normal* to E.

We denote by $\mathscr{Y}$ the class of all finite perimeter sets in $\mathbb{R}^3$ contained in Ω, and by $\mathscr{X}$ the class of all Borel sets Σ with finite area contained in S. The class $\mathscr{Y}$ can be immersed in $L^1(\Omega)$ by mapping every set E into its characteristic function 1_E, and similarly the class $\mathscr{X}$ can be immersed in $L^1(S)$; we then endow $\mathscr{Y}$ and $\mathscr{X}$ with the distances induced by these immersions, that is,

$$d_{\mathscr{Y}}(E,E') := |1_E - 1_{E'}|_{L^1(\Omega)} = |E \triangle E'| \,,$$
$$d_{\mathscr{X}}(\Sigma,\Sigma') := |1_\Sigma - 1_{\Sigma'}|_{L^1(S)} = |\Sigma \triangle \Sigma'| \,.$$

It is tacitly assumed in this definition that we identify sets in $\mathscr{Y}$ which differ by subsets of negligible volume; thus we say that two sets in $\mathscr{Y}$ agree, or one is contained in the other, if this is true up to negligible subsets. A similar convention applies to sets in $\mathscr{X}$.

Finally, we say that a set $E \in \mathscr{Y}$ is a *subgraph* if it agrees (up to negligible subsets) with the subgraph of a function $u : U \to \mathbb{R}$ such that $u \geq g$, that is,

$$E = \big\{ x \in U \times (0,+\infty) \ : \ g(x_1,x_2) < x_3 < u(x_1,x_2) \big\} \,.$$

For more details on the theory of finite perimeter sets see any of the standard references [22, 4, 18].

Remark. Note that the contact line γ, the contact angle θ, and the mean curvature H^f of the free surface Σ^f cannot be properly defined for finite perimeter sets. None of these objects occur in the definition of the capillary energy $\mathscr{E}$, but they do appear in the definition of solution: the mean curvature H^f is used to define the pressure p, while the contact line γ is used, together with its normal velocity v^c, to define the dissipation rate $\mathscr{R}$, and pressure and dissipation rate appear in the energy-dissipation balance.

To get around this problem, in §3.4 we give a formula for the pressure p which does not involve the mean curvature H^f and makes sense for all (energy-minimizing) finite perimeter sets, while in §3.6 we give a formula for the energy dissipated during the evolution which involves neither the contact line γ nor the normal velocity v^c, and indeed does not require any regularity for the solution $E(t)$.

3.3. A special family of deformations. For every $\lambda \in \mathbb{R}$, let Φ_λ be the deformation of $\mathbb{R}^3$ given by

$$\Phi_\lambda(x) := (x_1, x_2, e^\lambda(x_3 - g) + g) \,, \tag{3.4}$$

where $g = g(x_1,x_2)$ is the function that defines the container Ω (see §3.1). Each Φ_λ is a bi-Lipschitz diffeomorphism of $\mathbb{R}^3$ which maps the container Ω into itself and agrees with the identity on the bottom S.

For every set $E \in \mathscr{Y}$ we write

$$E_\lambda := \Phi_\lambda(E) \,. \tag{3.5}$$

Thus E_λ belongs to $\mathscr{Y}$,[8] and we write Σ^c_λ, Σ^f_λ, and η^f_λ for the contact surface, the free surface and the outer normal of E_λ, respectively. Note that $\Sigma^c_\lambda = \Sigma^c$ because Φ_λ leaves every point of S fixed.

The properties of Φ_λ and E_λ will be described in detail in Proposition 5.3, we just anticipate here two formulas that will be used in the next paragraph:

$$\frac{d}{d\lambda}|E_\lambda|\Big|_{\lambda=0} = |E| \quad \text{and} \quad \frac{d}{d\lambda}\mathscr{E}(E_\lambda)\Big|_{\lambda=0} = P^*(t,E) \,, \tag{3.6}$$

where

$$P^*(t,E) := \int_{\Sigma^f} \sigma_{\mathrm{LV}}(1 - (\bar\eta\cdot\eta^f)\eta^f_3) + \rho(x_3 - g)\eta^f_3 \,. \tag{3.7}$$

In this formula, $\bar\eta$ is the vector-field on Ω given by

$$\bar\eta(x) := (-\nabla g(x_1,x_2), 1) \,, \tag{3.8}$$

$\bar\eta\cdot\eta^f$ is the scalar product of $\bar\eta$ and the outer normal η^f to Σ^f, and η^f_3 is the third component of η^f.

3.4. Pressure for energy-minimizing sets. Let E be a set in $\mathscr{Y}$ which minimizes (at a given time t) the capillary energy $\mathscr{E}$—or equivalently the energy $\mathscr{E}'$ defined in (2.15)—among all sets with the same volume and the same contact surface. If E is sufficiently smooth, the pressure p is defined via Laplace's law (2.8); replacing $E(h)$ by E_λ in formula (2.16) and using (3.6) we get

$$p = \frac{\frac{d}{d\lambda}\mathscr{E}(E_\lambda)\Big|_{\lambda=0}}{\frac{d}{d\lambda}|E_\lambda|\Big|_{\lambda=0}} = \frac{P^*(t,E)}{|E|} \,. \tag{3.9}$$

Note that the last term in this identity makes sense for every set $E \in \mathscr{Y}$, regardless of its regularity. In the following we use (3.9) as definition of the pressure p.[9]

3.5. Maps with bounded variation. Let (X, d_X) be a metric space and I an interval in $\mathbb{R}$. The *variation* of a map $f : I \to X$ is

$$\mathrm{Var}(f;I) := \sup\left\{ \sum_{k=1}^n d_X(f(t_{k-1}), f(t_k)) \right\} \tag{3.10}$$

[8] E_λ is a finite perimeter set in $\mathbb{R}^3$ because Φ_λ is bi-Lipschitz, and is contained in Ω because Φ_λ maps Ω into Ω.

[9] The pressure p is a Lagrange multiplier for the minimization of the capillary energy $\mathscr{E}$ with a prescribed-volume constraint, and therefore should be defined only for those sets $E \in \mathscr{Y}$ which minimize $\mathscr{E}$ (or $\mathscr{E}'$) among all sets with the same volume and the same contact surface. For such sets, the regularity theory for almost minimal boundaries yields that Σ^f is a surface of class C^2 and therefore p can be defined by the usual formula (2.8). We prefer, nevertheless, to use formula (3.9). Firstly because it turns out to be particularly useful in certain proofs, and secondly because we make a point that regularity theory is not essential in proving the existence of solutions in the framework of finite perimeter sets.

where the supremum is taken over all positive integers n and all increasing finite sequences $t_0 \le t_1 \le \cdots \le t_n$ contained in I. When X is a subset of a normed space F and $f : I \to X$ is a map of class C^1, the variation of f is given by the well-known formula

$$\mathrm{Var}(f; I) = \int_I \left| \frac{df}{dt} \right|_F dt\,. \tag{3.11}$$

As usual, we say that f has *bounded variation* when $\mathrm{Var}(f; I)$ is finite.

3.6. Alternative definition of dissipation. The energy dissipated by friction by a moving drop $E(t)$ in the time interval $[t_0, t_1]$ is given by the integral from t_0 to t_1 of the dissipation rate $\mathscr{R}(E(t), v^c)$. In view of (2.18), the latter is μ times the norm of the derivative of $t \mapsto 1_{\Sigma^c(t)}$, viewed as a map with values in $L^1(\partial\Omega)$). In other words

$$\int_{t_0}^{t_1} \mathscr{R}(E(t), v^c)\, dt = \int_{t_0}^{t_1} \mu \left| \frac{d1_{\Sigma^c(t)}}{dt} \right|_{L^1} dt = \mu \, \mathrm{Var}(\Sigma^c(t); [t_0, t_1])$$

where the second equality follows from (3.11) (here and afterwards the variation of the map $t \mapsto \Sigma^c(t)$ is computed with respect to the distance $d_{\mathscr{X}}$ defined in §3.2).

While the first term in this formula makes sense only if $E(t)$ is sufficiently regular, the last term can be defined regardless of the regularity of $E(t)$. This motivates the following definition: given a map $t \mapsto E(t)$ from $[t_0, t_1]$ to $\mathscr{Y}$ we set

$$\mathrm{Diss}(E(t); [t_0, t_1]) := \mu \, \mathrm{Var}(\Sigma(t); [t_0, t_1])\,. \tag{3.12}$$

3.7. Stability and solutions, revisited. We say that a set $E' \in \mathscr{Y}$ is stable at time t if it minimizes $\mathscr{E}(t, E) + \mathscr{D}(E, E')$ among all sets $E \in \mathscr{Y}$ with $|E| = |E'|$.

Let w be a strictly positive function of class C^1 defined on the interval $[0, T]$. We say that a map $t \mapsto E(t)$ defined for $t \in [0, T]$ with values in $\mathscr{Y}$ is a solution satisfying the volume constraint $|E(t)| = w(t)$ if the following conditions are satisfied:

Global stability: $E(t)$ has volume $w(t)$ and is stable at time t for every $t \in [0, T]$.

Energy-dissipation balance: for every t_0, t_1 with $0 \le t_0 < t_1 \le T$ there holds

$$\begin{aligned} \mathscr{E}(t_1, E(t_1)) - \mathscr{E}(t_0, E(t_0)) &= \int_{t_0}^{t_1} \left[\int_{E(t)} \frac{\partial \rho}{\partial t}\, dx \right] dt \\ &\quad + \int_{t_0}^{t_1} p\dot{w}\, dt - \mathrm{Diss}(E(t); [t_0, t_1])\,, \end{aligned} \tag{3.13}$$

where the pressure $p(t)$ in the second integral is defined by (3.9) for each set $E(t)$.

3.8. Remark. (i) By Proposition 5.4 there exists a constant m (depending on the setting of the problem and on the function w, but not on t) such that, for every map $t \mapsto E(t)$ which satisfies the global stability condition, the quantities $|\Sigma^f(t)|$, $\mathscr{E}(t, E(t))$, and $p(t) = |P^*(t, E(t))|/|E(t)|$ are bounded from above by m and the sets $E(t)$ are contained in the bounded cylinder $U \times [0, m]$. Therefore the two addenda at the left-hand side of equation (3.13) and the two integrals at the right-hand side are all well-defined and finite. Hence this equation makes always sense, and when verified it implies that the dissipation $\mathrm{Diss}(E(t); [0, T])$ is also finite.

(ii) Lemma 4.4 shows that equation (3.13) holds with "=" replaced by "≥" for any map $t \mapsto E(t)$ that satisfies the global stability condition. Then, as pointed out in the proof of statement (ix) of Theorem 4.3 (end of Section 4), to obtain the equality it suffices to prove the opposite inequality for $t_0 := 0$ and $t_1 := T$.

(iii) The following semigroup property is (almost!) immediate: if $t \mapsto E(t)$ is a solution on the time intervals $[T_0, T_1]$ and $[T_1, T_2]$, then it is also a solution on $[T_0, T_2]$.

3.9. Theorem (existence of solutions). *Take w as in §3.7, and let $E_0 \in \mathscr{Y}$ be an initial configuration with volume $w(0)$ which is stable at time 0.*

Then there exists a solution $E(t)$ defined on $[0, T]$ which satisfies the initial condition $E(0) = E_0$ and the volume constraint $|E(t)| = w(t)$. Moreover the set $E(t)$ and the quantities $\mathscr{E}(E(t))$, $|\Sigma^f(t)|$, $p(t)$ are uniformly bounded in t, and the map $t \mapsto \Sigma^c(t)$ has bounded variation.

3.10. Remark. Let $E(t)$ be a solution.

(i) In general, $E(t)$ is not uniquely determined by the initial configuration $E(0)$ (see Examples 6.4, 6.6, and 6.8).

(ii) The regularity theory for minimal and almost minimal boundaries shows that the free boundary $\Sigma^f(t)$ is of class C^2 for every t (see [35, §1.5 and §1.9], or [26]).

(iii) The fact that the dissipation is finite means that the map $t \mapsto \Sigma^c(t)$ has bounded variation, and since jump discontinuities may occur (Example 6.6), the regularity of this map cannot be substantially higher. The map $t \mapsto E(t)$, on the other hand, may not even have bounded variation (see Example 6.8 and Remark 6.9(i)); this is due to the fact that in our model the dissipation is associated only to the movement of the contact surface, and not of the free surface.

3.11. Jump discontinuities. Let $t \mapsto E(t)$ be a solution. Since the map $t \mapsto \Sigma^c(t)$ has bounded variation and takes values in a complete metric space, for every $t_0 \in [0, T]$ there exist the left and right limits of $\Sigma^c(t)$ for $t \to t_0$, denoted by $\Sigma^c(t_0^-)$ and $\Sigma^c(t_0^+)$, and they both agree with $\Sigma^c(t_0)$ for all t_0 except at most countably many exceptions, called *jump discontinuities*.

Fix $t_0 \in [0, T]$ and consider a limit point E_+ of $E(t)$ for $t \to t_0^+$ and a limit point E_- for $t \to t_0^-$.[10] Clearly $|E_\pm| = |E(t_0)| = w(t_0)$, and Proposition 5.8 implies that the sets $E_\pm$ are stable at time t_0, and their contact surfaces agree with $\Sigma^c(t_0^\pm)$ respectively.

[10] The map $t \mapsto E(t)$ does not have bounded variation and therefore the left and right limit at t_0 may not exist; however, this map takes values in a compact subsets of $\mathscr{Y}$ (see Remark 3.8(i)), and therefore E_- and E_+ always exist.

Moreover, passing to the limit in the energy-dissipation balance (3.13), we obtain

$$\begin{aligned}\mathscr{E}(t_0, E_+) + \mathscr{D}(E_+, E(t_0)) &= \mathscr{E}(t_0, E(t_0))\,,\\ \mathscr{E}(t_0, E(t_0)) + \mathscr{D}(E_-, E(t_0)) &= \mathscr{E}(t_0, E_-)\,,\end{aligned} \tag{3.14}$$

and using (3.14) and the stability of $E(t_0)$ and E_- we get[11]

$$\begin{aligned}E_+ &\in \operatorname{argmin}\big\{\mathscr{E}(t_0, E) + \mathscr{D}(E, E(t_0)) : |E| = w(t_0)\big\}\,,\\ E(t_0) &\in \operatorname{argmin}\big\{\mathscr{E}(t_0, E) + \mathscr{D}(E, E_-) : |E| = w(t_0)\big\}\,.\end{aligned} \tag{3.15}$$

Assume now that the sets $E_\pm$ and $E(t_0)$ are sufficiently smooth. We have already shown in §2.11 that in this case stability implies that the contact angle belongs to the interval $[\theta_{\rm rec}, \theta_{\rm adv}]$. In a similar way we can use (3.15) to derive the following additional conditions, which should be viewed as extensions of conditions (i) and (ii) in §2.10 to jump discontinuities:

(i) the contact angle of $E(t_0)$ agrees with $\theta_{\rm rec}$ in the interior of $\Sigma^c(t_0^-)$, and with $\theta_{\rm adv}$ in the interior of the complement of $\Sigma^c(t_0^-)$;

(ii) the contact angle of E_+ agrees with $\theta_{\rm rec}$ in the interior of $\Sigma^c(t_0)$, and with $\theta_{\rm adv}$ in the interior of the complement of $\Sigma^c(t_0)$.

Another (almost straightforward) consequence of (3.14) is the following: if we modify the map $t \mapsto E(t)$ in t_0 by replacing $E(t_0)$ either by E_+ or E_-, the resulting map is still a solution.

3.12. Proposition (subgraph solutions). *Take g and ρ as in §3.1, and assume that $\rho(t,x)$ is increasing[12] in the variable x_3 and*

$$\operatorname{Lip}(g) \le \cot\theta_{\rm adv}\,. \tag{3.16}$$

Then every solution $E(t)$ is a subgraph for every $t \in [0,T]$.

This proposition is an immediate corollary of a more general result stating that, under these assumptions on ρ and g, every set $E \in \mathscr{Y}$ which is stable in the sense of §3.7 is a subgraph (see Corollary 5.11 and Remark 5.12(ii)).

4. Convergence of discretized solutions and proof of Theorem 3.9

In this section we follow the notation introduced in the previous one. In order to make the proof of Theorem 3.9 more transparent, we have postponed many auxiliary results to the next section.

In the following, the letter δ denotes a positive discretization parameter tending to 0. By "subsequence of δ" we mean any sequence (δ_n) which tends to 0. For simplicity we often omit to relabel subsequences, and write δ instead of δ_n.

[11] Here and in the following, $\operatorname{argmin}\{f(x) : P(x)\}$ denotes the set of all x that minimize $f(x)$ among those for which proposition $P(x)$ is true.

[12] We use the word "increasing" in the weak sense, that is, to mean "non-decreasing".

For the rest of this paper we fix the number T and the positive function w given in the statement of Theorem 3.9, and write v_m and v_M for the minimum and the maximum value of w, respectively.

Warning. By "constant" we always mean a positive finite number which depends only on the setting of the problem—that is, on the choice of the container Ω and of the parameters in $\mathscr{E}$ and $\mathscr{D}$—and on the values v_m, v_M (and therefore not on the initial configuration). With few exceptions, all constants will be denoted by the letter C; in particular the value of C may change at every occurrence, even within the same line.

4.1. Discretized solution. Let $E_0 \in \mathscr{Y}$ be the initial configuration chosen in Theorem 3.9. For every $\delta > 0$ we construct the *discretized solutions* $E_\delta(t)$ with $t \in [0,T]$ as follows: for every integer n such that $n\delta \le T$ we define $E_\delta(n\delta) \in \mathscr{Y}$ by the recursive formula $E_\delta(0) := E_0$ and

$$E_\delta(n\delta) \in \operatorname{argmin}\Big\{\mathscr{E}(n\delta, E) + \mathscr{D}(E_\delta(n\delta - \delta), E) : \ E \in \mathscr{Y},\ |E| = w(n\delta)\Big\}\ ;$$

and for every $t \in [0,T]$ we set

$$E_\delta(t) := E_\delta(t_\delta) \quad \text{where } t_\delta := \sup\{n\delta : n\delta \le t\}\,.$$

In the following we write $\Sigma^f_\delta(t)$ and $\Sigma^c_\delta(t)$ for the free and the contact surfaces of $E_\delta(t) = E_\delta(t_\delta)$. We write $p_\delta(t)$ for the pressure of $E_\delta(t)$ computed at time t_δ (and not t),[13] and set

$$q_\delta(t) := \int_{E_\delta(t)} \frac{\partial\rho}{\partial t}(t_\delta, x)\,dx + p_\delta(t_\delta)\,\dot w(t_\delta)\,. \tag{4.1}$$

4.2. Lemma. *The set $E_\delta(n\delta)$ in §4.1 exists for every δ and n with $n\delta \le T$.*

Proof. Let $\mathscr{F}$ be the functional given by (5.23) with $\rho := \rho(n\delta, x)$ and

$$\sigma(x) := \begin{cases}\cos\theta_{\rm rec} & \text{if } x \in S_\delta\\ \cos\theta_{\rm adv} & \text{if } x \in S \setminus S_\delta\end{cases} \qquad \text{where } S_\delta := \Sigma^c_\delta(n\delta - \delta)\,.$$

Then $E_\delta(n\delta)$ is a minimizer of $\mathscr{F}(E)$ under the constraint $|E| = w(n\delta)$; its existence is proved in Proposition 5.5. □

The next theorem contains all we need to know about the discretized solutions E_δ and their limit as $\delta \to 0$, and it implies Theorem 3.9 as a corollary.

4.3. Theorem. *For every $\delta > 0$ and every $t \in [0,T]$ take $E_\delta(t)$ as in §4.1, and let h_0 be the constant that appears in Proposition 5.4. Then there exist positive constants C such that the following statements hold:*

(i) *for every $\delta > 0$ and $t \in [0,T]$ we have that*
  (a) *$E_\delta(t)$ is contained in $U \times [0, h_0]$ (U is the cross-section of Ω),*
  (b) *$|\Sigma^f_\delta(t)| \le C$, $\mathscr{E}(t_\delta, E_\delta(t)) \le C$, $|p_\delta(t)| \le C$, $|q_\delta(t)| \le C$;*

[13] Thus $p_\delta(t) := P^*(t_\delta, E_\delta(t_\delta))/w(t_\delta)$, and $p_\delta(t) = p_\delta(t_\delta)$.

(ii) *for every $t \in [0,T]$, every subsequence of $E_\delta(t)$ admits a sub-subsequence (depending on t, and not relabelled) that converges to some set $E \in \mathscr{Y}$ contained in $U \times [0,h_0]$; moreover, if we denote by Σ^f, Σ^c, and p the free surface, the contact surface, and the pressure of E, then*
  (a) $\Sigma^c_\delta(t) \to \Sigma^c$ *in* $\mathscr{X}$, $|\Sigma^f_\delta(t)| \to |\Sigma^f|$, *and* $V(t_\delta, E_\delta(t)) \to V(t,E)$,
  (b) $\mathscr{E}(t_\delta, E_\delta(t)) \to \mathscr{E}(t,E)$,
  (c) *E has volume $w(t)$ and is stable at time t,*
  (d) $p_\delta(t) \to p$,
  (e) $|\Sigma^f| \le C$, $\mathscr{E}(t,E) \le C$, $|p| \le C$;

(iii) *for every $t \in [0,T]$, the family of all limit points of the sequence $E_\delta(t)$ is a non-empty compact subset of $\mathscr{Y}$, and for $t=0$ it consists only of the set E_0;*

(iv) *the variations* $\mathrm{Var}(\Sigma^c_\delta(t),[0,T])$ *are uniformly bounded in δ;*

(v) *every subsequence of δ admits a sub-subsequence (not relabelled) such that $\Sigma^c_\delta(t)$ converge to some set in $\mathscr{X}$ for every $t \in [0,T]$.*

Consider now a subsequence of δ (not relabelled) such that $\Sigma^c_\delta(t)$ converge to some $\Sigma(t)$ in $\mathscr{X}$ for every $t \in [0,T]$, and denote by $\mathscr{M}(t)$ the family of all limit points $E \in \mathscr{Y}$ of the subsequence $E_\delta(t)$. Then

(vi) $\mathrm{Var}(\Sigma(t),[0,T]) \le \liminf \mathrm{Var}(\Sigma^c_\delta(t),[0,T]) < +\infty$;

(vii) *$\mathscr{M}(t)$ is a non-empty compact set in $\mathscr{Y}$ for every $t \in [0,T]$, and the contact surface of every $E \in \mathscr{M}(t)$ agrees with $\Sigma^c(t)$; in particular it is always possible to choose $E(t) \in \mathscr{M}(t)$ so that $t \mapsto E(t)$ is a Borel map from $[0,T]$ to $\mathscr{Y}$;*

(viii) *if $\mathscr{E}(T_\delta, E_\delta(T))$ converge to some number in $[-\infty,+\infty]$,*[14] *the functions q_δ defined in (4.1) converge in measure on $[0,T]$ to*
$$q^*(t) := \limsup_{\delta\to 0} q_\delta(t)\,.$$

(ix) *if $\mathscr{E}(T_\delta, E_\delta(T))$ converge to some number in $[-\infty,+\infty]$ and q_δ converge to q^* a.e.,*[15] *every Borel map $t \mapsto E(t)$ from $[0,T]$ to $\mathscr{Y}$ such that $E(t) \in \mathscr{M}(t)$ for every t is a solution with initial configuration E_0 which satisfies $|E(t)| = w(t)$ for every t.*[16]

Remark. (i) The maps $t \mapsto E_\delta(t)$ constructed in §4.1 are not unique because the minimum problem that defines $E_\delta(n\delta)$ may have more than one solution.

(ii) In general it is not possible to find a subsequence of δ (independent of t) such that $E_\delta(t)$ converge for a.e. t, see Example 6.8 and Remark 6.9(ii). This is related to the fact that the movement of the free surface carries no dissipation, and therefore the variations of the maps $t \mapsto E_\delta(t)$ may be not uniformly bounded.

[14] This requirement can be met by refining the subsequence of δ.
[15] By statement (viii) these requirements can be met by refining the subsequence of δ.
[16] The existence of such maps is ensured by statement (vii).

PROOF OF THEOREM 3.9. Take a solution $t \mapsto E(t)$ as in statement (ix) of Theorem 4.3; statement (vi) shows that the map $t \mapsto \Sigma^c(t)$ has bounded variation, while statement (ii) yields uniform bounds on $E(t)$, $|\Sigma^f(t)|$, $\mathscr{E}(t,E(t))$, and $p(t)$. □

The rest of this section is devoted to the proof of Theorem 4.3. Every reference to a "statement #" without further specification, should be understood as "statement # of Theorem 4.3".

PROOF OF STATEMENT (i). It follows immediately from the definition that each $E_\delta(t)$ minimizes (at time t_δ) the energy $\mathscr{E}'$ defined in (2.15) among all sets with the same volume and the same contact surface, and therefore this statement becomes an immediate corollary of Proposition 5.4 (recall that $p_\delta(t) = P^*(t_\delta, E_\delta(t))/w(t_\delta)$ and $q_\delta(t)$ is given by (4.1)). □

PROOF OF STATEMENT (ii). Let t be fixed. For every $\delta > 0$ take t_δ as in §4.1, let $\mathscr{Y}_\delta$ be the class of all sets $E \in \mathscr{Y}$ with volume $v_\delta := w(t_\delta)$, and let $\mathscr{F}_\delta$ be the functional given by formula (5.23) with ρ and σ replaced by $\rho_\delta(x) := \rho(t_\delta, x)$ and
$$\sigma_\delta(x) := \begin{cases} \cos\theta_{\mathrm{rec}} & \text{for } x \in S_\delta \\ \cos\theta_{\mathrm{adv}} & \text{for } x \in S \setminus S_\delta \end{cases} \qquad \text{with } S_\delta := \Sigma^c_\delta(t_\delta - \delta)\,.$$
By definition, each $E_\delta(t)$ minimizes $\mathscr{F}_\delta$ on $\mathscr{Y}_\delta$.

As $\delta \to 0$, the volumes v_δ converge to $v_0 := w(t)$, the functions ρ_δ converge uniformly to $\rho_0(x) := \rho(t,x)$, and, possibly passing to a suitable subsequence of δ, we can assume that the functions σ_δ converge weakly* in $L^\infty(S)$ to some function σ_0. Therefore, if we denote by $\mathscr{F}_0$ the functional given by (5.23) with ρ and σ replaced by ρ_0, σ_0, then Theorem 5.6 implies that the sets $E_\delta(t)$ converge up to subsequences to some minimizer E of $\mathscr{F}_0$ on the class $\mathscr{Y}_0$ of all sets $E \in \mathscr{Y}$ with volume $v_0 := w(t)$.

Moreover Theorem 5.6 gives all the convergences in sub-statement (a), which in turn imply sub-statement (b).

Concerning sub-statement (c), the stability of E follows by Lemma 5.7(ii) and the fact that E minimizes $\mathscr{F}_0$ on $\mathscr{Y}_0$ (note that σ_0 satisfies the bounds $\cos\theta_{\mathrm{adv}} \le \sigma_0 \le \cos\theta_{\mathrm{rec}}$ because the functions σ_δ do so).

Let us prove sub-statement (d). By formula (3.9), the convergence of pressure reduces to the convergence of $P^*(t_\delta, E_\delta(t))$ to $P^*(t,E)$. To prove the latter, we consider the vector measure $\mu := \eta^f \cdot 1_{\Sigma^f} \cdot \mathscr{H}^2$; hence (3.7) becomes
$$P^*(t,E) = \int_\Omega f\Big(x, \frac{\mu}{|\mu|}(x)\Big)\, d|\mu|(x) + \int_\Omega \varphi(t,x)\, d\mu_3(x) \tag{4.2}$$
where $\mu/|\mu|$ is the Radon–Nikodym density of μ with respect to $|\mu|$, μ_3 is the third component of μ, and
$$f(x,v) := \sigma_{\mathrm{LV}}\Big(1 - \frac{(\bar\eta(x)\cdot v)v_3}{|v|^2}\Big)\,, \quad \varphi(t,x) := \rho(t,x)\,(x_3 - g(x_1,x_2))\,.$$

Since f is 0-homogeneous in v and continuous in (x,v) for $v \neq 0$, a well-known theorem by Yu.G. Reshetnyak (see [33] or the appendix in [24]) implies that the first integral in (4.2) is continuous with respect to convergence in mass of μ, while the second one is obviously continuous with respect to convergence of t and μ (in the sense of measures).

Therefore the convergence of $P^*(t_\delta, E_\delta(t))$ to $P^*(t,E)$ is implied by the convergence in mass of the measure μ_δ associated with the sets $E_\delta(t)$ to the measure μ associated with the limit E. Now, the convergence of μ_δ in the sense of measures follows by the fact that μ_δ is the distributional derivative of the characteristic function of $E_\delta(t)$ in Ω, while the convergence of masses is in sub-statement (a).

Finally, the bounds in (e) can be derived from the corresponding bounds in statement (i-b), or directly from Proposition 5.4 using the fact that E is stable, and therefore minimizes the energy $\mathscr{E}'$ (at time t) among all sets with the same volume and the same contact surface. □

PROOF OF STATEMENT (iii). This is a straightforward corollary of statement (ii). □

PROOF OF STATEMENT (iv). Let δ be fixed. We denote by n_δ the largest integer n such that $n\delta \leq T$, and for every $n = 1, \dots, n_\delta$ we set $E^n_\delta := E_\delta(n\delta)$ and $t_n := n\delta$. Taking into account the definition of $E_\delta(t)$ and formulas (2.17), (3.10), (3.12) we obtain

$$\mu \operatorname{Var}(\Sigma^c_\delta(t), [0,T]) = \operatorname{Diss}(E_\delta(t), [0,T]) = \sum_{n=1}^{n_\delta} \mathscr{D}(E^{n-1}_\delta, E^n_\delta)\,. \tag{4.3}$$

Let n be fixed for the time being. In order to estimate $\mathscr{D}(E^{n-1}_\delta, E^n_\delta)$ from above we set

$$\tilde{E_\delta}^{n-1} := \Phi_\lambda(E^{n-1}_\delta)\,,$$

where Φ_λ is defined in (3.4) and λ is chosen so that $|\tilde{E_\delta}^{n-1}| = |E^n_\delta|$, that is, $\lambda := \log w(t_n) - \log w(t_{n-1})$. Thus

$$|\lambda| \leq \min\{C, C'\delta\} \tag{4.4}$$

where $C := \log(v_M/v_m)$ and $C' := \operatorname{Lip}(\log w)$.[17]

Since $\tilde{E_\delta}^{n-1}$ has the same volume as E^n_δ, it can be used as a comparison set for the minimum problem that defines E^n_δ (see §4.1). Then, taking into account that $\mathscr{D}(\tilde{E_\delta}^{n-1}, E^{n-1}_\delta) = 0$ because $\tilde{E_\delta}^{n-1}$ and E^{n-1}_δ have the same contact surface, we get

$$\mathscr{E}(t_n, E^n_\delta) + \mathscr{D}(E^{n-1}_\delta, E^n_\delta) \leq \mathscr{E}(t_n, \tilde{E_\delta}^{n-1})\,.$$

Hence

$$\mathscr{D}(E^{n-1}_\delta, E^n_\delta) \leq I^n_\delta + II^n_\delta + \mathscr{E}(t_{n-1}, E^{n-1}_\delta) - \mathscr{E}(t_n, E^n_\delta)\,, \tag{4.5}$$

[17] We denote by C' any positive finite number which depends on the setting of the problem and on the choice of the function w and not just on its minimum and maximum (this is the case with $\operatorname{Lip}(\log w)$). As usual, the value of C' may vary at every occurrence.

where

$$I^n_\delta := \mathscr{E}(t_n, \tilde{E_\delta}^{n-1}) - \mathscr{E}(t_n, E^{n-1}_\delta)\,, \quad II^n_\delta := \mathscr{E}(t_n, E^{n-1}_\delta) - \mathscr{E}(t_{n-1}, E^{n-1}_\delta)\,.$$

Next we estimate I^n_δ and II^n_δ. Taking into account the definition of $\tilde{E_\delta}^n$, and in particular that it has the same contact surface as E^{n-1}_δ, we get

$$I^n_\delta = \mathscr{E}'(t_n, \tilde{E_\delta}^{n-1}) - \mathscr{E}'(t_n, E^{n-1}_\delta) = g(\lambda) - g(0)\,, \tag{4.6}$$

where we have set $g(s) := \mathscr{E}'(t_n, \Phi_s(E^{n-1}_\delta))$. Using estimate (5.8) and the bounds on the set E^{n-1}_δ and on the area of its free surface proved in statement (i), we obtain that $|\dot{g}| \leq C$ on the interval $[0,\lambda]$. Hence (4.6) and (4.4) yield

$$I^n_\delta \leq C|\lambda| \leq C'\delta\,. \tag{4.7}$$

Concerning II^n_δ, since the gradient of ρ is bounded and $|E^{n-1}_\delta| \leq v_M$, we have

$$II^n_\delta = V(t_n, E^{n-1}_\delta) - V(t_{n-1}, E^{n-1}_\delta) = \int_{E^{n-1}_\delta} \left[\int_{t_{n-1}}^{t_n} \frac{\partial \rho}{\partial t}\, dt \right] dx \leq C\delta\,. \tag{4.8}$$

We can now conclude: (4.5), (4.7) and (4.8) imply

$$\mathscr{D}(E^{n-1}_\delta, E^n_\delta) \leq C'\delta + \mathscr{E}(t_{n-1}, E^{n-1}_\delta) - \mathscr{E}(t_n, E^n_\delta)\,,$$

and taking the sum over all $n = 1, \dots, n_\delta$ we finally get (see (4.3))

$$\mu \operatorname{Var}(\Sigma^c_\delta(t), [0,T]) \leq C'T + \mathscr{E}(0, E_0) - \mathscr{E}(t_{n_\delta}, E^{n_\delta}_\delta) \leq C'\,.$$ □

PROOF OF STATEMENT (v). It suffices to apply Theorem 5.1 to the sequence of maps $t \mapsto \Sigma^c_\delta(t)$. To this end, note that the variations of these maps are uniformly bounded by statement (iv), and that for every $t \in [0,T]$ the set of values $\{\Sigma^c_\delta(t)\}$ is relatively compact in $\mathscr{X}$ by statement (ii). □

PROOF OF STATEMENT (vi). This is an immediate consequence of statement (iv) and of the lower semicontinuity of variation with respect to pointwise convergence. □

PROOF OF STATEMENT (vii). This is an immediate consequence of statements (iii), (ii-a), and of a standard measurable-selection theorem.[18] □

Given a map $t \mapsto E(t)$ as in statement (ix), the fact that each $E(t)$ is stable essentially follows from the statements we have already proved, and what remains to be shown is the energy-dissipation balance. For this we will need the next three lemmas and statement (viii).

[18] See for instance [34, Corollary 5.2.5]: the key point is that $t \mapsto \mathscr{M}(t)$ is a Borel map from $[0,T]$ into the space of compact subsets of $\mathscr{Y}$, endowed with the Hausdorff distance; this measurability property is an immediate consequence of the definition of $\mathscr{M}(t)$ as sets of limit points of $E_\delta(t)$.

4.4. Lemma. *Consider a Borel map $t \mapsto E(t)$ from $[0,T]$ to $\mathscr{Y}$ such that for every $t \in [0,T]$ the set $E(t)$ has volume $w(t)$ and is stable at time t. Then for every t_0, t_1 with $0 \le t_0 < t_1 \le T$ there holds*

$$\mathscr{E}(t_1, E(t_1)) - \mathscr{E}(t_0, E(t_0)) \ge \int_{t_0}^{t_1} q(t)\, dt - \mathrm{Diss}(E(t); [t_0, t_1])\,, \tag{4.9}$$

where

$$q(t) := \int_{E(t)} \frac{\partial \rho}{\partial t}(t,x)\, dx + p(t)\, \dot w(t) \tag{4.10}$$

and $p(t)$ is the pressure at time t for the set $E(t)$, as given by (3.9).

PROOF. We can restrict our attention to the case $t_0 = 0$ and $t_1 = T$. Since each $E(t)$ is stable, it minimizes the energy $\mathscr{E}'$ among all sets with the same volume and the same contact surface. Then Proposition 5.4 yields $E(t) \subset U \times [0, h_0]$, $|\Sigma^f(t)| \le C$, and $|p(t)| \le C$.

We fix for the time being $\delta > 0$ and a partition $0 = t_0 < t_1 < \dots < t_m = T$ of the interval $[0,T]$ such that $\delta_n := t_n - t_{n-1} \le \delta$ for $n = 1, \dots, m$. Moreover we set $E_n := E(t_n)$ and

$$\hat E_n := \Phi_\lambda(E_n)\,,$$

where λ is chosen so that $|\hat E_n| = |E_{n-1}|$, that is, $\lambda := \log w(t_{n-1}) - \log w(t_n)$.

Thus the first-order Taylor expansion of $\log w$ around the point t_n yields [19]

$$\lambda = -\frac{\dot w(t_n)}{w(t_n)} \delta_n + o(\delta_n)\,. \tag{4.11}$$

We first establish a lower bounded for $\mathscr{E}(t_n, E_n) - \mathscr{E}(t_{n-1}, E_{n-1})$. Since $\hat E_n$ has the same volume as E_{n-1}, it can be used as a comparison set in the definition of stability of $E_{n-1} = E(t_{n-1})$, and taking into account that $\hat E_n$ and E_n have the same contact surface, we get

$$\mathscr{E}(t_{n-1}, E_{n-1}) \le \mathscr{E}(t_{n-1}, \hat E_n) + \mathscr{D}(E_{n-1}, E_n)\,.$$

Hence

$$\mathscr{E}(t_n, E_n) - \mathscr{E}(t_{n-1}, E_{n-1}) \ge I_n + II_n - \mathscr{D}(E_{n-1}, E_n)\,, \tag{4.12}$$

where

$$I_n := \mathscr{E}(t_{n-1}, E_n) - \mathscr{E}(t_{n-1}, \hat E_n)\,, \quad II_n := \mathscr{E}(t_n, E_n) - \mathscr{E}(t_{n-1}, E_n)\,.$$

To estimate I_n and II_n from below we closely follow the estimates of I^n_δ and II^n_δ in the proof of statement (iv). Thus

$$I_n := \mathscr{E}'(t_{n-1}, E_n) - \mathscr{E}'(t_{n-1}, \hat E_n) = g(0) - g(\lambda) \tag{4.13}$$

[19] We write $o(\delta^\alpha)$ for every function g depending on δ and possibly on other variables which satisfies $|g| \le \tilde g$ where $\tilde g$ is a positive function of δ which depends only on the setting of the problem and on the choice of the function w, and satisfies $\tilde g(\delta)/\delta^\alpha \to 0$ as $\delta \to 0$. Similarly, we write $O(\delta^\alpha)$ for every function g which satisfies $|g| \le C'\delta^\alpha$ for some positive constant C' as in footnote 4.

where $g(s) := \mathscr{E}'(t_{n-1}, \Phi_s(E_n))$. Next we replace $g(\lambda)$ in (4.13) by its first-order Taylor expansion at 0: using estimate (5.9) and the usual bounds on Σ^f_n and λ, we obtain that $|\ddot g| \le C$ on $[0, \lambda]$, while formula (3.6) yields $\dot g(0) = P^*(t_{n-1}, E_n)$. Moreover formula (3.7) shows that

$$P^*(t_{n-1}, E_n) = P^*(t_n, E_n) + O(\delta_n)\,,$$

while (3.9) yields $P^*(t_n, E_n) = p(t_n)\, w(t_n)$. Hence (4.13) and (4.11) yield

$$\begin{aligned} I_n = -\dot g(0)\, \lambda + O(\lambda^2) &= -p(t_n)\, w(t_n)\, \lambda + O(\delta_n)\, |\lambda| + O(\lambda^2) \\ &= p(t_n)\, \dot w(t_n)\, \delta_n + o(\delta_n)\,. \end{aligned} \tag{4.14}$$

Concerning II_n we have

$$\begin{aligned} II_n &= V(t_n, E_n) - V(t_{n-1}, E_n) \\ &= \int_{E_n} \Big[\int_{t_{n-1}}^{t_n} \frac{\partial \rho}{\partial t}\, dt \Big]\, dx = \Big[\int_{E_n} \frac{\partial \rho}{\partial t}(t_n, x)\, dx \Big] \delta_n + o(\delta_n)\,. \end{aligned} \tag{4.15}$$

We can now conclude the proof of (4.9): using that $o(\delta_n) = o(1) \cdot \delta_n$, from formulas (4.10), (4.12), (4.14) and (4.15) we obtain

$$\mathscr{E}(t_n, E_n) - \mathscr{E}(t_{n-1}, E_{n-1}) \ge q(t_n)\, \delta_n - o(1)\, \delta_n - \mathscr{D}(E_{n-1}, E_n)\;;$$

then, taking the sum over all n and recalling the definition of dissipation ((3.12) and (3.10)), we get

$$\mathscr{E}(T, E(T)) - \mathscr{E}(0, E(0)) \ge \sum_{n=1}^m q(t_n)\, \delta_n - o(1) - \mathrm{Diss}(E(t), [0,T])\,. \tag{4.16}$$

To recover inequality (4.9) from (4.16), it suffices to notice that it is always possible to choose first δ and then the partition points t_n so that the term $o(1)$ at the right-hand side of (4.16) is arbitrarily close to 0, and the sum of $q(t_n)\, \delta_n$ is arbitrarily close to, or even larger than, the integral of q from 0 to T. □

4.5. Lemma. *Under the assumption of statement* (viii)*, let $t \mapsto E(t)$ be a map such that $E(t)$ belongs to $\mathscr{M}(t)$ for every t. Then*

$$\mathscr{E}(T, E(T)) - \mathscr{E}(0, E(0)) \le \liminf_{\delta \to 0} \int_0^T q_\delta(t)\, dt - \mathrm{Diss}(E(t), [0,T])\,. \tag{4.17}$$

PROOF. We start from inequality (4.5) in the proof of statement (iv). Following the notation of that proof, we rewrite (4.5) as

$$\mathscr{E}(t_n, E^n_\delta) - \mathscr{E}(t_{n-1}, E^{n-1}_\delta) \le I^n_\delta + II^n_\delta - \mathscr{D}(E^{n-1}_\delta, E^n_\delta)\,. \tag{4.18}$$

Proceeding as in the proof of Lemma 4.4 we obtain the following expressions for I^n_δ and II^n_δ:

$$I^n_\delta = p_\delta(t_{n-1})\, \dot w(t_{n-1})\, \delta + o(\delta)\,, \quad II^n_\delta = \Big[\int_{E^{n-1}_\delta} \frac{\partial \rho}{\partial t}(t_{n-1}, x)\, dx \Big] \delta + o(\delta)\,.$$

Therefore (4.18) becomes

$$\mathscr{E}(t_n, E^n_\delta) - \mathscr{E}(t_{n-1}, E^{n-1}_\delta) \le q_\delta(t_{n-1})\, \delta - \mathscr{D}(E^{n-1}_\delta, E^n_\delta) + o(\delta)\,,$$

and summing over all $n = 1, \dots, n_\delta$ we obtain

$$\mathscr{E}(t_{n_\delta}, E_\delta^{n_\delta}) - \mathscr{E}(t_0, E_\delta^0) = \int_0^{t_{n_\delta}} q_\delta(t)\,dt - \mathrm{Diss}(E_\delta(t), [0,T]) + o(1)$$

(use (4.3) and the fact that $q_\delta(t) = q_\delta(t_{n-1})$ for every $t \in [t_{n-1}, t_n)$).

Taking into account the definition of E_δ^n and the fact that $t_0 = 0$, $t_{n_\delta} = T_\delta$, $T - t_{n_\delta} < \delta$ and $|q_\delta(t)| \le C$ (statement (i-b)), the previous inequality yields

$$\mathscr{E}(T_\delta, E_\delta(T)) - \mathscr{E}(0, E_\delta(0)) \le \int_0^T q_\delta(t)\,dt - \mathrm{Diss}(E_\delta(t), [0,T]) + o(1)\,.$$

To obtain (4.17) we pass to the limit as $\delta \to 0$ and use the following facts:

(a) $E_\delta(0) = E(0)$;

(b) $\mathrm{Diss}(E(t), [0,T]) \le \liminf \mathrm{Diss}(E_\delta(t), [0,T])$;

(c) $\mathscr{E}(T_\delta, E_\delta(T)) \to \mathscr{E}(T, E(T))$.

Assertion (b) follows from statement (vi) and the fact that the dissipation of $E(t)$ is μ times the variation of $\Sigma(t)$—recall that $\Sigma(t)$ is the contact surface of $E(t)$ by statement (vii). Assertion (c) follows from the assumption that $\mathscr{E}(T_\delta, E_\delta(T))$ converges, and therefore the limit must be $\mathscr{E}(T, E(T))$ by statement (ii-b). □

PROOF OF STATEMENT (viii). For every $t \in [0,T]$ we can choose $E(t) \in \mathscr{M}(t)$ such that the number $q(t)$ defined in (4.10) agrees with $q^*(t)$. Indeed the definition of $q^*(t)$ and statement (ii) yield a subsequence δ_k such that $q_{\delta_k}(t)$ converge to $q^*(t)$ and $E_{\delta_k}(t)$ converge to a set $E(t) \in \mathscr{M}(t)$.[20]

Moreover the sets $E_{\delta_k}(t)$ are uniformly bounded (statement (i-a)) and $p_{\delta_k}(t)$ converge to the pressure $p(t)$ associated to $E(t)$ (statement (ii-d)). Thus (4.10) and (4.1) imply that $q_{\delta_k}(t) \to q(t)$, and therefore $q(t) = q^*(t)$.

Now, the map $t \mapsto E(t)$ satisfies the hypothesis of Lemma 4.5 by assumption, and those of Lemma 4.4 by statement (ii-c). Therefore, putting together inequality (4.9) with $t_0 := 0$ and $t_1 := T$, inequality (4.17), and the identity $q = q^*$, we finally get

$$\int_0^T q^*(t)\,dt \le \liminf_{\delta\to 0} \int_0^T q_\delta(t)\,dt\,. \tag{4.19}$$

It remains to show that (4.19) implies that q_δ converge in measure to q^*. Since q^* is the upper limit of q_δ, we have only to prove that for every $\varepsilon > 0$ the measure of the set

$$A_\delta := \big\{t : \ q_\delta(t) \le q^*(t) - \varepsilon\big\}$$

converges to 0 as $\delta \to 0$. If we set $q'_\delta(t) := \max\{q_\delta(t), q^*(t)\}$ we have $q_\delta + \varepsilon 1_{A_\delta} \le q'_\delta$ and therefore

$$\int_0^T q_\delta(t)\,dt + \varepsilon|A_\delta| \le \int_0^T q'_\delta(t)\,dt\,. \tag{4.20}$$

Moreover, since the functions q'_δ converge pointwise to q^* and are uniformly bounded (because the functions q_δ are, see statement (i-b)), the integrals of q'_δ converge to the integral of q^* by the dominated convergence theorem, and passing to the limit in (4.20), we get

$$\liminf_{\delta\to 0} \int_0^T q_\delta(t)\,dt + \varepsilon \limsup_{\delta\to 0} |A_\delta| \le \int_0^T q^*(t)\,dt\,,$$

which together with (4.19) implies $\limsup |A_\delta| = 0$. □

[20] We assume that the map $t \mapsto E(t)$ is Borel measurable, see footnote 4.

4.6. Lemma. *Let $t \mapsto E(t)$ be taken as in statement* (ix)*. Then the function q defined in* (4.10) *agrees with q^* a.e. in* $[0,T]$.

PROOF. For every t there exists a subsequence δ_k such that $E_{\delta_k}(t)$ converge to $E(t)$, and therefore $q_{\delta_k}(t) \to q(t)$. On the other hand $q_\delta(t) \to q^*(t)$ for a.e. t by assumption. □

PROOF OF STATEMENT (ix). Statements (iii) and (ii-c) show that $E(0) = E_0$, and that each $E(t)$ has volume $w(t)$ and is stable at time t.

It remains to prove the energy-dissipation balance (3.13). Note that this balance can be re-written as $I(t_0, t_1) = 0$ for every $0 \le t_0 \le t_1 \le T$, where $I(t_0, t_1)$ is the difference between the left and the right-hand side of (4.9),[21] and we already know from Lemma 4.4 that $I(t_0, t_1) \ge 0$.

By Lemma 4.6 the functions q_δ converge a.e. to q, and therefore the lower limit of their integrals, which appears in (4.17), is actually a limit and agrees with the integral of q.[22] Thus (4.17) becomes $0 \ge I(0,T)$. Hence

$$0 \ge I(0,T) = I(0,t_0) + I(t_0,t_1) + I(t_1,T)\,, \text{[23]}$$

and since the three addenda in the last term are all non-negative, they must be null. In particular $I(t_0, t_1) = 0$. □

5. AUXILIARY RESULTS

In this section we collect some technical lemmas used in Section 4. We follow the notation introduced in the previous two sections; in particular concerning constants (see the "warning" at the beginning of Section 4).

We first recall a generalization of a classical result by E. Helly on monotone functions.

5.1. Helly's Selection Theorem (see [25], Theorem 3.2). *Let I be an interval, X a complete metric space, and $f_n : I \to X$ a sequence of maps with uniformly bounded variations (in the sense of §3.5) such that for every $t \in I$ the set of values $\{f_n(t)\}$ is relatively compact in X. Then, upon extraction of a suitable subsequence, the maps f_n converge pointwise to some limit map $f : I \to X$.*

[21] $I(t_0, t_1)$ is well-defined because all terms in (4.9) are finite with the only possible exception of the dissipation.

[22] Apply the dominated convergence theorem and the fact the functions q_δ are uniformly bounded, see statement (i-b).

[23] Use the additivity of variation: $\mathrm{Var}(f, [t_0, t_2]) = \mathrm{Var}(f, [t_0, t_1]) + \mathrm{Var}(f, [t_1, t_2])$ whenever $t_0 \le t_1 \le t_2$.

5.2. Lemma. *For every set Σ in $\mathscr{X}$ and every $\delta > 0$ there exists a set E in $\mathscr{Y}$ with contact surface equal to Σ such that $|E| \le \delta|\Sigma|$ and $|\Sigma^f| \le (1+\delta)|\Sigma|$ ($\mathscr{X}$ and $\mathscr{Y}$ are defined in §3.2) . Moreover E is contained in $U \times (0, h_0)$ where U is the cross-section of Ω and h_0 is any number strictly greater than the supremum of g over U.*

PROOF. Consider the open set $A := \Omega \cap (U \times (0, h_0))$. By a well-known result of E. Gagliardo (see for instance [22], Theorem 2.16 and Remark 2.17) there exists a smooth function $u : A \to [0, 1]$ such that $\|u\|_1 \le \delta|\Sigma|$ and $\|\nabla u\|_1 \le (1+\delta)|\Sigma|$, and whose trace on ∂A agrees with the characteristic function of Σ. We then take E equal to a suitable superlevel set of u (to choose the right level one can use the coarea formula, see the proof of Theorem 1.24 in [22]). □

5.3. Proposition. *Given $E \in \mathscr{Y}$ and $\lambda \in \mathbb{R}$, take E_λ as in §3.3 and define Σ^f_λ and η^f_λ accordingly. Then for every $t \in [0, T]$ there holds*

$$|E_\lambda| = e^\lambda |E| \quad \text{and then} \quad \frac{d}{d\lambda}|E_\lambda| = |E_\lambda|\,, \tag{5.1}$$

$$|\Sigma^f_\lambda| = \int_{\Sigma^f} |e^\lambda \eta^f + (1-e^\lambda)\eta^f_3 \bar\eta| \le e^{C|\lambda|}|\Sigma^f|\,, \tag{5.2}$$

$$\frac{d}{d\lambda}|\Sigma^f_\lambda| = \int_{\Sigma^f_\lambda} 1 - \eta^f_{\lambda,3}\bar\eta \cdot \eta^f_\lambda\,, \tag{5.3}$$

$$V(t, E_\lambda) = e^\lambda \int_E \rho(t, \Phi_\lambda(x))\,dx \le e^{C|\lambda|} V(t, E)\,, \tag{5.4}$$

$$\mathscr{E}'(t, E_\lambda) \le e^{C|\lambda|}\mathscr{E}'(t, E) \tag{5.5}$$

(in the previous equations η^f_3 and $\eta^f_{\lambda,3}$ are the third components of η^f and η^f_λ, $\bar\eta$ is defined in (3.8), and $\mathscr{E}'$ is defined in (2.15)).

If, in addition, the set E is bounded, then

$$\frac{d}{d\lambda}V(t, E_\lambda) = \int_{E_\lambda} \rho + (x_3 - g)\frac{\partial\rho}{\partial x_3} = \int_{\Sigma^f_\lambda} (x_3 - g)\rho\,\eta^f_{\lambda,3}\,, \tag{5.6}$$

$$\frac{d}{d\lambda}\mathscr{E}'(t, E_\lambda) = \int_{\Sigma^f_\lambda} \sigma_{\mathrm{LV}}\big(1 - \eta^f_{\lambda,3}\bar\eta\cdot\eta^f_\lambda\big) + (x_3 - g)\rho\,\eta^f_{\lambda,3}\,, \tag{5.7}$$

where g is the function that defines Ω, see §3.1. Moreover there exists a positive function $C(h, s)$, increasing both in h and s, with the following property: for every set $E \in \mathscr{Y}$ contained in $U \times [0, h]$, and every $|\lambda| \le s$ there holds

$$\left|\frac{d}{d\lambda}\mathscr{E}'(t, E_\lambda)\right| \le C(h, s)\,|\Sigma^f|\,, \tag{5.8}$$

$$\left|\frac{d^2}{d\lambda^2}\mathscr{E}'(t, E_\lambda)\right| \le C(h, s)\,|\Sigma^f|\,. \tag{5.9}$$

Remark. Some statements of the previous proposition concern the energy $\mathscr{E}'$, but remain valid even if $\mathscr{E}'$ is replaced by $\mathscr{E}$, because $\mathscr{E}$ differs from $\mathscr{E}'$ only by a boundary contribution which does not depend on λ (recall that E and E_λ have

the same contact surface) and is bounded from above by a constant. A similar remark applies to Proposition 5.4.

PROOF OF PROPOSITION 5.3. Recall that $E_\lambda := \Phi_\lambda(E)$ where Φ_λ is defined in (3.4). Then the first identity in (5.1) follows by the fact that the Jacobian determinant of Φ_λ is e^λ; the second one is obtained by differentiating the first one with respect to λ.

Since the map Φ_λ is bi-Lipschitz, if E has finite perimeter then $E_\lambda := \Phi_\lambda(E)$ has finite perimeter, too, and the *essential* boundary ∂E_λ is equal to $\Phi_\lambda(\partial E)$. Hence the free surface Σ^f_λ is equal to $\Phi_\lambda(\Sigma^f)$. Then a lengthy but straightforward computation show that the (approximate) outer normal to E_λ at $y := \Phi_\lambda(x)$ is given by

$$\eta^f_\lambda = H\big(e^\lambda\eta^f + (1-e^\lambda)\eta^f_3\bar\eta\big) \tag{5.10}$$

where $H(v) := v/|v|$ for every $v \ne 0$, and η^f and $\bar\eta$ are computed at x. Moreover the Jacobian determinant of the restriction of Φ_λ to the rectifiable set Σ^f is

$$J = \big|e^\lambda\eta^f + (1-e^\lambda)\eta^f_3\bar\eta\big|\,. \tag{5.11}$$

The equality in (5.2) follows from (5.11), and the inequality follows from the estimate

$$\begin{aligned} J = \big|\eta^f + (e^\lambda - 1)(\eta^f - \eta^f_3\bar\eta)\big| &\le |\eta^f| + |e^\lambda - 1|\,|\eta^f - \eta^f_3\bar\eta| \\ &= 1 + C|e^\lambda - 1| \le e^{C|\lambda|}\,. \end{aligned}$$

Differentiating the identity in (5.2) with respect to λ we get

$$\frac{d}{d\lambda}|\Sigma^f_\lambda|\Big|_{\lambda=0} = \int_{\Sigma^f} 1 - \eta^f_3\bar\eta\cdot\eta^f\ ; \tag{5.12}$$

we then obtain (5.3) using the semi-group property of Φ_λ, namely that

$$\Phi_{\lambda_1+\lambda_2} = \Phi_{\lambda_1}\circ\Phi_{\lambda_2} \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R}. \tag{5.13}$$

The equality in (5.4) is obtained by applying the change of variable $x' = \Phi_\lambda(x)$ to the integral that gives $V(t, E_\lambda)$; the inequality follows from the estimate

$$\begin{aligned} \rho(t, \Phi_\lambda(x)) &\le \rho(t, x) + \mathrm{Lip}(\rho)\,|\Phi_\lambda(x) - x| \\ &= \rho(t, x) + \mathrm{Lip}(\rho)\,|e^\lambda - 1|\,(x_3 - g) \\ &\le (1 + C|e^\lambda - 1|)\,\rho(t, x) \le e^{C|\lambda|}\rho(t, x) \end{aligned}$$

(to obtain the second inequality we used assumption (3.3) and the fact that $g \ge 0$).

Estimate (5.5) follows immediately from the inequalities in (5.2) and (5.4).

Differentiating the identity in (5.4) with respect to λ we obtain

$$\begin{aligned} \frac{d}{d\lambda}V(t, E_\lambda)\Big|_{\lambda=0} &= \int_E \rho(t, x) + \frac{\partial\rho}{\partial x_3}(t, x)\,(x_3 - g)\,dx \\ &= \int_E \frac{\partial}{\partial x_3}((x_3 - g)\rho) = \int_{\Sigma^f} (x_3 - g)\rho\,\eta^f_3\ ; \end{aligned} \tag{5.14}$$

the last identity is a consequence of the usual representation of the distributional partial derivatives of the characteristic function of E in terms of integration on the essential boundary ∂E, namely

$$\frac{\partial}{\partial x_i} 1_E = -\eta_i \cdot 1_{\partial E} \cdot \mathscr{H}^2 ,$$

and the fact that the test function $(x_3 - g)\rho$ vanishes on $\partial E \setminus \Sigma^f$.[24]

We obtain (5.6) from (5.14) using the semi-group property of Φ_λ.

Identity (5.7) follows from (5.3) and (5.6).

Estimate (5.8) follows from (5.7) using estimate (5.2), and the fact that the integrand in (5.7) is bounded on bounded sets.

To prove (5.9) we must first compute the second derivative of $\mathscr{E}'(t, E_\lambda)$ with respect to λ. To this purpose, we start from (5.7) and re-write the first derivative as

$$\frac{d}{d\lambda} \mathscr{E}'(t, E_\lambda) = \sigma_{\rm LV} |\Sigma^f_\lambda| + \int_{\Sigma^f_\lambda} \big[-\sigma_{\rm LV} \bar\eta \cdot \eta^f_\lambda + (y_3 - g)\rho \big] \eta^f_{\lambda,3} \, dy ,$$

where $\bar\eta$, η^f_λ, g, and ρ are computed at y. Next we apply the change of variable $y = \Phi_\lambda(x)$ and use (5.10) and (5.11) to prove that the Jacobian determinant J of this transformation satisfies $\eta^f_{\lambda,3}(y) = \eta^f_3(x)/J(x)$. Hence

$$\frac{d}{d\lambda} \mathscr{E}'(t, E_\lambda) = \sigma_{\rm LV} |\Sigma^f_\lambda| + \int_{\Sigma^f} \big[-\sigma_{\rm LV} \bar\eta \cdot \eta^f_\lambda + e^\lambda (x_3 - g)\rho \big] \eta^f_3 \, dx , \tag{5.15}$$

where η^f_λ and ρ are computed at $y = \Phi_\lambda(x)$, while $\bar\eta$, g and η^f_3 are computed at x.[25] Starting from formula (5.10), a straightforward computation yields

$$\frac{d\eta^f_\lambda}{d\lambda}\Big|_{\lambda=0} = -\eta^f_3 \, p(\bar\eta) \tag{5.16}$$

where p denotes the projection from $\mathbb{R}^3$ onto the plane orthogonal to η^f. Using (5.12) and (5.16) we can compute the derivative of (5.15) with respect to λ:

$$\begin{aligned} &\frac{d^2}{d\lambda^2} \mathscr{E}'(t, E_\lambda)\Big|_{\lambda=0} \\ &= \int_{\Sigma^f} \sigma_{\rm LV} \big[1 - \eta^f_3 \bar\eta \cdot (\eta^f - \eta^f_3 p(\bar\eta)) \big] + (x_3 - g)\rho + (x_3 - g)^2 \frac{\partial\rho}{\partial x_3} . \end{aligned} \tag{5.17}$$

Using the semi-group property of Φ_λ we obtain a similar formula for the second derivative of $\mathscr{E}'(t, E_\lambda)$ at any λ, and then we derive estimate (5.9) using estimate (5.2) and the fact that the integrand in (5.17) is bounded on bounded sets. □

[24] Since $(x_3 - g)\rho$ is not a function of class C^1 with compact support on $\mathbb{R}^3$, a correct derivation of the last identity in (5.14) is a bit more delicate: the key points are that E is bounded and $(x_3 - g)\rho$ is locally bounded on $U \times \mathbb{R}$ and of class C^1 with respect to the variable x_3.

[25] The identity $y = \Phi_\lambda(x)$ implies $y_i = x_i$ for $i = 1, 2$, and both $\bar\eta$ and g depend only on the first two variables (since g is originally defined as a function of two variables, here there is a slight abuse of notation).

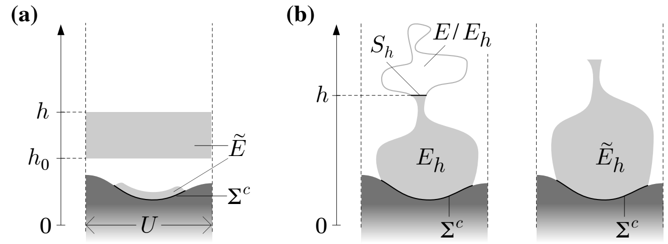

FIGURE 4. The comparison sets $\tilde E$ and $\tilde E_h$ in the proof of Proposition 5.4.

5.4. Proposition. *There exist constants h_0 and C with the following property: if $E \in \mathscr{Y}$ satisfies $v_m \le |E| \le v_M$ and minimizes $\mathscr{E}'$ (at a given time t) among all sets in $\mathscr{Y}$ with the same volume and the same contact surface, then*

- (i) *$|\Sigma^f| \le C$ and $|\mathscr{E}'(t, E)| \le C$ for every t;*
- (ii) *E is contained (up to a negligible subset) in $U \times [0, h_0]$;*
- (iii) *$|P^*(t, E)| \le C$ for every t.*

PROOF. To prove the energy bound in (i) it suffices to construct a comparison set $\tilde E \in \mathscr{Y}$ with contact surface $\tilde\Sigma^c$ equal to the contact surface Σ^c of E, volume $|\tilde E|$ equal to $|E|$, and energy $\mathscr{E}'(t, \tilde E)$ bounded from above by a constant which depends neither on E nor on t.

Let $h_1 := 1 + \sup g$. By Lemma 5.2 we can find a set $E_0 \in \mathscr{Y}$ such that $E_0 \subset U \times (0, h_1)$, $\Sigma^c_0 = \Sigma^c$, $|E_0| < v_m$, and $|\Sigma^f_0| \le 2|\Sigma^c|$. We then set (see Fig. 4a)

$$\tilde E := E_0 \cup (U \times (h_1, h))$$

where h is chosen in such a way that $|\tilde E| = v$, that is, $h := h_1 + (v - |E_0|)/|U|$; it is straightforward to check that $\tilde E$ meets all requirements. To conclude the proof of statement (i), note that a bound on $\mathscr{E}'$ implies also a bound on $|\Sigma^f|$ (for example, because V is positive).

Now we define, for every $h > 0$,

$$E_h := E \cap (U \times (0, h)) \quad \text{and} \quad a(h) := |E \setminus E_h| .$$

In order to prove statement (ii) we must show that there exists a constant h_0 such that $a(h) = 0$ for $h \ge h_0$. This will follow from the fact that the Lipschitz function a satisfies the differential inequality (5.22); in turn, this differential inequality will be obtained by estimating the energy of the comparison sets $\tilde E_h$ defined in (5.19).

We first prove a decay estimate for a. The energy bound in (i) and assumption (3.3) yield

$$C \ge \mathscr{E}'(t, E) \ge V(t, E) \ge V(t, E \setminus E_h) \ge c_0 \, a(h) \, h ,$$

and therefore

$$a(h) \le C/h \quad \text{for every } h > 0. \tag{5.18}$$

By (5.18) there exists a constant h_2 such that $a(h) \le v_m/2$ for every $h \ge h_2$. For every such h we define the comparison set (see Fig. 4b)

$$\tilde{E}_h := \Phi_\lambda(E_h) \tag{5.19}$$

where Φ_λ is the deformation defined in (3.4) and the parameter $\lambda = \lambda(h)$ is chosen so that $|\tilde{E}_h| = |E|$. Taking into account (5.1), (5.18), and the choice of h_2 we get

$$1 \le e^\lambda = \frac{|E|}{|\tilde{E}_h|} = \frac{1}{1 - a(h)/|E|} \le \frac{1}{1 - a(h)/v_m} \le 2\,. \tag{5.20}$$

Let S_h be the section of E at height h, that is, the set of all $x \in E$ such that $x_3 = h$. Note that

$$a(h) = \int_h^{+\infty} |S_t|\,dt \quad \text{and then} \quad |S_h| = -\dot{a}(h) \text{ for a.e. } h,$$

where $\dot{a}$ is the derivative of the Lipschitz function a.

Next we estimate $\mathscr{E}'(t, E_h)$ and $\mathscr{E}'(t, \tilde{E}_h)$. The key point is that the section of the free surface of E_h at height h agrees with S_h (up to a subset of negligible area) for a.e. h, and therefore

$$\Sigma^f_h = [\Sigma^f \cap (U \times (0,h))] \cup S_h\,.$$

Hence

$$|\Sigma^f_h| = |\Sigma^f| + |S_h| - |\Sigma^f \cap (U \times [h,\infty))|\,,$$

and since the inclusion $E_h \subset E$ implies $V(t,E_h) \le V(t,E)$, for a.e. $h \ge h_2$ we get

$$\begin{aligned}
\mathscr{E}'(t,E_h) &\le \mathscr{E}'(t,E) + \sigma_{\rm LV}\big[|S_h| - |\Sigma^f \cap (U\times[h,\infty))|\big] \\
&= \mathscr{E}'(t,E) + \sigma_{\rm LV}\big[2|S_h| - |\partial(E\setminus E_h)|\big] \\
&\le \mathscr{E}'(t,E) + \sigma_{\rm LV}\big[2|S_h| - C|E\setminus E_h|^{2/3}\big] \\
&= \mathscr{E}'(t,E) - C\dot{a} - Ca^{2/3}\,,
\end{aligned}$$

where the last inequality is obtained by applying the isoperimetric inequality to the set $E \setminus E_h$. Using the previous estimate and (5.5) we get

$$\begin{aligned}
\mathscr{E}'(t,\tilde{E}_h) &\le e^{C\lambda}\mathscr{E}'(t,E_h) \\
&\le e^{C\lambda}\mathscr{E}'(t,E) - Ce^{C\lambda}\dot{a} - Ce^{C\lambda}a^{2/3} \\
&\le \mathscr{E}'(t,E) + Ca - C\dot{a} - Ca^{2/3}\,,
\end{aligned} \tag{5.21}$$

where the last inequality was obtained using the following estimates, derived from statement (i) and (5.20): $\mathscr{E}'(t,E) \le C$, $1 \le e^{C\lambda} \le C$, and

$$e^{C\lambda} = (1 - a/|E|)^{-C} \le 1 + Ca\,.$$

The minimality of E implies $\mathscr{E}'(t,E) \le \mathscr{E}'(t,\tilde{E}_h)$, thus (5.21) yields

$$\dot{a} \le Ca - Ca^{2/3} \quad \text{a.e. in } (h_2, +\infty).$$

Now, whatever the constants C are in this equation,[26] by (5.18) there exist constants C and h_3, with $h_3 \ge h_2$, such that the right-hand side is smaller than $-Ca^{2/3}$ for every $h \ge h_3$, and therefore

$$\dot{a} \le -Ca^{2/3} \quad \text{a.e. in } (h_3, +\infty). \tag{5.22}$$

We conclude the proof of statement (ii) by a standard argument. Let $\bar{h}$ be the supremum of all h such that $a(h) > 0$; then (5.22) implies

$$-C \ge \frac{1}{3}a^{-2/3}\dot{a} \quad \text{a.e. in } (h_3, \bar{h}).$$

Integrating this inequality from h_3 to $\bar{h}$ we finally get

$$C(h_3 - \bar{h}) \ge a(\bar{h})^{1/3} - a(h_3)^{1/3} \ge -a(h_3)^{1/3} \ge -|U|^{1/3}$$

(use that $0 \le a(\bar{h}) \le a(h_3) \le |U|$), which implies $\bar{h} \le h_0 := h_3 + \frac{1}{C}|U|^{1/3}$, concluding the proof of statement (ii).

Statement (iii) follows immediately from the bounds on the set E and on the area $|\Sigma^f|$, and from formula (3.7). □

The next two statements concern the existence and convergence of minimizers of energies of capillary type under a prescribed-volume constraint. More precisely, given $\rho : \Omega \to [0,\infty)$ and $\sigma : S \to [-1,1]$, where ρ satisfies the growth condition (3.3) and S is the bottom of the container (see §3.1), we consider the following variant of the capillary energy $\mathscr{E}$:

$$\mathscr{F}(E) := \sigma_{\rm LV}\underbrace{\Big[|\Sigma^f| - \int_{\Sigma^c}\sigma\Big]}_{\mathscr{F}'(E)} + \underbrace{\int_E \rho(x)\,dx}_{V(E)}\,. \tag{5.23}$$

5.5. Proposition. *For every real number $v \ge 0$, the functional $\mathscr{F}$ admits a minimizer in the class $\mathscr{Y}_v$ of all sets in $\mathscr{Y}$ with volume v.*

PROOF. The existence of a minimizer is obtained as usual by showing that the energy $\mathscr{F}$ is lower semicontinuous on $\mathscr{Y}$, and that every sequence in $\mathscr{Y}_v$ with uniformly bounded energy admits a limit point in $\mathscr{Y}_v$ (compactness).

Step 1: compactness. Consider a sequence of sets E_n in $\mathscr{Y}_v$ such that $\mathscr{F}(E_n)$ is uniformly bounded. Then the perimeters $|\partial E_n| = |\Sigma^f_n| + |\Sigma^c_n|$ are uniformly bounded, and the Sobolev embedding for the space $BV(\mathbb{R}^3)$ implies that the characteristic functions 1_{E_n} converge up to subsequence in $L^1_{\rm loc}(\mathbb{R}^3)$, and clearly the limit must be the characteristic function of some set E_∞ contained in Ω, with locally finite volume and finite perimeter. Moreover, the upper bound on the energy and assumption (3.3) imply an upper bound on

$$\int_{E_n} |x|\,dx\,,$$

[26] Remember that they are not necessarily equal.

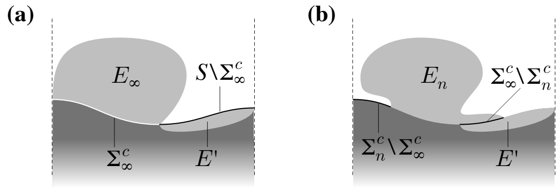

FIGURE 5. The sets E_∞, E' and E_n in the proof of Proposition 5.5.

which, by a standard argument, implies that E_n converge to E_∞ in $\mathscr{Y}$. Finally, convergence in $\mathscr{Y}$ implies the convergence of volume, and therefore E_∞ belongs to $\mathscr{Y}_v$.

Step 2: semicontinuity. Since $V(E)$ is clearly lower semicontinuous in E, it is enough to prove that also $\mathscr{F}'(E)$ is lower semicontinuous in E, namely that $\liminf \mathscr{F}'(E_n) \geq \mathscr{F}'(E_\infty)$ for every sequence of sets E_n which converge to E_∞ in $\mathscr{Y}$. Using Lemma 5.2 we construct a finite perimeter set E' in $\mathbb{R}^3$ with the following properties: (a) E' is contained in $\mathbb{R}^3 \setminus \Omega$; (b) the intersection of the essential boundary $\partial E'$ and the boundary of the container Ω agrees with $S \setminus \Sigma^c_\infty$ (see Fig. 5a).

It is then easy to check that (see Fig. 5b)

$$|\partial(E_n \cup E')| - |\partial(E_\infty \cup E')| = |\Sigma^f_n| - |\Sigma^f_\infty| - |\Sigma^c_\infty \triangle \Sigma^c_n|$$

and therefore, taking into account that $|\sigma| \leq 1$,

$$\begin{aligned} \mathscr{F}'(E_n) - \mathscr{F}'(E_\infty) &= |\Sigma^f_n| - |\Sigma^f_\infty| - \int_{\Sigma^c_n \setminus \Sigma^c_\infty} \sigma + \int_{\Sigma^c_\infty \setminus \Sigma^c_n} \sigma \\ &\geq |\partial(E_n \cup E')| - |\partial(E_\infty \cup E')| \,. \end{aligned} \tag{5.24}$$

We then conclude using the lower semicontinuity in E of the perimeter $|\partial E|$. $\square$

We consider now sequences of functions $\sigma_n : S \to [-1,1]$ and $\rho_n : \Omega \to [0,+\infty)$ and assume that all σ_n satisfy $|\sigma_n| \leq 1-\delta$ a.e. for some fixed $\delta > 0$ and converge in the weak* topology of $L^\infty(S)$ to some limit σ_∞, and that all ρ_n satisfy the growth condition (3.3) and converge uniformly on compact sets to some limit ρ_∞. Then we define the functionals $\mathscr{F}_n$, $\mathscr{F}'_n$ and V_n as in (5.23), with σ and ρ replaced by σ_n and ρ_n.

We also consider a sequence of non-negative real numbers v_n converging to some v_∞, and denote by $\mathscr{Y}_n$ the class of all sets E in $\mathscr{Y}$ with volume $|E| = v_n$.

5.6. Theorem. *For every integer n, let E_n be a minimizer of $\mathscr{F}_n$ on $\mathscr{Y}_n$. Then, upon extraction of a suitable subsequence (not relabelled), the following statements hold:*

(i) *the sets E_n converge in $\mathscr{Y}$ to a set $E_\infty \in \mathscr{Y}_\infty$;*
(ii) *E_∞ minimizes $\mathscr{F}_\infty$ on $\mathscr{Y}_\infty$, and $\mathscr{F}_n(E_n)$ converge to $\mathscr{F}_\infty(E_\infty)$;*
(iii) *the contact surfaces Σ^c_n converge to Σ^c_∞ in $\mathscr{X}$, and $|\Sigma^c_n| \to |\Sigma^c_\infty|$;*

(iv) *$V_n(E_n) \to V(E_\infty)$ and $|\Sigma^f_n| \to |\Sigma^f_\infty|$.*

PROOF. Statements (i) and (ii) are direct consequences of the fact that the restrictions of the functionals $\mathscr{F}_n$ to $\mathscr{Y}_n$ are equicoercive and Γ-converge to the restriction of $\mathscr{F}_\infty$ to $\mathscr{Y}_\infty$.[27] The proof of this fact is divided in Steps 1-3.

Step 1: (equicoercivity) every sequence of sets $E_n \in \mathscr{Y}_n$ with uniformly bounded energies $\mathscr{F}_n(E_n)$ admits a subsequence which converges in $\mathscr{Y}$ to a set $E_\infty \in \mathscr{Y}_\infty$. The proof is the same as Step 1 in the proof of Proposition 5.5.

Step 2: (lower-bound inequality) for every sequence of sets E_n which converge to E_∞ in $\mathscr{Y}$ there holds $\liminf \mathscr{F}_n(E_n) \geq \mathscr{F}_\infty(E_\infty)$. Indeed, since $\liminf V_n(E_n) \geq V_\infty(E_\infty)$ by Fatou's lemma, it only remains to show that $\liminf \mathscr{F}'_n(E_n) \geq \mathscr{F}'_\infty(E_\infty)$. The proof is a modification of Step 2 in the proof of Proposition 5.5: having defined E' in the same way, we obtain that (see (5.24))

$$\begin{aligned} &\mathscr{F}'_n(E_n) - \mathscr{F}'_\infty(E_\infty) \\ &\qquad = |\Sigma^f_n| - |\Sigma^f_\infty| - \int_{\Sigma^c_n \setminus \Sigma^c_\infty} \sigma_n + \int_{\Sigma^c_\infty \setminus \Sigma^c_n} \sigma_n + \int_{\Sigma^c_\infty} \sigma_\infty - \sigma_n \\ &\qquad \geq |\partial(E_n \cup E')| - |\partial(E_\infty \cup E')| + \int_{\Sigma^c_\infty} \sigma_\infty - \sigma_n \,. \end{aligned} \tag{5.25}$$

We conclude the proof using the semicontinuity in E of the perimeter $|\partial E|$ and the fact that the integral in the last line of formula (5.25) converges to 0 because σ_n converges to σ_∞ in the weak* topology of $L^\infty(S)$.

Step 3: (upper-bound inequality) every set $E_\infty \in \mathscr{Y}_\infty$ can be approximated by a sequence of sets $E_n \in \mathscr{Y}_n$ so that $\mathscr{F}_n(E_n)$ converges to $\mathscr{F}_\infty(E_\infty)$. Indeed the sets E_n can be obtained by perturbing E_∞ away from S in order to meet the volume constraint $|E_n| = v_n$; for this purpose one can use for example the deformations Φ_λ defined in §3.3 and the estimates in Proposition 5.3 (we omit the details).

Step 4: proof of statement (iii). We already know that for an arbitrary sequence E_n converging to E_∞ in $\mathscr{Y}$ there holds

$$\liminf \mathscr{F}'_n(E_n) \geq \mathscr{F}'_\infty(E_\infty) \quad \text{and} \quad \liminf V_n(E_n) \geq V_\infty(E_\infty)\,.$$

It follows immediately that if $\mathscr{F}_n(E_n)$ converges to $\mathscr{F}_\infty(E_\infty)$—which is the case when the sets E_n minimize $\mathscr{F}_n$ on $\mathscr{Y}_n$—then $V_n(E_n)$ converges to $V_\infty(E_\infty)$ and $\mathscr{F}'_n(E_n)$ converges to $\mathscr{F}'_\infty(E_\infty)$.

On the other hand, using the fact that $|\sigma_n| \leq 1-\delta$ for every n, we can sharpen inequality (5.25) as follows:

$$\begin{aligned} &\mathscr{F}'_n(E_n) - \mathscr{F}'_\infty(E_\infty) \\ &\qquad \geq |\partial(E_n \cup E')| - |\partial(E_\infty \cup E')| + \delta|\Sigma^c_n \triangle \Sigma^c_\infty| + \int_{\Sigma^c_\infty} \sigma_\infty - \sigma_n \,, \end{aligned}$$

[27] For the definition and basic properties of Γ-convergence, see for instance [1]; for a more detailed treatment see [5, 9].

and passing to the limit we get

$$0 \geq \delta \limsup_{n\to+\infty} |\Sigma_n^c \triangle \Sigma_\infty^c| \,,$$

which means that Σ_n^c converges to Σ_∞^c in $\mathscr{X}$. In particular $|\Sigma_n^c| \to |\Sigma_\infty^c|$.

Step 5: proof of statement (iv). We know from the previous step that $\mathscr{F}'_n(E_n)$ converges to $\mathscr{F}'_\infty(E_\infty)$, while the convergence of Σ_n^c to Σ_∞^c in $\mathscr{X}$ implies

$$\lim_{n\to+\infty} \int_{\Sigma_n^c} \sigma_n = \int_{\Sigma_\infty^c} \sigma_\infty \,.$$

Hence

$$\lim_{n\to+\infty} |\Sigma_n^f| = \lim_{n\to+\infty} \Big[\mathscr{F}'_n(E_n) + \int_{\Sigma_n^c} \sigma_n \Big] = \mathscr{F}'_\infty(E_\infty) + \int_{\Sigma_\infty^c} \sigma_\infty = |\Sigma_\infty^f| \,. \quad \square$$

Using Theorem 5.6 and the next lemma we can prove that limits of stable sets are stable (Proposition 5.8).

5.7. Lemma. *Let be given a set $E_0 \in \mathscr{Y}$ and denote by $\mathscr{Y}_0$ the class of all sets $E \in \mathscr{Y}$ with $|E| = |E_0|$. Then*

(i) *E_0 is stable at time t (see §3.7) if and only if it minimizes on $\mathscr{Y}_0$ the functional $\mathscr{F}_0$ given by formula (5.23) with ρ and σ replaced by $\rho_0(x) := \rho(t,x)$ and*

$$\sigma_0(x) := \begin{cases} \cos\theta_{\rm rec} & \text{for } x \in \Sigma_0^c \\ \cos\theta_{\rm adv} & \text{for } x \in S \setminus \Sigma_0^c. \end{cases}$$

(ii) *E_0 is stable at time t if it minimizes on $\mathscr{Y}_0$ any functional $\mathscr{F}$ of the form (5.23) with $\rho(x) := \rho(t,x)$ and $\cos\theta_{\rm adv} \leq \sigma(x) \leq \cos\theta_{\rm rec}$.*

PROOF. Statement (i) follows from the straightforward identity

$$\mathscr{E}(t,E) + \mathscr{D}(E,E_0) = \mathscr{F}_0(E) + \mu|\Sigma_0^c| \,.$$

Statement (ii) is an immediate consequence of statement (i) and the following claim: if E_0 minimizes $\mathscr{F}$ on $\mathscr{Y}_0$, then it minimizes $\mathscr{F}_0$ on $\mathscr{Y}_0$. To prove the claim it suffices to show that for every set E in $\mathscr{Y}_0$ there holds

$$\mathscr{F}_0(E) - \mathscr{F}_0(E_0) \geq \mathscr{F}(E) - \mathscr{F}(E_0) \,,$$

which in turn reduces to

$$\int_{\Sigma_0^c\setminus\Sigma^c} \sigma_0 - \int_{\Sigma^c\setminus\Sigma_0^c} \sigma_0 \geq \int_{\Sigma_0^c\setminus\Sigma^c} \sigma - \int_{\Sigma^c\setminus\Sigma_0^c} \sigma \,.$$

This is true because $\sigma_0 = \cos\theta_{\rm rec} \geq \sigma$ in Σ_0^c, and $\sigma_0 = \cos\theta_{\rm adv} \leq \sigma$ in $S \setminus \Sigma_0^c$. $\square$

5.8. Proposition. *For every positive integer n, let $E_n \in \mathscr{Y}$ be a set with volume $v_n \in [v_m, v_M]$. Assume that each E_n is stable at a certain time $t_n \in [0,T]$ (see §3.7), and that t_n and v_n converge to t_∞ and v_∞, respectively.*

Then, possibly passing to a subsequence, the sets E_n converge to some limit $E_\infty \in \mathscr{Y}$. Moreover

(i) *$\Sigma_n^c \to \Sigma_\infty^c$ in $\mathscr{X}$, $|\Sigma_n^f| \to |\Sigma_\infty^f|$, and $V(t_n,E_n) \to V(t_\infty,E_\infty)$;*

(ii) *$\mathscr{E}(t_n,E_n) \to \mathscr{E}(t_\infty,E_\infty)$;*

(iii) *E_∞ has volume v_∞ and is stable at time t_∞.*

PROOF. For every n, let $\mathscr{Y}_n$ be the class of all sets $E \in \mathscr{Y}$ with volume v_n, and define the functional $\mathscr{F}_n$ by formula (5.23) with σ and ρ replaced by

$$\sigma_n(x) := \begin{cases} \cos\theta_{\rm rec} & \text{for } x \in \Sigma_n^c \\ \cos\theta_{\rm adv} & \text{for } x \in S \setminus \Sigma_n^c, \end{cases}$$

and $\rho_n(x) := \rho(t_n,x)$ for every $x \in \Omega$.

By Lemma 5.7(i), the stability of E_n means that E_n minimizes $\mathscr{F}_n$ on $\mathscr{Y}_n$.

Since ρ is Lipschitz, the functions ρ_n converge uniformly on bounded sets to $\rho_\infty(x) := \rho(t_n,x)$. Moreover, passing to a suitable subsequence, we can assume that the functions σ_n converge weakly* in $L^\infty(S)$ to some function σ_∞. Therefore, if we denote by $\mathscr{F}_\infty$ the functional given by formula (5.23) with σ, ρ replaced by σ_∞, ρ_∞, then Theorem 5.6 implies that the sets E_n converge up to subsequences to a minimizer E_∞ of $\mathscr{F}_\infty$ on $\mathscr{Y}_\infty$, and that all convergences in statement (i) hold.

Statement (ii) follows immediately from (i).

Concerning statement (iii), the stability of E_∞ follows by Lemma 5.7(ii) and the fact that E_∞ minimizes $\mathscr{F}_\infty$ on $\mathscr{Y}_\infty$ (σ_∞ satisfies $\cos\theta_{\rm adv} \leq \sigma_\infty \leq \cos\theta_{\rm rec}$ because the functions σ_n do so). $\square$

Let $\mathscr{F}$ be the functional defined in (5.23); the next results show that, under suitable assumptions on the functions ρ, σ and g, every set $E \in \mathscr{Y}$ which minimizes $\mathscr{F}$ among all sets with the same volume must be a subgraph (in the sense of §3.2). The key tool for the proof is the following notion of volume-preserving rearrangement for sets in $\mathscr{Y}$.

5.9. Vertical rearrangement. For every $y \in U$, let R_y be the vertical line in $\mathbb{R}^3$ passing through the point $(y,0)$, namely the set of all (y,t) with $t \in \mathbb{R}$. Given a set $E \in \mathscr{Y}$, the *vertical rearrangement* of E is

$$\widehat{E} := \big\{ x = (y,t) \in U \times (0,+\infty) \,:\, g(y) < t < u(y) \big\} \,,$$

where

$$u(y) := g(y) + \mathscr{H}^1(E \cap R_y) \quad \text{for every } y \in U. \tag{5.26}$$

Thus $\widehat{E}$ is the unique (up to negligible subsets) subgraph in Ω with the property that the length of $\widehat{E} \cap R_y$ is equal to the length of $E \cap R_y$ for a.e. $y \in U$ (see Fig. 6). Therefore E is a subgraph if and only if it agrees with $\widehat{E}$.

5.10. Proposition. *Take $\mathscr{F}'$ and V as in (5.23). The following statements hold for every set $E \in \mathscr{Y}$:*

(i) *$\widehat{E}$ has finite perimeter in $\mathbb{R}^3$ and therefore it belongs to $\mathscr{Y}$;*

(ii) *E and $\widehat{E}$ have the same volume;*

(iii) *if $\rho(x)$ is increasing in the variable x_3 then $V(\widehat{E}) \leq V(E)$;*

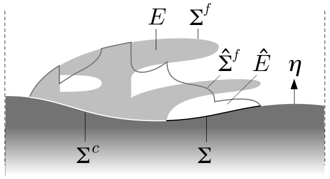

FIGURE 6. A set E and its vertical rearrangement $\widehat E$.

(iv) *if σ is non-negative and satisfies*

$$|\nabla g| \le \sigma/\sqrt{1-\sigma^2} \quad \text{a.e. on } U,^{28} \tag{5.27}$$

then $\mathscr{F}'(\widehat E) \le \mathscr{F}'(E)$; moreover the equality holds if and only if E is a subgraph.

SKETCH OF PROOF. A standard computation shows that the function u in (5.26) is Borel measurable and belongs to the space $BV(U)$, and this is enough to ensure that the set $\widehat E$ is Borel measurable and has finite perimeter in $\mathbb{R}^3$, and statement (i) is proved.

Statement (ii) follows from the definition of $\widehat E$ and Fubini's theorem.

Statement (iii) can be obtained using Fubini's theorem and the following one-dimensional result: for every set A with finite length contained in the half-line $\mathbb{R}^+ := [0,+\infty)$, let $\widehat A := (0,|A|)$; then for every increasing function r on $\mathbb{R}^+$ there holds

$$\int_{\widehat A} r(t)\,dt \le \int_A r(t)\,dt\,. \tag{5.28}$$

To prove this inequality it suffices to apply to the integral at the left-hand side of (5.28) the change of variable $t=\varphi(\tau)$ where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is the Lipschitz function defined by $\varphi(0):=0$ and $\dot\varphi := 1_A$.

Note that the same argument implies that

$$\int_E g(x)\,dx = \int_{\widehat E} g(x)\,dx \tag{5.29}$$

for every bounded function $g(x)$ which is constant with respect to x_3.

We now prove statement (iv). Given a set $E\in\mathscr{Y}$ we write $\Sigma := \widehat\Sigma^c \setminus \Sigma^c$, and denote by η the restriction to S of the inner normal of $\partial\Omega$, and by $\eta_h := (\eta_1,\eta_2,0)$ its horizontal component. Statement (iv) is a direct consequence of the following claim: *For every E there holds*

$$|\Sigma^f| \ge |\widehat\Sigma^f| - \int_\Sigma |\eta_h|\,, \tag{5.30}$$

and we have equality if and only if E agrees with $\widehat E$.

[28] The right-hand side of this inequality is set equal to $\pm\infty$ for $\sigma=\pm1$.

Indeed assumption (5.27) can be re-written as $|\eta_h|\le\sigma$ a.e. in S. Therefore (5.30) and the fact that $\widehat\Sigma^c = \Sigma^c\cup\Sigma$ (see Fig. 6) yield

$$\begin{aligned}\mathscr{F}'(\widehat E) = |\Sigma^f| - \int_{\Sigma^c}\sigma &\ge |\widehat\Sigma^f| - \Big(\int_{\Sigma^c}\sigma + \int_\Sigma |\eta_h|\Big)\\ &\ge |\widehat\Sigma^f| - \int_{\widehat\Sigma^c}\sigma = \mathscr{F}'(E)\,,\end{aligned}$$

and the inequality in the first line is an equality only if $E=\widehat E$.

It remains to prove the claim. To this end, we first establish the auxiliary formulas (5.31) and (5.32). Let $\phi(x)$ be a vector-field of class C^1 on $\mathbb{R}^3$ which is constant with respect to x_3 and assume that E is bounded. Then the divergence theorem and identity (5.29) yield

$$\int_{\Sigma^f}\phi\cdot\eta^f = \int_E \mathrm{div}\phi + \int_{\Sigma^c}\phi\cdot\eta = \int_{\widehat E}\mathrm{div}\phi + \int_{\widehat\Sigma^c}\phi\cdot\eta - \int_\Sigma \phi\cdot\eta\,,$$

and by applying once again the divergence theorem we obtain

$$\int_{\Sigma^f}\phi\cdot\eta^f = \int_{\widehat\Sigma^f}\phi\cdot\hat\eta^f - \int_\Sigma\phi\cdot\eta\,. \tag{5.31}$$

Then identity (5.31) can be extended by approximation to every set $E\in\mathscr{Y}$ and every bounded Borel vector-field ϕ on $\mathbb{R}^3$ which is constant with respect to x_3.

Let $\varphi(x)$ be a bounded function on $\mathbb{R}^3$ which is constant with respect to x_3. Then

$$\begin{aligned}\int_{\Sigma^f}\varphi|\eta_3^f| &= \int_U \#(\Sigma^f\cap R_y)\,\varphi(y)\,dy\\ &\ge \int_U \#(\widehat\Sigma^f\cap R_y)\,\varphi(y)\,dy = \int_{\widehat\Sigma^f}\varphi\hat\eta_3^f\,,\end{aligned} \tag{5.32}$$

where the first equality follows by applying the coarea formula to the vertical projection $p:\Sigma^f\to U$,[29] the last equality is obtained in the same way (note that $\hat\eta_3^f$ is non-negative a.e. on $\widehat\Sigma^f$), and the inequality follows by the fact that $\widehat\Sigma^f\cap R_y$ contains at most one point for a.e. $y\in U$, and is empty when $\Sigma^f\cap R_y$ is empty (see Fig. 6). This argument shows, in addition, that if φ is a.e. positive and the inequality in (5.32) is an equality, then $\Sigma^f\cap R_y$ contains at most one point for a.e. $y\in U$, that is, $E=\widehat E$.

Consider now a Borel vector-field $\phi(x)$ which is constant with respect to x_3 and satisfies $|\phi(x)|\le 1$ everywhere and $\phi(x)=\hat\eta^f(x)$ for a.e. $x\in\widehat\Sigma^f$ (such a vector-field exists because $\widehat E$ is a subgraph). Then, denoting by ϕ_h the horizontal

[29] Thus the Jacobian determinant of p is $|\eta_3^f|$.

component of ϕ,

$$
\begin{aligned}
|\Sigma^f| &\geq \int_{\Sigma^f} \phi_h \cdot \eta^f + \int_{\Sigma^f} \phi_3 |\eta_3^f| \\
&\geq \int_{\widehat{\Sigma}^f} \phi_h \cdot \hat{\eta}^f - \int_{\Sigma} \phi_h \cdot \eta + \int_{\widehat{\Sigma}^f} \phi_3 \hat{\eta}_3^f \\
&\geq \int_{\widehat{\Sigma}^f} \phi \cdot \hat{\eta}^f - \int_{\Sigma} \phi_h \cdot \eta \geq |\widehat{\Sigma}^f| - \int_{\Sigma} |\eta_h| \,,
\end{aligned}
$$

and (5.30) is proved (the second inequality follows by applying (5.31) with ϕ_h instead of ϕ, and (5.32) with ϕ_3 instead of φ, and the last inequality follows from the estimate $\phi_h \cdot \eta = \phi_h \cdot \eta_h \leq |\eta_h|$).

Moreover, if equality holds in (5.30), then equality must hold in particular in the second of the chain of inequalities above, and therefore also in (5.32) with φ replaced by ϕ_3. As pointed out above, this implies $E = \widehat{E}$ (note that ϕ_3 agrees with the third component of $\hat{\eta}^f$ and therefore is a.e. positive). □

The following statement is a straightforward consequence of Proposition 5.10.

5.11. Corollary. *Given a positive number v, let $\mathscr{Y}_v$ be the class of all sets $E \in \mathscr{Y}$ with volume $|E| = v$. If $\rho(x)$ is increasing in the variable x_3 and condition (5.27) holds, then every minimizer of $\mathscr{F}$ on $\mathscr{Y}_v$ is a subgraph.*

5.12. Remark. (i) Since $t/\sqrt{1-t^2}$ is increasing in t, condition (5.27) is verified whenever

$$
\operatorname{Lip}(g) \leq m/\sqrt{1-m^2}\,, \tag{5.33}
$$

where m is the essential infimum of σ.

(ii) An immediate consequence of Corollary 5.11 is the following: if the function $\rho(t,x)$ in the definition of the capillary energy $\mathscr{E}$ is increasing in the variable x_3 and (3.16) is verified, then every set $E \in \mathscr{Y}$ which is stable (in the sense of §3.7) is a subgraph. Indeed "stable" means that E minimizes the functional $\mathscr{F}$ with

$$
\sigma(x) := \begin{cases} \cos\theta_{\rm rec} & \text{if } x \in \Sigma^c \\ \cos\theta_{\rm adv} & \text{if } x \in S \setminus \Sigma^c, \end{cases}
$$

and for this choice of σ, inequality (5.33) reduces to (3.16).

(iii) Assumption (5.27) admits a nice geometric interpretation, which shows that it is in some sense natural (and optimal). Assume that E is sufficiently regular, and for every point of the contact line let ϑ be the angle between the vertical direction $e_3 := (0,0,1)$ and the normal η^f to the free surface: then $\vartheta \leq \alpha + \theta$ where α is the angle between e_3 and the inner normal η of the bottom of the container, and θ is the contact angle, that is, the angle between η and η^f. Now, if E minimizes $\mathscr{F}'$ in $\mathscr{Y}_v$ then θ must satisfy a suitable variant of Young's law, namely $\cos\theta = \sigma$, and therefore

$$
\begin{aligned}
\cos\vartheta \geq \cos(\alpha+\theta) &= \cos\alpha\,\sin\theta\,(\cot\theta - \tan\alpha) \\
&= \cos\alpha\,\sin\theta\,\big[\sigma/\sqrt{1-\sigma^2} - |\nabla g|\big]\,.
\end{aligned}
$$

Thus (5.27) is the weakest condition we can impose on σ and g to ensure that $\cos\vartheta \geq 0$ (that is, $\vartheta \leq \pi/2$) at every point of the contact line, which is clearly necessary if E were to be a subgraph. And we have just shown above that it is also sufficient.

The last result of this section provides an effective way to check if a given map $t \mapsto E(t)$ is a solution, and will be used in the examples in Section 6. We take the function w as in §3.7, and for every $t \in [0,T]$ we denote by $\mathscr{Y}_{w(t)}$ the class of all sets $E \in \mathscr{Y}$ with volume $|E| = w(t)$.

5.13. Proposition. *Consider a map $t \mapsto E(t)$ which is left-continuous at every $t \in [0,T]$ and satisfies $E(t) \in \mathscr{Y}_{w(t)}$. Assume moreover that*

(i) *$E(t) \in \operatorname{argmin}\big\{\mathscr{E}(t,E) + \mathscr{D}(E(0),E) : E \in \mathscr{Y}_{w(t)}\big\}$ for every $t \in [0,T]$;*

(ii) *$\operatorname{Diss}(E(t);[0,T]) = \mathscr{D}(E(0),E(T))$.*

Then the map $E(t)$ is a solution in the sense of §3.7.

PROOF. We first prove that for every t and t' such that $0 \leq t' \leq t \leq T$ there holds

$$
E(t) \in \operatorname{argmin}\big\{\mathscr{E}(t,E) + \mathscr{D}(E(t'),E) : E \in \mathscr{Y}_{w(t)}\big\}\,. \tag{5.34}
$$

Take indeed an arbitrary set $E \in \mathscr{Y}_{w(t)}$. Then assumption (i) implies

$$
\mathscr{E}(t,E(t)) + \mathscr{D}(E(0),E(t)) \leq \mathscr{E}(t,E) + \mathscr{D}(E(0),E)\,,
$$

while assumption (ii) implies

$$
\mathscr{D}(E(0),E(t)) = \mathscr{D}(E(0),E(t')) + \mathscr{D}(E(t'),E(t))\,,
$$

and therefore

$$
\begin{aligned}
\mathscr{E}(t,E(t)) + \mathscr{D}(E(t'),E(t)) &\leq \mathscr{E}(t,E(t)) + \mathscr{D}(E(0),E(t)) - \mathscr{D}(E(0),E(t')) \\
&\leq \mathscr{E}(t,E) + \mathscr{D}(E(0),E) - \mathscr{D}(E(0),E(t')) \\
&\leq \mathscr{E}(t,E) + \mathscr{D}(E(t'),E)\,,
\end{aligned}
$$

where the last inequality follows by the triangle inequality for the distance $\mathscr{D}$.

Now, for every $\delta > 0$ and every $t \in [0,T]$, set $E_\delta(t) := E(t_\delta)$ where t_δ is the supremum of all $n\delta \leq t$ with n integer. Then (5.34) implies that the map E_δ is one of the discretized solutions with initial configuration $E(0)$ defined in §4.1. Moreover $E_\delta(t)$ converge to $E(t)$ as $\delta \to 0$ for every t because of the continuity assumptions on $E(t)$, and therefore statement (ix) of Theorem 4.3 implies that $E(t)$ is a solution. □

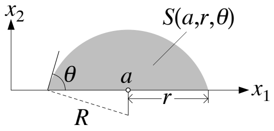

FIGURE 7. The circular segment $S(a,r,\theta)$.

6. EXAMPLES

In this section we present a few concrete examples of solutions, illustrating strengths and weaknesses of the approach discussed in the paper. It should be underlined that the proofs of the various claims concerning these examples are only briefly sketched.

Throughout this section, to prove that a certain map $t \mapsto E(t)$ is a solution in the sense of §3.7, we use Proposition 5.13. Note that this statement gives slightly more, namely that $E(t)$ can be obtained as limit of the discretized solutions defined in §4.1.

6.1. Notation. For the sake of simplicity, the examples in this section are given in two space dimensions only,[30] even though some of them (indeed all except Example 6.6) can be easily extended to three dimensions. We write $x = (x_1, x_2)$ for points in $\mathbb{R}^2$, and refer to the x_2-axis as the "vertical axis".

We also assume that there is no bulk contribution V in the capillary energy, that is, the potential ρ vanishes. It is then convenient to renormalize the capillary energy setting $\sigma_{\rm LV} = 1$; that is,

$$\mathscr{E}(E) = |\Sigma^f| - \cos\theta_Y |\Sigma^c| .$$

Given a set E_0, it is easily checked (see the proof of statement (i) of Lemma 5.7) that minimizing $\mathscr{E}(E) + \mathscr{D}(E, E_0)$ on the class $\mathscr{Y}_v$ of all sets $E \in \mathscr{Y}$ with area $|E| = v$ is equivalent to minimizing

$$\mathscr{F}_0(E) := |\Sigma^f| - \int_{\Sigma^c} \sigma_0 \ , \quad \sigma_0(x) := \begin{cases} \cos\theta_{\rm rec} & \text{for } x \in \Sigma_0^c \\ \cos\theta_{\rm adv} & \text{for } x \in S \setminus \Sigma_0^c. \end{cases} \tag{6.1}$$

6.2. Circular segments. For every $a \in \mathbb{R}$, $r \in (0, +\infty)$ and $\theta \in (0, \pi)$, we denote by $S(a, r, \theta)$ the (open) circular segment described in Fig. 7. In the following the term "circular segment" always refers to circular segments of this type.

If we consider as container the upper half-plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$, the contact line of $S = S(a, r, \theta)$ has length $2r$ and the contact angle is equal to θ at both contact points. Thus we refer to S as the circular segment centered at a with contact length $2r$ and contact angle θ.

[30] We therefore use the two dimensional terminology and write "area", "free line", "contact points" instead of "volume", "free surface", "contact line". The notation, however, remains unchanged.

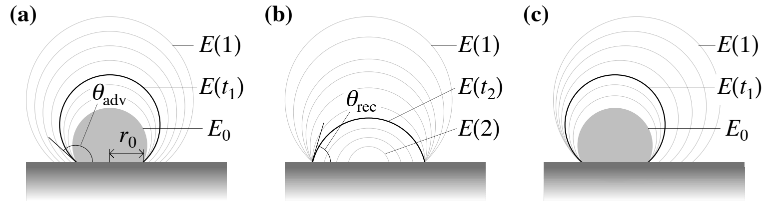

FIGURE 8. The solutions in Example 6.3 (left and center) and Example 6.4 (right).

The free line of S is an arc of radius $R = r/\sin\theta$ and length $|\Sigma^f| = r\, f(\theta)$ while the area is $|S| = r^2 g(\theta)$, where

$$f(\theta) := \frac{2\theta}{\sin\theta} \quad \text{and} \quad g(\theta) := \frac{\theta - \sin\theta\,\cos\theta}{\sin^2\theta} . \tag{6.2}$$

6.3. Example. Let the container Ω be the upper half-plane as before. We choose an initial configuration of the form $E_0 = S(0, r_0, \theta_0)$ with $r_0 \in (0, +\infty)$ and $\theta_0 \in (\theta_{\rm rec}, \theta_{\rm adv})$, and consider the area constraint $|E(t)| = w(t)$ where w is a function of class C^1 on the time interval $[0, 2]$ satisfying the following assumptions:

(a) w is strictly increasing in $[0, 1]$ and strictly decreasing in $[1, 2]$;
(b) $w(0) = |E_0| = r_0^2 g(\theta_0)$;
(c) $w(1) = r_1^2 g(\theta_{\rm adv})$ for some r_1 with $r_1 > r_0$;
(d) $w(2) = r_2^2 g(\theta_{\rm rec})$ for some r_2 with $r_2 < r_1$.

Note that the assumptions (c) and (d) and the continuity of w imply that there exist $t_1 \in (0, 1)$ and $t_2 \in (1, 2)$ such that $w(t_1) = r_0^2 g(\theta_{\rm adv})$ and $w(t_2) = r_1^2 g(\theta_{\rm rec})$.

A solution with initial configuration E_0 is given as follows: $E(t)$ is a circular segment centered at 0 for all t; as t increases from 0 to t_1 the contact line remains fixed while the contact angle increases from θ_0 to the largest admissible value $\theta_{\rm adv}$; as t increases from t_1 to 1 the contact line grows while the contact angle remains fixed ($= \theta_{\rm adv}$), see Fig. 8a; as t increases from 1 to t_2 the contact line remains fixed while the contact angle decreases from $\theta_{\rm adv}$ to the smallest admissible value $\theta_{\rm rec}$; as t increases from t_2 to 2 the contact line shrinks while the contact angle remains fixed ($= \theta_{\rm rec}$), see Fig. 8b.

In other words $E(t) = S(0, r(t), \theta(t))$ where[31]

$$r(t) := \begin{cases} r_0 \\ (w(t)/g(\theta_{\rm adv}))^{1/2} \\ r_1 \\ (w(t)/g(\theta_{\rm rec}))^{1/2} \end{cases} \qquad \theta(t) := \begin{cases} g^{-1}(w(t)/r_0^2) & \text{for } 0 \le t \le t_1 \\ \theta_{\rm adv} & \text{for } t_1 < t \le 1 \\ g^{-1}(w(t)/r_1^2) & \text{for } 1 < t \le t_2 \\ \theta_{\rm rec} & \text{for } t_2 < t \le 2. \end{cases} \tag{6.3}$$

[31] The function g is smooth and has strictly positive derivative on the interval $[0, \pi)$; here and in the following we denote by g^{-1} the inverse of its restriction to this interval.

PROOF. It is easy to check that $E(t)$ satisfies the flow rules in §2.10, but unfortunately these conditions are necessary but not sufficient to be a solution. To prove that $E(t)$ is a solution on the time interval $[0,1]$ we will use instead Proposition 5.13. In the same way it can be proved (we omit the details) that $E(t)$ is a solution also on the time interval $[1,2]$, and then the proof is completed by Remark 3.8(iii).

Let us check that the assumptions of Proposition 5.13 are verified. The continuity of $E(t)$ with respect to t is obvious. Assumption (ii) can be re-written as

$$\mathrm{Var}(\Sigma^c(t);[0,1]) = |\Sigma^c(0) \triangle \Sigma^c(1)|$$

and is clearly verified whenever the set-valued map $\Sigma^c(t)$ is either increasing or decreasing on the interval $[0,1]$, as in this case.

To check assumption (i) we have to show that $E(t)$ minimizes the functional $\mathscr{F}_0$ defined in (6.1) on the class $\mathscr{Y}_{w(t)}$ for every t (see §6.1). This is achieved by showing that if E is a minimizer of $\mathscr{F}_0$ in $\mathscr{Y}_{w(t)}$ and E' is the Steiner symmetrization of E with respect to the vertical axis (see for instance [6, §9.2]), then

(a) E' is a minimizer of $\mathscr{F}_0$ on $\mathscr{Y}_{w(t)}$;
(b) E' agrees with $E(t)$.

Claim (a) follows by the fact that Steiner symmetrization preserves the area and does not increase the value of $\mathscr{F}_0$, the latter assertion being an (almost) straightforward consequence of well-known properties of Steiner symmetrization.

To prove (b) we first show that E' is a circular segment (centered at 0). Since E' minimizes $\mathscr{F}_0$, its free line is a smooth curve with constant curvature. Hence the connected components of E' are either circular segments or discs, all with equal radii and centers on the vertical axis. In particular, at most one of the connected components is a circular segment. Assuming by contradiction that there is one which is a disc, we can move it till it touches the wall of the container, obtaining, therefore, a new minimizer which does not satisfy Young's law, and this is impossible.

Thus E' is of the form $S(0,r,\theta)$, and then the stability conditions associated with the minimization of $\mathscr{F}_0$ yield the following: if $r > r_0$ then $\theta = \theta_{\mathrm{adv}}$; if $r < r_0$ then $\theta = \theta_{\mathrm{rec}}$. Using these implications and the fact that E' and $E(t)$ have the same area one readily obtains that they are the same set. □

6.4. Example. Take Ω, E_0, and $w(t)$ as in the previous example; we want to show that there are infinitely many solutions with initial configuration E_0 besides the one given there.

We focus for simplicity on the time interval $[0,1]$, and for every t we consider a horizontal translation $\tilde{E}(t)$ of $E(t)$ chosen in such way that the map $\tilde{\Sigma}^c(t)$ is increasing. In other words $\tilde{\Sigma}^c(t) = S(a(t), r(t), \theta(t))$ where $r(t)$ and $\theta(t)$ are taken as in (6.3) and $a(t)$ satisfies

$$|a(t') - a(t)| \le r(t') - r(t) \quad \text{for } 0 \le t \le t' \le 1,$$

or, equivalently, a is a Lipschitz function on $[0,1]$ which satisfies $|\dot{a}| \le \dot{r}$ a.e.[32]

Then $\tilde{E}(t)$ is a solution with initial configuration E_0.

PROOF. We apply Proposition 5.13 as in the proof of Example 6.3. To this end, it suffices to notice that the map $\tilde{\Sigma}^c(t)$ is increasing by construction, and $\tilde{E}(t)$ minimizes $\mathscr{F}_0$ in $\mathscr{Y}_{w(t)}$ for every t because it is obtained from $E(t)$ by a horizontal translation that preserves the value of $\mathscr{F}_0$. □

6.5. Remark. (i) It seems plausible that every solution with initial configuration E_0 must agree with $E(t)$ up to time t_1. This is certainly true for all solutions obtained as limit of discretized solutions via Theorem 4.3.

(ii) The setting and the initial configuration E_0 described in Example 6.4 present an evident axial symmetry which is not preserved by the solutions given there. In this case, the lack of symmetry of solutions is related to the fact that distance in space of contact surfaces $\mathscr{X}$ is defined in terms of the L^1 norm, which is convex but (very much) not strictly convex.

(iii) Examples 6.3 and 6.4 suggest the following question: are there axially symmetric settings and initial configurations such that *no* solution is symmetric? We can easily force symmetry-breaking by adding a bulk contribution V to the capillary energy, with V given by a suitably chosen *symmetric and time-dependent* potential ρ.

6.6. Example. In this example the parameters related to the capillary energy and the dissipation must be carefully chosen. We require that[33]

$$\theta_{\mathrm{adv}} = \frac{\pi}{2} \quad \text{and} \quad \theta_{\mathrm{rec}} \in (\theta_*, \theta^*)\,, \tag{6.4}$$

where the angles $\theta_*, \theta^* \in (0, \pi/2)$ are defined by the relations

$$\cos\theta_* = \frac{\pi}{4}(2 - \sqrt{2}) \quad \text{and} \quad \theta^* - \sin\theta^* \cos\theta^* = \frac{\pi}{4} \tag{6.5}$$

(thus $\theta_* = 1.093 \pm 10^{-3}$ and $\theta^* = 1.155 \pm 10^{-3}$).

The container is the upward half-band $\Omega := (-s,s) \times (0,+\infty)$. Through this paragraph we denote by $Q(r)$ the open quarter-disc contained in Ω with center in the lower left corner of the container, $(s,0)$, and radius r (see Fig. 9b).

We consider the initial configuration

$$E_0 := S(0, r_0, \theta_{\mathrm{rec}}) \quad \text{with } r_0 < s/3,$$

and a function $w(t)$ on the time interval $[0,1]$ such that

(a) w is of class C^1 and strictly increasing;
(b) $w(0) = |E_0| = r_0^2 g(\theta_{\mathrm{rec}})$;
(c) $w(1) = \pi r_0^2/2$.

[32] This condition is verified by infinitely many functions a because $\dot{r} > 0$ for a.e. $t \in (t_1, 1)$.

[33] Formula (2.19) shows that every choice of θ_{rec} and θ_{adv} such that $0 < \theta_{\mathrm{rec}} < \theta_{\mathrm{adv}} < \pi$ is compatible with a suitable choice of the physical parameters $\mu, \sigma_{LV}, \ldots$

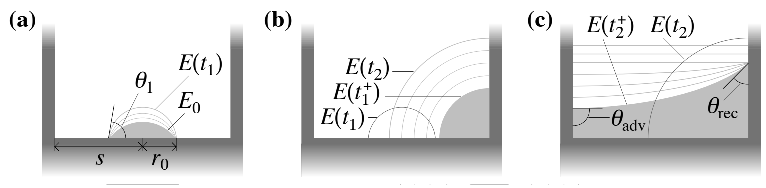

FIGURE 9. The solution $E(t)$ in Example 6.6.

A solution with initial configuration E_0 is as follows: as t increases from 0 to a certain critical time $t_1 \in (0,1)$, the set $E(t)$ is a circular segment centered at 0 with the same contact line as E_0, and contact angle increasing from $\theta_{\rm rec}$ to a certain $\theta_1 \in (\theta_{\rm rec}, \pi/2)$, see Fig. 9a;[34] at $t = t_1$ a discontinuity occurs, and, as t increases from t_1 to 1, the set $E(t)$ is a quarter-disc with increasing radius, see Fig. 9b. More precisely

$$E(t) = \begin{cases} S(0, r_0, \theta(t)) & \text{with } \theta(t) := g^{-1}(w(t)/r_0^2) \quad \text{for } 0 \le t \le t_1 \\ Q(r(t)) & \text{with } r(t) := 2\sqrt{w(t)/\pi} \quad \text{for } t_1 < t \le 1. \end{cases} \tag{6.6}$$

Remark. If we keep increasing $w(t)$ after the time $t = 1$, it seems plausible that the set $E(t)$ remains a quarter-disc for a while, and then another discontinuity occurs at a certain time t_2, after which $E(t)$ evolves as shown in Fig. 9c. We do not attempt a more precise description of this solution (nor a proof).

PROOF. We use Proposition 5.13 to prove that $E(t)$ is a solution in the time interval $[0,1]$ provided that t_1 is carefully chosen. The map $\Sigma^c(t)$ is indeed left-continuous, and it is easy to check that assumption (ii) of Proposition 5.13 is verified. To verify assumption (i) we must show that the set $E(t)$ minimizes $\mathscr{F}_0$ on $\mathscr{Y}_{w(t)}$ for every t, where $\mathscr{F}_0$ is defined as in (6.1), that is

$$\mathscr{F}_0(E) = |\Sigma^f| - \cos\theta_{\rm rec} |\Sigma^c \cap \Sigma_0^c| \tag{6.7}$$

(recall that $\theta_{\rm adv} = \pi/2$). The proof of this claim is divided into four steps.

Step 1. We recall a known fact: a set E which minimizes the quantity $|\Sigma^f|$ in the class $\mathscr{Y}_v$ is a quarter-disc of the form $Q(r)$, or its reflection with respect to the vertical axis, when $v < 4s^2/\pi$.

Let indeed E_i, $i = 1, \dots, n$, be the connected components of E and denote by v_i the corresponding area. Since the free line has constant curvature and the contact angle is $\pi/2$ at every contact point, we have the following possibilities: (a) E_i is a disc and $|\Sigma_i^f| = \sqrt{4\pi v_i}$; (b) E_i is a half-disc and $|\Sigma_i^f| = \sqrt{2\pi v_i}$; (c) E_i is a quarter-disc and $|\Sigma_i^f| = \sqrt{\pi v_i}$; (d) E_i is a rectangle of the form $(-s,s) \times (0,h)$ and $|\Sigma_i^f| = 2s$.

Thus $|\Sigma^f|$ is a concave function of the variables $v_1, \dots, v_n$ and the minimality of E implies that $v_1, \dots, v_n$ minimize f on the simplex T defined by the constraints $v_1 + \dots + v_n = v$ and $v_i > 0$ for every i. But, since f is *strictly* convex on T,

[34] The values of t_1 and θ_1 will be made more precise in the proof below.

this is impossible unless $n = 1$. Thus E has only one connected component, and then we can easily conclude by comparing the values of $|\Sigma^f|$ for the four configurations.

Step 2. Let E be a minimizer of $\mathscr{F}_0$ in $\mathscr{Y}_v$ with $r_0^2 g(\theta_{\rm rec}) \le v \le \pi r_0^2/2$. We claim that E is either a circular segment of the form $S(0, r_0, \theta)$ with $\theta_{\rm rec} \le \theta \le \pi/2$ or a quarter-disc of the form $Q(r)$, or the reflection of the latter with respect to the vertical axis.

Let indeed $\mathscr{G}$ be the class of all connected components of E whose contact line does not intersect Σ_0^c. Then (6.7) and Step 1 imply that $\mathscr{G}$ consists of just one element, which must be a quarter-disc because v is smaller than $\pi r_0^2/2$, which is smaller than $4s^2/\pi$ (recall that we assumed $r_0 < s/3$).

Let $\mathscr{G}'$ be the class of all connected components of E whose contact line intersects Σ_0^c. Because of Young's law, the contact line Σ^c of an element of $\mathscr{G}'$ must satisfy one of the following: (a) Σ^c strictly contains Σ_0^c, (b) Σ^c is strictly contained in Σ_0^c, (c) Σ^c agrees with Σ_0^c. Note that case (a) can be excluded because it would imply an area larger than $\pi r_0^2/2$, while case (b) can be excluded using a concavity argument similar to that used in Step 1. Hence it remains (c), and this implies that $\mathscr{G}'$ contains at most one component of the form $S(0, r_0, \theta)$, and $\theta_{\rm rec} \le \theta \le \pi/2$ because of Young's law.

It remains to exclude the case that both $\mathscr{G}$ and $\mathscr{G}'$ are not empty. Were this not true, E would be of the form

$$E = S(0, r_0, \theta) \cup Q(r)$$

for some $\theta \in [\theta_{\rm rec}, \pi/2]$ and $r = r_0/\sin\theta$ (because the free line of E has constant curvature). Hence

$$|E| = r_0^2 \left[g(\theta) + \frac{\pi}{4\sin^2\theta} \right] ,$$

but it can be shown that the right-hand side of this equality is larger than $\pi r_0^2/2$ for every θ, contradicting the assumption $|E| = v \le \pi r_0^2/2$.

Step 3. Let us compare the values of $\mathscr{F}_0$ for the circular segment $S = S(0, r_0, \theta)$ with $\theta \in [\theta_{\rm rec}, \pi/2]$ and for the quarter-disc $Q = Q(r)$, where r is taken so that the area of Q is the same as that of S, that is $\pi r^2/4 = r_0^2 g(\theta)$. By (6.7) we have

$$\mathscr{F}_0(S) = r_0 (f(\theta) - 2\cos\theta_{\rm rec}) \quad \text{and} \quad \mathscr{F}_0(Q) = r_0 \sqrt{\pi\, g(\theta)} ,$$

and therefore $\mathscr{F}_0(S) < \mathscr{F}_0(Q)$ if and only if

$$h(\theta) < 0 \quad \text{where } h(\theta) := (f(\theta) - 2\cos\theta_{\rm rec})^2 - \pi\, g(\theta) .$$

Now, the upper and lower bounds on $\theta_{\rm rec}$ in (6.4) are equivalent to $h(\theta_{\rm rec}) < 0$ and $h(\pi/2) > 0$, respectively, and since the function h is strictly convex on the interval $[\theta_{\rm rec}, \pi/2]$, there exists a critical angle θ_1 in the interior of this interval such that $h(\theta) < 0$ if and only if $\theta < \theta_1$.

Taking into account the previous step and the fact that the function g is strictly increasing, we conclude that a minimizer E of $\mathscr{F}_0$ on the class $\mathscr{Y}_v$ is a circular segment of the form $S(0, r_0, \theta)$ when v satisfies $r_0^2 g(\theta_{\rm rec}) \le v < r_0^2 g(\theta_1)$, and is a

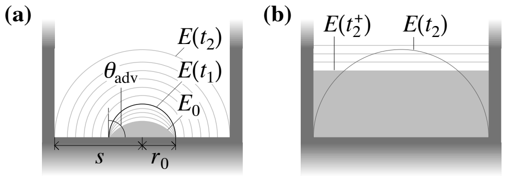

FIGURE 10. A physically plausible solution for Example 6.6.

quarter-disc of the form $Q(r)$, or its reflection, when $r_0^2 g(\theta_1) < v \leq \pi r_0^2/2$; clearly both solutions coexist for $v = r_0^2 g(\theta_1)$.

Step 4. From Step 3 we immediately infer that $E(t)$ is a minimizer of $\mathscr{F}_0$ on the class $\mathscr{Y}_{w(t)}$ for every $t \in [0,1]$ provided that we have chosen t_1 so that $w(t_1) = r_0^2 g(\theta_1)$. □

6.7. Remark. (i) The solution $E(t)$ in the previous example does not preserve the axial symmetry of the setting and of the initial configuration E_0. Therefore a different solution with initial configuration E_0 can be obtained by reflecting $E(t)$ with respect to the vertical axis. It seems plausible that these are the only solutions with initial configuration E_0, at least for $0 \leq t \leq 1$.[35]

(ii) Choose the setting in Examples 6.3 and 6.6 so that we have the same initial configuration and the same prescribed area $w(t)$ at every time, and only the walls of the containers are different far away from the initial contact region Σ_0^c. Then the solutions described in these examples agree up to the moment when the difference between the geometry of the containers starts playing a role. However, this happens when the drop is still far away from the vertical part of the wall, which seems rather unphysical.

A more plausible behaviour would be the one sketched in Fig. 10: at first the contact line is fixed while the contact angle grows until it reaches the largest admissible value $\theta_{\rm adv} = \pi/2$ (at $t = t_1$ with t_1 s.t. $w(t_1) = r_0^2 g(\pi/2)$). Then the contact angle remains constant ($= \pi/2$) and the contact line grows until the contact points reach the vertical part of the wall (at $t = t_2$ with t_2 s.t. $w(t_2) = \pi s^2/2$). At that moment a discontinuity occurs, and the drop becomes a rectangle and remains a rectangle at later times.

This solution is only *locally stable*, in the sense that $E(t)$ minimizes $\mathscr{E}(t,E) + \mathscr{D}(E, E(t))$ among all sets E with $|E| = w(t)$ in a neighbourhood of $E(t)$, but not necessarily among all sets E in $\mathscr{Y}_{w(t)}$.

Moreover, this solution violates the energy-dissipation balance. More precisely, the energy instantly dissipated at the discontinuity t_2 is *strictly larger* than the one prescribed by the energy-dissipation balance: $(\pi - 2)s + \frac{\pi}{2} s \cos\theta_y$ instead of $\frac{\pi}{2} s \cos\theta_y$. This extra dissipation is associated with time scales much faster than

[35] It is not difficult to show that every solution obtained as limit of discretized solutions is of this type; as usual the problem is that we cannot exclude the existence of other solutions.

those characterizing the slowly changing $t \mapsto w(t)$, which are not accounted for by the quasistatic dissipation rate $\mathscr{R}$.

6.8. Example. We conclude this section with an example of purely mathematical interest. Let the container Ω be the upper half-plane as in Example 6.3, and choose $\theta_{\rm rec}$ and $\theta_{\rm adv}$ so that

$$\theta_{\rm rec} < \pi/2 < \theta_{\rm adv}\,, \quad g(\theta_{\rm adv}) > \pi\,. \tag{6.8}$$

Then we take E_0 of the form

$$E_0 := S(d_0, r_0, \theta_0) \cup S(-d_0, r_0, \pi - \theta_0) \tag{6.9}$$

with $r_0, d_0 \in (0, +\infty)$ and $\theta_0 \in (\pi/2, \pi)$ chosen so that

$$\sin\theta_0 > \max\left\{ \sqrt{\frac{\pi}{g(\theta_{\rm adv})}}\,, \frac{2\pi}{\pi + 2\cos\theta_{\rm rec}} \right\}\,, \quad d_0 > r_0\left(1 + \frac{1}{\sin\theta_0}\right)\,. \tag{6.10}$$

Finally, we denote by $\tilde{E}_0$ the reflection of E_0 with respect to the vertical axis, that is, the union of $S(d_0, r_0, \pi - \theta_0)$ and $S(-d_0, r_0, \theta_0)$.

Then $E_0 \neq \tilde{E}_0$ because $\theta_0 \neq \pi/2$, and

$$|E_0| = |\tilde{E}_0| = \frac{\pi r_0^2}{\sin^2\theta_0}\,, \quad |\Sigma_0^f| = |\tilde{\Sigma}_0^f| = \frac{2\pi r_0}{\sin\theta_0}\,. \tag{6.11}$$

Moreover both E_0 and $\tilde{E}_0$ are stable, and *every* map $E(t)$ of the form

$$E(t) := \begin{cases} E_0 & \text{if } t \in A \\ \tilde{E}_0 & \text{if } t \in [0,1] \setminus A, \end{cases} \tag{6.12}$$

where A is a Borel subset of $[0,1]$ which contains 0, is a solution with initial configuration E_0 and constant area $|E(t)| = |E_0|$.

PROOF. We apply Proposition 5.13. Since $\tilde{E}_0$ is the reflection of E_0 with respect to to the vertical axis and has the same contact line as E_0, it suffices to show that E_0 minimizes the functional $\mathscr{F}_0$ defined in (6.1) in the class $\mathscr{Y}_v$ where $v := |E_0|$.

This will be achieved by showing that if E is a minimizer of $\mathscr{F}_0$ on $\mathscr{Y}_v$, then, up to a modification which preserves the value of $\mathscr{F}_0$, it agrees with E_0 or $\tilde{E}_0$. The proof of this claim is divided into several steps.

Step 1. Let E_+ be the union of all connected components of E whose contact line intersects the right component of Σ_0^c, namely the interval $[d_0 - r_0, d_0 + r_0]$, and let $E_- := E \setminus E_+$. We modify E by replacing E_+ and E_- by their Steiner symmetrizations with respect to the vertical lines $x_1 = d_0$ and $x_1 = -d_0$. Then, arguing as in the proof of Example 6.3, we can show that the modified set, also denoted by E, is still a minimizer of $\mathscr{F}_0$, and its components E_+ and E_- are circular segments centered respectively at d_0 and $-d_0$.[36]

Thus $E_\pm$ are of the form $S(\pm d_0, r_\pm, \theta_\pm)$.

[36] To make this argument work it is essential that the contact line of E_+ does not intersect the left component of Σ_0^c.

Step 2: we have $r_\pm \le r_0$, and if r_+ (resp., r_-) is equal to r_0 then θ_+ (resp., θ_-) is strictly smaller than $\theta_{\rm adv}$. Assume for instance that $r_+ > r_0$. Then Young's law implies $\theta_+ = \theta_{\rm adv}$, which leads to the contradiction $|E| > |E_0|$. Indeed

$$|E| \ge |E_+| \ge r_0^2\, g(\theta_{\rm adv}) > \frac{\pi r_0^2}{\sin^2\theta_0} = |E_0|\,,$$

where the last inequality follows from the first inequality in (6.10), and the equality follows from (6.11). The same argument proves the rest of the claim.

Step 3: we have either $\theta_+ > \pi/2$ or $\theta_- > \pi/2$. The contrary would lead to the contradiction $|E| < |E_0|$; indeed

$$|E| = r_+^2\, g(\theta_+) + r_-^2\, g(\theta_-) \le 2r_0^2\, g(\pi/2) = \pi r_0^2 < |E_0|\,,$$

where the last inequality follows from (6.11).

From now on we assume $\theta_+ > \pi/2$; the other case can be reduced to this one by reflection. Note that this implies $\theta_+ > \theta_{\rm rec}$, and therefore $r_+ = r_0$ by Young's law and Step 2.

Step 4: the set E_- is not empty. Assume the contrary: by Step 3 and (6.11) we get

$$\mathscr{F}_0(E) = |\Sigma^f| - \cos\theta_{\rm rec}|\Sigma^c| \ge \pi r_0 - 2r_0\cos\theta_{\rm rec}\,,$$

$$\mathscr{F}_0(E_0) = |\Sigma_0^f| - \cos\theta_{\rm rec}|\Sigma_0^c| = \frac{2\pi r_0}{\sin\theta_0} - 4r_0\cos\theta_{\rm rec}\,,$$

and then the first inequality in (6.10) yields the contradiction $\mathscr{F}_0(E) > \mathscr{F}_0(E_0)$.

Step 5: we have $r_- = r_0$. This follows by a concavity argument similar to the one used in Step 1 of the proof of Example 6.6: let v_+ and v_- denote the areas of E_+ and E_-. Assume indeed that $r_- < r_0$. Then $\theta_- = \theta_{\rm rec}$ and one easily checks that $\mathscr{F}_0(E_-)$ is a strictly concave function of v_-. On the other hand, we know from Step 3 and Step 1 that $r_+ = r_0$ and $\pi/2 < \theta_+ < \theta_{\rm adv}$, and then $\mathscr{F}_0(E_+)$ is a strictly concave function of v_+.[37] Thus $\mathscr{F}_0(E)$ is a strictly concave function of (v_-, v_+) with a minimum point in the *interior* of the segment of all admissible (v_-, v_+), and this is impossible.

Step 6. A variant of the concavity argument used in Step 5 yields $\theta_- \le \pi/2$.

Step 7. We know from the previous steps that $E_+ = S(d_0, r_0, \theta_+)$ with $\theta_+ \in (\pi/2, \theta_{\rm adv})$, and $E_- = S(-d_0, r_0, \theta_-)$ with $\theta_- \in [\theta_{\rm rec}, \pi/2]$. Moreover, the fact that the free line of E has constant curvature implies $r_0/\sin\theta_+ = r_0/\sin\theta_-$, and then $\theta_- = \pi - \theta_+$. Then the constraint $|E| = |E_0|$ implies $\theta_+ = \theta_0$, and this concludes the proof. □

6.9. Remark. (i) The solution $E(t)$ defined in (6.12) has bounded variation (as a map from $[0,1]$ to $\mathscr{Y}$) if and only if the topological boundary of the set A is finite.

[37] The key point is to show that $f(g^{-1}(s))$ is strictly concave for $s \ge \pi/2$; this follows by a lengthy but straightforward computation.

(ii) In the proof above it has been shown that the sets E_0 and $\tilde E_0$ minimize $\mathscr{F}_0$ on $\mathscr{Y}_0$. It follows immediately that for every $\delta > 0$ the map

$$E_\delta(t) := \begin{cases} E_0 & \text{if } t \in [k\delta, k\delta+\delta) \text{ with } k \text{ even} \\ \tilde E_0 & \text{if } t \in [k\delta, k\delta+\delta) \text{ with } k \text{ odd} \end{cases}$$

is a discretized solution in the sense of §4.1. Note that the maps $E_\delta(t)$ admit no subsequence which converge pointwise for every t (and not even for almost every t), and indeed the variations of these maps (on any bounded interval) tend to $+\infty$ as $\delta \to 0$.

(iii) Obviously, the initial configuration E_0 admits also discretized solutions which converge at every time (for instance $E_\delta(t) := E_0$ for every t and every δ). However, it seems plausible that by adding a suitably chosen, time-dependent bulk-contribution V to the capillary energy, we can enforce a highly oscillatory behaviour in *all* discretized solutions, so that no subsequence can possibly converge pointwise.

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