A Lusin Type Theorem for Gradients

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We prove that for every Borel vector field \( f \), there exists a function \( u \) of class \( C^1 \) whose gradient \( Du \) agrees with \( f \) outside a set of arbitrary small measure.

INTRODUCTION

It is well-known that given any vector field \( f \) of class \( C^1 \) on a simply connected open set \( \Omega \subset \mathbb{R}^N \), there exists a function whose gradient is \( f \) if and only if \( \text{curl} \ f = 0 \), where \( \text{curl} \ f \) is the function of \( \Omega \) into \( \mathbb{R}^{N	imes N} \) defined by

\[
(\text{curl} \ f)_{j,i} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \quad \text{for all } j, i = 1, \ldots, N.
\]

By using convolutions, the analogous result may be easily proved when \( f \) is a distribution and \( \text{curl} \ f = 0 \) in the distributional sense.

In this paper we prove that if \( f \) is a Borel vector field on \( \Omega \) and \( \varepsilon \) is a positive real number, then there exists a function \( u \) of class \( C^1 \) such that \( f \) agrees with \( Du \) outside an open set \( A \) with measure less than \( \varepsilon \). Notice that this holds even if \( f \) is a field such that \( \text{curl} \ f \neq 0 \) everywhere; it may easily be proved that in this case the set \( A \) must be dense in \( \Omega \).

Our main result is the following.

THEOREM 1. Let \( \Omega \) be a open subset of \( \mathbb{R}^N \) \((N > 1)\) with finite measure, and let \( f : \Omega \to \mathbb{R}^N \) be a Borel function. Then, for every \( \varepsilon > 0 \), there exist an open set \( A \subset \Omega \) and a function \( u \in C^1_0(\Omega) \) such that

\[
\begin{align*}
|A| & \leq \varepsilon |\Omega| \quad \text{(1a)} \\
f & = Du \quad \text{in } \Omega \setminus A, \quad \text{(1b)} \\
\|Du\|_p & \leq C \varepsilon^{1/p-1} \|f\|_p \quad \text{for all } p \in [1, \infty], \quad \text{(1c)}
\end{align*}
\]

where \( C \) is a constant which depends on \( N \) only.
We add some remarks and further results.

**Remark 2.** Notice that when $p = 1$ the condition $|\Omega| < \infty$ may be dropped and Theorem 1 may be stated as follows:

Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^N$ be a Borel function. Then, for every $\varepsilon > 0$, there exists a function $u \in \mathcal{C}_0^1(\Omega)$ such that $f = Du$ outside an open set with measure less than $\varepsilon$ and $\|Du\|_1 \leq C\|f\|_1$ ($C$ is the same constant of Theorem 1).

If the function $u$ in the statement of Theorem 1 is allowed to be taken in the space $BV$, (1a), (1b) and (1c) may be strengthened as follows.

**Theorem 3.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}^N$ be a function in $L^1$. Then there exists a function $u \in BV(\mathbb{R}^N)$ and a Borel function $g : \Omega \to \mathbb{R}^N$ such that

\begin{align*}
Du &= f \cdot \mathcal{L}^N + g \cdot \mathcal{H}^{N-1}, \quad (2a) \\
\int |g| d\mathcal{H}^{N-1} &\leq C\|f\|_1, \quad (2b)
\end{align*}

where $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$, $\mathcal{H}^{N-1}$ is the $(N-1)$ dimensional Hausdorff measure, and $C$ is a constant which depends on $N$ only.

**Remark 4.** In Theorem 1, (1c) gives an upper bound of the $L^p$ norm of the gradient of $u$ which essentially depends on the measure of the set $A$. We may ask whether this is the best estimate we can get in general, that is, whether for some $p$ formula (1c) may be replaced with

$$
\|Du\|_p \leq \phi(\varepsilon) \|f\|_p,
$$

where $\phi$ is a function such that $\lim_{\varepsilon \to 0} \phi(\varepsilon) \varepsilon^{1-1/p} = 0$.

The answer is “no” as the following proposition shows.

**Proposition 5.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure and let $f : \Omega \to \mathbb{R}^N$ be a Borel function. Let $\{u_n\}$ be a sequence in $W^{1,p}(\Omega)$ and let $A_n = \{x \in \Omega : f(x) \neq Du_n(x)\}$. If we have that

\begin{align*}
\lim_{n \to \infty} |A_n| = 0, \quad \text{and} \quad \liminf_{n \to \infty} |A_n|^{1-1/p}\|Du_n\|_p = 0,
\end{align*}

(3)

then $\text{curl } f = 0$ as a distribution on $\Omega$.

The proposition above shows that if $\text{curl } f \neq 0$ as a distribution on $\Omega$ (for example, take $N = 2$ and $f(x, y) = (y, 0)$), then no sequence $\{u_n\} \subset W^{1,p}(\Omega)$ can satisfy (3).
Theorem 1 can be applied to study integral functionals on Sobolev space of the form (cf. [2])

$$F(u, A) = \int_A g(x, Du(x))dx$$

where $\Omega$ is an open subset of $\mathbb{R}^N$, $g : \Omega \times \mathbb{R}^N \to [-\infty, \infty]$ is a Borel function, $A$ varies among all open subsets of $\Omega$ and $u$ varies in the space $W^{1,p}(\Omega)$. We may ask in which sense the function $g$ which represents $F$ is determined.

**Corollary 6.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $h$ and $g$ be two Borel functions of $\Omega \times \mathbb{R}^N$ into $[-\infty, \infty]$ such that for every $u \in C^1_c(\Omega)$

$$h(x, Du(x)) = g(x, Du(x)) \quad \text{a.e. in } \Omega,$$

that is, $h$ and $g$ represent the same integral functional. Then there exists a negligible Borel set $N \subset \Omega$ such that

$$h(x, s) = g(x, s) \quad \text{for all } x \in \Omega \setminus N \text{ and } s \in \mathbb{R}^N.$$

**Proof of the results**

To begin with, we prove the following auxiliary lemma.

**Lemma 7.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure, let $f : \Omega \to \mathbb{R}^N$ be a continuous function and let $\eta$ and $\varepsilon$ be positive real numbers. Then there exist a compact set $K \subset \Omega$ and a function $u \in C^1_c(\Omega)$ such that

$$|\Omega \setminus K| \leq \varepsilon|\Omega| \quad (5a)$$

$$|f - Du| \leq \eta \quad \text{on } K, \quad (5b)$$

$$\|Du\|_p \leq C'\varepsilon^{1/p-1}\|f\|_p \quad \text{for all } p \in [1, \infty], \quad (5c)$$

where $C'$ is a constant which depends on $N$ only.

**Proof.** Of course we may suppose $\varepsilon < 1$. Let $K'$ be a compact subset of $\Omega$ such that $|\Omega \setminus K'| < |\Omega|\varepsilon/2$ ; there exists a positive $\delta$ such that, for all $x \in K'$, $y \in \Omega$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \eta \quad \text{and } \quad Q(x, 4\delta) \subset \Omega \quad (6)$$

where $Q(x, 4\delta)$ is the cube with center $x$ and side $4\delta$.

Let $\{T_i\}_{i \in I}$ be the (finite) family of all closed cubes $T$ whose sides’ length is $\delta$, whose centers $y_i$ belong to lattice $(\delta\mathbb{Z})^N$ and which intersect $K$ by the choice of $\delta$, each $T_i$ is included in $\Omega$. For all $i \in I$, let $Q_i$ be the closed cube with the same center of $T_i$ and side $(1 - \varepsilon/(2N))\delta$ ; let $a_i$ be the mean value
of \( f \) on \( T_i \) and let \( \phi_i \) be a function of class \( \mathcal{C}^1 \) such that \( \phi_i \equiv 1 \) in \( Q_i \), \( \phi_i \equiv 0 \) outside \( T_i \) and

\[
\|D\phi_i\|_{\infty} \leq \frac{8N}{\delta\varepsilon}. \tag{7}
\]

For all \( x \in \mathbb{R}^N \) set

\[
u(x) = \sum_i \phi_i(x) < a_i, x - y_i >. \tag{8}
\]

It is easy to see that \( u \) is a function of class \( \mathcal{C}^1 \) whose support is included in \( \bigcup_i T_i \subset \Omega \) and whose gradient is \( a_i \) within each cube \( Q_i \). Finally we set \( K = \bigcup_i Q_i \). We have to prove that \( u \) and \( K \) satisfy (5a), (5b) and (5c).

- (5a): By the choice of each \( Q_i \) we have that

\[
| T_i \setminus Q_i | \leq \left[ 1 - \left( 1 - \frac{\varepsilon}{2N} \right)^N \right] | T_i | \leq \frac{\varepsilon}{2} | T_i |. \tag{9}
\]

and then, as each \( T_i \) is a subset of \( \Omega \) by (6),

\[
| \Omega \setminus K | \leq | \Omega \setminus K' | + \sum_i | T_i \setminus Q_i | \leq \varepsilon | \Omega |.
\]

- (5b): By (8), \( Du \) is equal to the mean value of \( f \) on \( T_i \) within each \( Q_i \) and then \( |Du(x) - f(x)| \leq \eta \) within each \( Q_i \) by (6).

- (5c): By (8) we have that

\[
Du(x) = \sum_i D\phi_i(x) < a_i, x - y_i > + \sum_i a_i \phi_i(x);
\]

and then, for all \( p \in [1, \infty] \), taking into account (6), (7) and recalling that \( D\phi_i = 0 \) outside \( T_i \setminus Q_i \) and that \( a_i \) is the mean value of \( f \) on \( T_i \),

\[
\|Du\|_p \leq \left[ \sum_i \left( \|D\phi_i\|_{\infty} |a_i| \sqrt{N} \delta \right)^p | T_i \setminus Q_i | \right]^{1/p} + \left[ \sum_i |a_i|^p | T_i | \right]^{1/p}
\]

\[
\leq \left[ \sum_i \left( 8N^{3/2} |a_i| \varepsilon^{-1} \varepsilon | T_i | \right)^p | T_i | \right]^{1/p} + \left[ \sum_i |a_i|^p | T_i | \right]^{1/p}
\]

\[
\leq \left( 8N^{3/2} \varepsilon^{-1/2} + 1 \right) \left[ \sum_i \frac{1}{| T_i |} \int_{T_i} f \, dx \right]^{1/p}
\]

\[
\leq \left( 8N^{3/2} \varepsilon^{-1/2} + 1 \right) \left[ \int_{\Omega} |f|^p \, dx \right]^{1/p}.
\]
As the same inequality hold when \( p = \infty \) and \( \varepsilon < 1 \), Lemma 7 is proved.

Proof of Theorem 1. Of course we may suppose \( \varepsilon < 1 \) and that \( f \) is not almost everywhere 0.

First Case. \( f \) is a continuous bounded function.

Let \( \{ \eta_n \} \) be a sequence of positive real numbers; by induction on \( n \) we build a sequence \( \{ u_n, K_n, f_n \} \) as follows: set \( u_0 = 0, K_0 = \emptyset \) and \( f_0 = f \). Let \( n > 0 \) and let \( u_{n-1}, K_{n-1} \) and \( f_{n-1} \) be chosen. Apply Lemma 7 to obtain a compact set \( K_n \subset \Omega \) and a function \( u_n \in \mathcal{C}^1(\Omega) \) such that

\[
\begin{align*}
|\Omega \setminus K_n| &\leq |\Omega|2^{-n}\varepsilon \quad \text{(10a)} \\
|f_{n-1} - Du_{n}| &\leq \eta_n \quad \text{on } K_n, \quad \text{(10b)} \\
\|Du_n\|_p &\leq C'(2^{-n}\varepsilon)^{1/p-1}\|f_{n-1}\|_p \quad \text{for all } p \in [1, \infty]. \quad \text{(10c)}
\end{align*}
\]

Define \( f_n(x) = f_{n-1}(x) - Du_n(x) \) for all \( x \in K_n \) and apply Titze’s lemma to extend \( f_n \) to the whole of \( \Omega \) so that

\[
\sup_{x \in \Omega} |f_n(x)| = \sup_{x \in K_n} |f_n(x)| \leq \eta_n. \quad \text{(11)}
\]

We set \( A = \Omega \setminus \bigcap_n K_n, u = \sum_n u_n \) and then choose a sequence \( \{ \eta_n \} \) so that these definitions make sense and satisfy (1a), (1b) and (1c). By (10a) we obtain

\[
|A| \leq \sum_{i=1}^{\infty} |\Omega \setminus K_n| \leq \sum_{i=1}^{\infty} |\Omega|2^{-n}\varepsilon = |\Omega|\varepsilon
\]

and (1a) holds. For all \( p \in [1, \infty] \), (10c) and (11) yield

\[
\sum_{i=1}^{\infty} \|Du_n\|_p \leq \sum_{i=1}^{\infty} C'(2^{-n}\varepsilon)^{1/p-1}2^n\|f_{n-1}\|_p
\]

\[
\leq 2C'\varepsilon^{1/p-1}\left(\|f_0\|_p + \sum_{i=1}^{\infty} 2^n\|f_n\|_{\infty}|\Omega|^{1/p}\right)
\]

\[
\leq 2C'\varepsilon^{1/p-1}\|f\|_p \left[1 + \frac{|\Omega|^{1/p}}{\|f\|_p} \sum_{i=1}^{\infty} 2^n\eta_n\right].
\]

As \( f \) is bounded and not almost everywhere 0, an easy computation shows that the function \( p \mapsto |\Omega|^{1/p}/\|f\|_p \) is continuous and positive on \([1, \infty]\), hence it has a positive upper bound \( a \) and we may choose all \( \eta_n \) small enough to have that \( \sum_{i=1}^{\infty} 2^n\eta_n \leq 1/a \) and then

\[
\sum_{i=1}^{\infty} \|Du_n\|_p \leq 4C'\varepsilon^{1/p-1}\|f\|_p.
\]
Poincaré’s inequality (cf. [1, Chap. 9]) shows that the series \( \sum_n u_n \) converges in the \( \mathcal{C}_0^1(\Omega) \) norm to a function \( u \) that satisfies (1c) with \( C = 4C' \). By the definition of \( f_n \) we have that, for all \( x \) in \( \Omega \setminus A \) and for all integers \( m \), \( f(x) - \sum_1^m D u_n(x) = f_m(x) \) and then by (10b)

\[
|f(x) - D u(x)| \leq |f_m(x)| + \sum_{m+1}^\infty |D u_n(x)| \leq \eta_m + \sum_{m+1}^\infty |D u_n(x)|.
\]

Hence (1b) immediately follows because the sequences \( \eta_m \) and \( \sum_m^\infty \|D u_n\|_\infty \) converge to 0.

**Second Case.** \( f \) is a Borel function.

Let \( \varepsilon > 0 \) be fixed. There exists a positive \( r \) such that \( |B| < \varepsilon/4 \), where \( B = \{x : \|f(x)\| > r\} \). By Lusin’s theorem there exists a continuous function \( f_1 : \Omega \to \mathbb{R}^N \) which agrees with \( f \) outside a Borel set \( C \) with \( |C| < |B| \). Set

\[
f_2(x) = \begin{cases} f_1(x) & \text{if } |f_1(x)| \leq r, \\ r f_1(x)/|f_1(x)| & \text{if } |f_1(x)| > r. \end{cases}
\]

The function \( f_2 \) is bounded and continuous, agrees with \( f \) outside \( C \cup B \) and since \( |C \cup B| < \varepsilon/2 \), there exists an open set \( A_1 \) such that \( |A_1| < \varepsilon/2 \) and \( f_2 \) agrees with \( f \) outside \( A_1 \). Moreover, for all \( p \in [1, \infty] \),

\[
\int_\Omega |f_2|^p dx \leq \int_{\Omega \setminus (B \cup C)} |f|^p dx + \int_{B \cup C} r^p dx \\
\leq \int_{\Omega \setminus (B \cup C)} |f|^p dx + 2 \int_B |f|^p dx \leq 2 \int_\Omega |f|^p dx,
\]

that is, \( \|f_2\|_p \leq 2 \|f\|_p \) for all \( p \) (in fact that the same inequality holds for \( p = \infty \)).

As \( f_2 \) is bounded and continuous we may apply Theorem 1 to obtain an open set \( A_2 \) with \( |A_2| \leq \varepsilon/2 \) and a function \( u \in \mathcal{C}_c^1(\Omega) \) such that \( D u = f_2 \) outside \( A_2 \) and \( \|D u\|_p \leq 4C'(\varepsilon/2)^{1/p-1}\|f_2\|_p \) for all \( p \in [1, \infty] \).

Hence \( D u = f \) outside the set \( A_1 \cup A_2 \), \( |A_1 \cup A_2| \leq \varepsilon \), and for all \( p \in [1, \infty] \),

\[
\|D u\|_p \leq 4C'(\varepsilon/2)^{1/p-1}\|f_2\|_p \leq 16C'\varepsilon^{1/p-1}\|f\|_p.
\]

Then Theorem 1 holds with \( A = A_1 \cup A_2 \).

The proof of Theorem 3 is quite similar to the one of Theorem 1; with no loss in generality we may suppose that \( \Omega = \mathbb{R}^N \).
To begin with, we prove an auxiliary lemma that will be used instead of Lemma 7.

**Lemma 8.** Let $f \in L^1(\mathbb{R}^N, \mathbb{R}^N)$ and let $\eta > 0$. Then there exist a function $u \in BV(\mathbb{R}^N)$ and two Borel functions $g^a$ and $g^s$ such that $Du = g^a \cdot \mathcal{L}^N + g^s \cdot \mathcal{H}^{N-1}$ and

\begin{align}
\|u\|_1 & \leq \|f\|_1 \quad \text{(12a)} \\
\|f - g^a\|_1 & \leq \eta \quad \text{(12b)} \\
\int |g^s| d\mathcal{H}^{N-1} & \leq C'\|f\|_1. \quad \text{(12c)}
\end{align}

where $C'$ is a constant which depends on $\eta$ only.

**Proof.** Let $\delta$ be a fixed positive number. Let $\{T_i\}_{i \in I}$ be the family of all open cubes whose sides’ length is $\delta$ and whose centers $y_i$ belong to lattice $(\delta\mathbb{Z})^N$. For all $i \in I$ let $a_i$ be the mean value of $f$ on $T_i$, let $\chi_i$ be the characteristic function of the set $T_i$, let $\nu_i$ be the inner normal of $\partial T_i$ (namely, if $x$ is a smooth point for $\partial T_i$ then $\nu_i(x)$ is the inner normal of $\partial T_i$ in $x$, otherwise $\nu_i(x)$ is 0). For all $x \in \mathbb{R}^N$ set

$$u_\delta(x) = \sum_i <a_i, x - y_i> \chi_i(x)$$

An easy computation shows that $u_\delta$ belongs to $BV$ and $Du_\delta = g^a_\delta \cdot \mathcal{L}^N + g^s_\delta \cdot \mathcal{H}^{N-1}$ where $g^a_\delta(x) = \sum_i a_i \chi_i(x)$ and $g^s_\delta(x) = \sum_i <a_i, x - y_i> \nu_i(x)$. Then

\begin{align}
\|u_\delta\|_1 & \leq \sum_i \sqrt{N} \delta |a_i| \cdot |T_i| \leq \sqrt{N} \delta \|f\|_1 \\
\|g^a_\delta\|_1 & \leq \sum_i |a_i| \cdot |T_i| \leq \|f\|_1 \\
\int |g^s_\delta| d\mathcal{H}^{N-1} & \leq \sum_i \sqrt{N} \delta |a_i| \mathcal{H}^{N-1}(\partial T_i) \leq \sum_i |a_i|2N^{3/2}|T_i| \leq 2N^{3/2}\|f\|_1.
\end{align}

Now it is enough to show that $\delta$ may be chosen so that (12a), (12b) and (12c) hold. Hence the proof is complete if we show that

$$\lim_{\delta \to 0} \|f - g^a_\delta\|_1 = 0. \quad \text{(13)}$$

Let $\Gamma_\delta : L^1 \to L^1$ be the linear operator taking each $f$ into $g^a_\delta$. By construction we have that $\|\Gamma_\delta\| \leq 1$ for all $\delta$ and an easy computation shows that
\lim_{n \to 0} \| \Gamma \delta f - f \|_1 = 0 \text{ whenever } f \in C_c. \text{ Hence (13) follows because } C_c \text{ is dense in } L^1.

Proof of Theorem 3. As in the proof of Theorem 1 we build by induction on \( n \) a sequence \( \{ u_n, f_n \} \) as follows.

Set \( u_0 = 0 \) and \( f_0 = f \). Let \( n > 0 \) and suppose that \( u_{n-1} \) and \( f_{n-1} \) has been chosen. Apply Lemma 8 to obtain a function \( u_n \in BV \) such that
\[
|u_n|_1 \leq |f_{n-1}|_1, \quad \| g_n - f_{n-1} \|_1 \leq 2^{-n} \| f \|_1, \text{ and }
\int |g_n^a| \, d\mathcal{H}^{N-1} \leq C' \| f_{n-1} \|_1.
\]
Set \( f_n = f_{n-1} - g_n^a \).

Hence the series \( \sum_n u_n \) converges in \( BV \) norm to a function \( u \) and \( Du = g^a \cdot \mathcal{L}^N + g^s \cdot \mathcal{H}^{N-1} \) with \( g^a = \sum_n g_n^a, g^s = \sum_n g_n^s \). Arguing as in the proof of Theorem 1 we get \( \| u \|_1 \leq 2 \| f \|_1 \), \( g^a = f \) almost everywhere and \( \int |g^s| \, d\mathcal{H}^{N-1} \leq 2C' \| f \|_1 \).

Proof of Proposition 5. Possibly passing to a subsequence we may assume
\[
\lim_{n \to \infty} |A_n|^{1-1/p} \| Du_n \|_p = 0. \tag{14}
\]
For all \( n \) set
\[
g_n(x) = \begin{cases} |Du_n(x)| & \text{if } x \in A_n, \\
0 & \text{if } x \notin A_n. \end{cases}
\]
Then \( |Du_n| \leq |f| + g_n \) everywhere by definition of \( A_n \) and \( \| g_n \|_1 \leq |A_n|^{1-1/p} \| Du_n \|_p \) by Schwartz-Hölder inequality. Now (14) implies that \( \| g_n \|_1 \) converges to 0; Hence \( \{ Du_n \} \) is a sequence of uniformly integrable functions and Dunford-Pettis theorem (cf. [4, Theorem II.25]) ensures that it has at least one limit point in \( w - L^1(\Omega, \mathbb{R}^N) \). This limit point must be \( f \), that is, \( Du_n \) converges to \( f \) in the weak topology of \( L^1 \).

Then curl \( f = \lim_n \text{curl} \, Du_n \) in the sense of distributions and the conclusion follows immediately because curl \( Du = 0 \) for any distribution \( \mathcal{D}'(\Omega) \) (cf. [5, Chap. 6]).

Proof of Corollary 6. Set \( B = \{(x, s) : h(x, s) \neq g(x, s)\} \) and let \( \pi \) be the projection of \( \Omega \times \mathbb{R}^N \) on \( \Omega \). By the Aumann measurable selection theorem (cf. [3, Theorems III.22 and III.23]) we have
\begin{enumerate}
\item \( \pi(B) \) is Lebesgue measurable
\item there exists a Lebesgue measurable function \( f : \pi(B) \to \mathbb{R}^N \) whose graph is a subset of \( B \).
\end{enumerate}
As $\pi(B)$ is Lebesgue measurable, it is enough to show that $|\pi(B)| = 0$. By contradiction, suppose that $|\pi(B)| > 0$; then, by (ii) and Theorem 1 there exists a function $u \in C^1(\mathbb{R}^N)$ such that $f = Du$ in a compact set $C$ of positive measure. Therefore

$$h(x, Du(x)) \neq g(x, Du(x)) \text{ for every } x \in C,$$

and this contradicts the assumption (4).

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\section*{References}