GMT 19/20 lecture 21 25/5/20

General Currents (continued)

Recap:

$$\mathcal{D}^k(\mathbb{R}^d) = \{ k\text{-forms on } \mathbb{R}^d \text{ of class } C^\infty_c \}$$

$$\mathcal{D}_k(\mathbb{R}^d) = \{ k\text{-dim. currents on } \mathbb{R}^d \} := \left(\mathcal{D}^k(\mathbb{R}^d)\right)^*.$$

$\mathbb{R}^d$ can be replaced with Ω open set in $\mathbb{R}^d$ or M d-dimensional Riem. manifold.

Given $T \in \mathcal{D}_k(\mathbb{R}^d)$, $k \geq 1$, the boundary $\partial T \in \mathcal{D}_{k-1}(\mathbb{R}^d)$ is defined by

$$\cancel{\partial T(\omega)} = \langle \partial T, \omega \rangle := \langle T, d\omega \rangle$$

and the mass of T is

$$M(T) := \sup_{\substack{\omega \in \mathcal{D}^k \text{ s.t.} \\ |\omega(x)| \leq 1 \ \forall x}} \langle T, \omega \rangle$$

what norm? comass!

If S is a closed, oriented k-surf. then T_S is defined by

$$\langle T_S ; \omega \rangle := \int_S \omega$$

and

$$\partial T_S = T_{\partial S} \qquad \text{(Stokes th.)}$$

and

$$\mathbb{M}(T_S) := \mathrm{Vol}_k(S) = \mathcal{H}^k(S).$$

more about this soon!

Topology on $\mathcal{D}^k(\mathbb{R}^d)$ and $\mathcal{D}_k(\mathbb{R}^d)$

We work only with LOCALLY CONVEX VECTOR SPACES !!

$\forall$ K compact in $\mathbb{R}^d$, let

$$\mathcal{D}^k(K) := \{ \omega \in \mathcal{D}^k(\mathbb{R}^d) : \mathrm{supp}(\omega) \subset K \}$$

This is a Frechet space with the seminorms

$$\|\omega\|_{C^h} := \sum_{\underline{i}} \|\omega_{\underline{i}}\|_{C^h} = \sum_{\underline{i}} \sum_{|\underline{j}| \leq h} \|D^{\underline{j}} \omega_{\underline{i}}\|_\infty$$

The topology on $\mathcal{D}^k(\mathbb{R}^d)$ is the weakest (the smallest) s.t.

$$i : \mathcal{D}^k(K) \longrightarrow \mathcal{D}^k(\mathbb{R}^d)$$

is continuous $\forall K$ (that is, the so-called direct-limit topology).

$\mathcal{D}_k(\mathbb{R}^d) := \{$ continuous linear funct. on $\mathcal{D}^k \}$ is endowed with the weak* topology.

A sequence of currents T_n converges (in the sense of currents) to T iff

$$\langle T_n, \omega \rangle \longrightarrow \langle T, \omega \rangle \quad \forall \omega \in \mathcal{D}^k(\mathbb{R}^d)$$

<u>Prop. 1</u> If $T_n \longrightarrow T$ then

(i) $\partial T_n \longrightarrow \partial T$;

(ii) $\mathbb{M}(T) \leq \liminf_{n \to \infty} \mathbb{M}(T_n)$.

Proof is immediate!

Remarks

(1) ∂ is the adjoint of d, i.e.

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle$$

(2) $\forall \omega \in \mathcal{C}^2$, $d^2\omega = 0$

Proof
$$d^2\omega = d\left(\sum_{j=1}^{d} dx_j \wedge \frac{\partial \omega}{\partial x_j} \right)$$

$$= \sum_{e=1}^{d} \sum_{j=1}^{d} dx_e \wedge dx_j \wedge \frac{\partial}{\partial x_e}\left(\frac{\partial \omega}{\partial x_j} \right)$$

$$= \sum_{1 \leq e < j \leq d} dx_e \wedge dx_j \wedge \frac{\partial^2 \omega}{\partial x_e \partial x_j} + dx_j \wedge dx_e \wedge \frac{\partial^2 \omega}{\partial x_j \partial x_e}$$

$$= 0$$

(3) $\forall T \in \mathcal{D}_k(\mathbb{R}^d)$, $\partial^2 T = 0$

Proof : (i) + (ii).

Significant (?) subclasses of currents

(no symbols given)

1| Currents with finite mass

$$\{T : \mathbb{M}(T) < +\infty\}$$

For such T,

$$|\langle T, \omega\rangle| \leq \mathbb{M}(T)\, \|\omega\|_\infty \quad \forall \omega \in \mathscr{D}^k$$

- hence T can be extented by density to all $\omega \in \mathscr{C}_0(\mathbb{R}^d, \Lambda^k(\mathbb{R}^d))$
- hence T can be represended by a measure with values in $\Lambda_k(\mathbb{R}^d)$:

$\exists\ \mu$ positive (locally) finite measure on $\mathbb{R}^d$

$\exists\ \tau : \mathbb{R}^d \longrightarrow \Lambda_k(\mathbb{R}^d) \in L^1(\mu)$ s.t.

$$\langle T, \omega\rangle := \int_{\mathbb{R}^d} \langle \tau(x) ; \omega(x)\rangle \, d\mu(x)$$

Then

$$\mathbb{M}(T) = \sup_{\substack{\omega \in \mathscr{D}^k \\ \|\omega\|_\infty \leq 1}} \langle T, \omega\rangle = \sup_{\substack{\omega \in \mathscr{C}_0 \\ \|\omega\|_\infty \leq 1}} \langle T, \omega\rangle = \int_{\mathbb{R}^d} |\tau(x)| \, d\mu(x)$$

mass norm

Notation: I write $T = \tau\mu$.

τ and μ are uniquely determined if you request that $|\tau(x)| = 1$ μ-a.e.

Rem if $T = T_S$ with S compact oriented C^1-surface then $T = \tau_S \cdot \mathcal{H}^k \llcorner S$ and $\mathbb{M}(T) = \mathcal{H}^k(S)$.

Prop. 2 (compactness): If $\mathbb{M}(T_n) \leq C < +\infty$ then (up to subseq.) T_n converges to some T (in the sense of currents) with

$$\mathbb{M}(T) \leq \liminf_{n\to\infty} \mathbb{M}(T_n) < +\infty .$$

indeed $\langle T_n, \omega \rangle \longrightarrow \langle T, \omega \rangle$ $\forall \omega \in \mathcal{E}_0(\mathbb{R}^d, \dots)$.

2| Normal currents.

if $k \geq 1$, for $k=0$ normal currents are just currents with $\mathbb{M}(T) < +\infty$

$T \in \mathcal{D}_k(\mathbb{R}^d)$ is normal if $\mathbb{M}(T), \mathbb{M}(\partial T) < +\infty$.

($T = \tau\mu$, $\partial T = \tau'\mu'$ ....)

Prop. 3 (compactness): if $\mathbb{M}(T_n), \mathbb{M}(\partial T_n) \leq C < +\infty$ then (up to subseq.) T_n converges to a normal current T.

Moreover $\partial T_n \to \partial T$, $\mathbb{M}(T) \leq \liminf_{n\to\infty} \mathbb{M}(T_n)$,

$\mathbb{M}(\partial T) \leq \liminf_{n\to\infty} \mathbb{M}(\partial T_n)$.

Proof Apply Prop. 2 & Prop. 1.

Coroll. (Solution of Plateau Problem in the class of normal currents). Fix T_0 normal k-current: then

$$\min \{ \mathbb{M}(T) \ ; \ \underset{\text{normal}}{T} \text{ s.t. } \partial T = \partial T_0 \}$$

exists.

NOT SATISFACTORY, because the class of normal currents is too large!

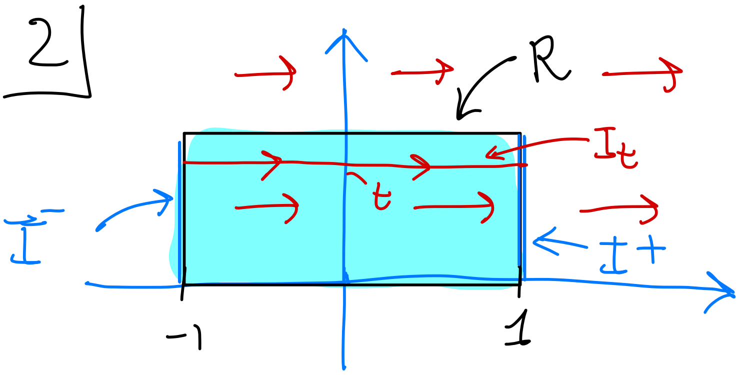
Examples of normal (and not) currents

1) Σ compacts surface of class C^1

$\Rightarrow$ T_Σ is normal, $\mathbb{M}(T_\Sigma) = \mathcal{H}^k(\Sigma)$,

$\mathbb{M}(\partial T_\Sigma) = \mathcal{H}^{k-1}(\partial \Sigma)$, $T = \tau_\Sigma \cdot \mathcal{H}^k \llcorner \Sigma$

2]



in $\mathbb{R}^2$

$\mu := \mathcal{L}^2 \llcorner R$

$e_1 := (1,0)$

then $T = e_1 \cdot \mu$ is a normal <u>1-dimensional</u> current

Indeed let $I^\pm := \{\pm 1\} \times [0,1]$,

then $\partial T = \mathcal{H}^1 \llcorner I^+ - \mathcal{H}^1 \llcorner I^-$

$= \tau' \mu'$ with $\mu' := \mathcal{H}^1 \llcorner (I^+ \cup I^-)$

$\tau' := \begin{cases} +1 & \text{on } I^+ \\ -1 & \text{on } I^- \end{cases}$

Direct proof: it's an exercise: $\langle T, d\phi \rangle =$

$= \int_R \langle e_1, d\phi \rangle \, d\mathcal{L}^2$

$= \int_R \frac{\partial \phi}{\partial x_1} \, d\mathcal{L}^2$

.....

Same proof in different word:

$\forall t \in [0,1]$, let $I_t := [-1,1] \times \{t\}$

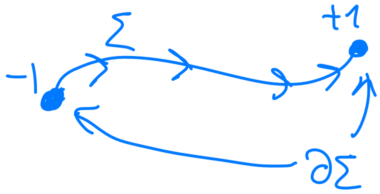
oriented by e_1, let T_t $= T(I_t)$ be the

associated 1-current.

Then

$$T = \int_0^1 T_t \, dt \quad \left(\text{i.e. } \langle T, \omega \rangle = \int_0^1 \langle T_t, \omega \rangle \, dt \right)$$

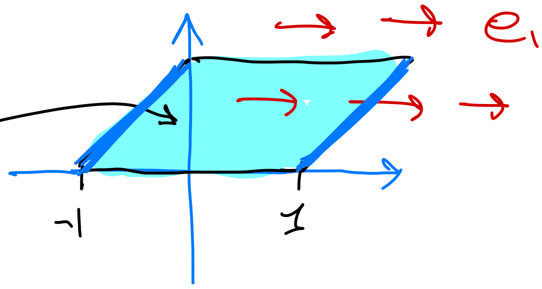
and
$$\partial T = \int_0^1 \partial T_t \, dt$$



then
$$\partial T = \int_0^1 \delta_{(1,t)} - \delta_{(-1,t)} \, dt$$
$$= \mathcal{H}^1 \llcorner I^+ - \mathcal{H}^1 \llcorner I^-$$

(fill the details!!)

2bis) What if R



?

3) Let T be the 1-current on $\mathbb{R}$ given by $T = e \cdot \delta_0$ where e is the standard orient. (basis?) of $\mathbb{R}$: $e = 1$

Then ∂T is NOT a measure!

Indeed let φ be a 0-form on $\mathbb{R}$ that is, a (scalar) function. Then

$$\langle \partial T, \varphi \rangle = \langle T, d\varphi \rangle = \int_{\mathbb{R}} \underbrace{\langle e, d\varphi \rangle}_{\dot\varphi} \, d\delta_0 = \dot\varphi(0)$$

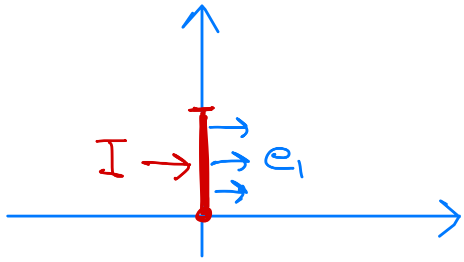
and

$$\mathbb{M}(\partial T) = \sup_{\|\varphi\|_\infty \le 1} \langle \partial T, \varphi \rangle = \sup_{\substack{\varphi \in C_c^\infty \\ |\varphi| \le 1}} \dot\varphi(0) = +\infty.$$

4) In $\mathbb{R}^2$ let T_1, T_2 be the 1-currents defined by

$$T_1 := e_1 \cdot \delta_0 , \quad T_2 := e_1 \cdot \mathcal{H}^1 \llcorner I$$

$\shortparallel$
$\{0\} \times [0,1]$



In both cases $\mathbb{M}(\partial T_i) = +\infty$!

Continuous vectorf. on I

Indeed let $T := \tau \cdot \mathcal{H}^1 \llcorner I$:

if $\mathbb{M}(\partial T) < +\infty$ then τ is

TANGENT to I !!