GMT 19/20 lecture 20 22/5/20

Now we construct a vector space (space of k-vectors on V) whose dual is $\Lambda^k(V)$ so that simple k-vectors are naturally embedded in this space.

We rely on the canonical identification of V and V^{**} (V is finite dim.)

Def. ($\forall\, k = 0, 1, 2, \ldots$)
The space of k-vectors on V is

$$\Lambda_k(V) := \text{space of } k\text{-covectors on } V^* = \Lambda^k(V^*)$$

Then $\Lambda_1(V) = (V^*)^* = V$

And we have the wedge product

$$\wedge : \Lambda_h(V) \times \Lambda_k(V) \longrightarrow \Lambda_{h+k}(V)$$

Given a basis $e_1, \dots, e_n$ of V we write

$$e_{\underline{i}} := e_{i_1} \wedge \dots \wedge e_{i_k} \quad \forall \underline{i} \in I_{n,k}$$

$\in \Lambda_k(V)$

We already proved that $\{e_{\underline{i}} : \underline{i} \in I_{n,k}\}$ is a basis of $\Lambda_k(V)$

We define the duality pairing $\langle\, ; \rangle$ of $\Lambda^k(V)$ and $\Lambda_k(V)$ by setting

$$\langle e^*_{\underline{i}} \, ; e_{\underline{j}} \rangle := \delta_{\underline{i}\,\underline{j}} \qquad (*)$$

(Thus $\{e^*_{\underline{i}}\}$ is the dual basis associated to $\{e_{\underline{i}}\}$ that is, the $\underline{i}$ coord. of $w \in \Lambda_k(V)$ wrt $\{e_{\underline{i}}\}$ is given by $\langle e^*_{\underline{i}}, w \rangle$!)

Everything works as it should ...

Lemma 7 $\forall \alpha \in \Lambda^k(V)$, $v_1, \dots, v_k \in V$ there holds

$$\langle \alpha ; v_1 \wedge \dots \wedge v_k \rangle = \alpha(v_1, \dots, v_k) \quad (**)$$

Proof

$(*)$ says that $(**)$ holds for

$$\alpha \in \{e_i^*\} , \quad v_j \in \{e_i\}$$

using linearity in α we get that $(**)$ holds for

$$\alpha \in \Lambda^k(V) , \quad v_j \in \{e_i\}$$

using linearity in each v_j we get that $(**)$ holds for

$$\alpha \in \Lambda^k(V), \quad v_j \in V$$

Corollary 8

The duality pairing $\langle \, ; \, \rangle$ does NOT depend on the choice of the basis $\{e_i\}$

Corollary 9

$$(v_1,\dots,v_k) \sim (\tilde{v}_1,\dots,\tilde{v}_k)$$
$$\Updownarrow$$
$$v_1\wedge\dots\wedge v_k = \tilde{v}_1\wedge\dots\wedge\tilde{v}_k$$

Thus we identify $\underbrace{[v_1,\dots,v_k]}$ with $v_1\wedge\dots\wedge v_k$

this notation disappears from now on!!

Proof

$$(v_1,\dots,v_k) \sim (\tilde{v}_1,\dots,\tilde{v}_k)$$
$$\Updownarrow$$
$$\alpha(v_1,\dots,v_k) = \alpha(\tilde{v}_1,\dots,\tilde{v}_k) \quad \forall \alpha\in\Lambda^k(V)$$
$$\Updownarrow$$
$$\langle \alpha, v_1\wedge\dots\wedge v_k\rangle = \langle\alpha, \tilde{v}_1\wedge\dots\wedge\tilde{v}_k\rangle \quad \forall\alpha\in\Lambda^k(V)$$
$$\Updownarrow \quad \text{(use that } \Lambda^k \text{ is dual to } \Lambda_k)$$
$$v_1\wedge\dots\wedge v_k = \tilde{v}_1\wedge\dots\wedge\tilde{v}_k$$

$\square$

Assume now that V is endowed with a scalar product $(\cdot)$ and $e_1,\dots,e_n$ is an orthonormal basis.

Then we can endow $\Lambda^k(V)$ and $\Lambda_k(V)$ with scalar products s.t. $\{e_{\underline{i}}^*\}$ and $\{e_{\underline{i}}\}$ are orthonormal basis.

I never use these scalar products, only the associated norm (on $\Lambda_k(V)$)

Given $w = \sum_{\underline{i}} w_{\underline{i}} \cdot e_{\underline{i}}$, $w \in \Lambda_k(V)$, $w_{\underline{i}} = \langle e_{\underline{i}}^*, w\rangle$

then

$$|w| := \sqrt{\sum_{\underline{i}} w_{\underline{i}}^2}$$

Proposition 10

$$|v_1 \wedge \cdots \wedge v_k| = \mathcal{H}^k(R(v_1, \dots, v_k)) = |[v_1, \dots, v_k]|$$

Proof

$$|v_1 \wedge \cdots \wedge v_k| = \sqrt{\sum_{\underline{i}} (\langle e_{\underline{i}}^* ; v_1 \wedge \cdots \wedge v_k\rangle)^2}$$

lemma 7

$$= \sqrt{\sum_{\underline{i}} (e_{\underline{i}}^*(v_1, \dots, v_k))^2}$$

two lectures ago:

$W :=$ matrix of coordinates of $v_1, \dots, v_k$

$W_{\underline{i}} :=$ minor of W corresp. to rows $i_1, \dots, i_k$

$$= \sqrt{\sum_{\underline{i}} (\det(W_{\underline{i}}))^2}$$

$$= \sqrt{\det(W^t W)}$$

Binet formula

T linear map given by

$T : \hat{e}_i \to v_i$

$\mathbb{R}^k \ni \hat{e}_i$, $\; v_i \in V$

$$= |\det T|$$

unit cube in $\mathbb{R}^k$

$$= |\det T| \cdot \mathrm{vol}_k(\underbrace{R(\hat{e}_1, \ldots, \hat{e}_k)})$$

$$= \mathcal{H}^k(T(R(\hat{e}_1, \ldots, \hat{e}_k)))$$

$$= \mathcal{H}^k(R(v_1, \ldots, v_k)) \,. \qquad \square$$

Concluding remarks

(1) There are two natural choices of norm on $\Lambda_k(V)$:

a) "Euclidean norm" defined above: $|\cdot|$

b) "mass norm" ϕ, namely the largest norm of $\Lambda_k(V)$ s.t.

$$\phi(v_1 \wedge \ldots \wedge v_k) = |v_1 \wedge \ldots \wedge v_k| \quad \forall v_j \in V$$

that is, convex envelope of the restriction of $|\cdot|$ to simple k-vectors

$$\phi(w) := \inf \Big\{ \sum t_i |w_i| : \; w = \underline{\sum t_i w_i} \Big\}$$

convex combination of simple vect.

— the dual norm is called "comass":

$$\phi^*(\alpha) := \sup_{\phi(w)\leq 1} \langle \alpha, w\rangle$$

$\alpha \in \Lambda^k(V)$

$$= \sup_{|v_1\wedge\cdots\wedge v_k|\leq 1} \langle \alpha, v_1\wedge\cdots\wedge v_k\rangle$$

There only a couple of results where you see the difference between Euclidean norm and mass (or comass)

(2) Are there non-simple k-vectors?

Yes: $e_1\wedge e_2 + e_3\wedge e_4$ (in V, $\dim V \geq 4$)

(exercise)

If $n := \dim(V)$,

1-vectors are all simple (obvious)

$(n-1)$-vectors are all simple (not so obvious)

k-vectors with $1<k<n-1$ are never all simple

Theory of general currents (de Rham)

Definition similar to that of "generalized functions" or "distributions".

BASIC OBSERVATION(S)

Let Σ be a closed, oriented k-dim. surface of class C^1 in $\mathbb{R}^n$. Then

$$T_\Sigma : \omega \longmapsto \int_\Sigma \omega$$

is a linear functional on the space of k-forms (continuous + compact support)

Moreover

1) Σ is uniquely determined by T_Σ

$$(\Sigma \neq \tilde{\Sigma} \Rightarrow T_\Sigma \neq T_{\tilde{\Sigma}})$$

2) Stokes theorem becomes

$$T_{\partial\Sigma}(\underset{\substack{\uparrow \\ k-1 \text{ form}}}{\omega}) = T_\Sigma(\underset{\substack{\uparrow \\ k\text{-form}}}{d\omega})$$

3) $$\mathrm{vol}_k(\Sigma) = \sup\left\{T_\Sigma(\omega) : \overbrace{|\omega(x)| \leq 1 \ \forall x}^{\omega \in C_c^\infty}\right\}$$

$$T_\Sigma(\omega) = \int_\Sigma \omega := \int_\Sigma \langle \omega(x); \tau(x) \rangle \, d\mathcal{H}^k(x)$$

$$\leq \int_\Sigma |\omega(x)| \cdot \underbrace{|\tau(x)|}_{=1} \, d\mathcal{H}^k(x)$$

if $|\omega(x)| \leq 1$ $\forall x$

$$\leq \int_\Sigma 1 \, . \, d\mathcal{H}^k(x) = \mathrm{vol}_k(\Sigma)$$

and you get $=$ if $\omega(x)$ is such that

$$\langle \omega(x); \tau(x) \rangle = |\tau(x)| = 1 \quad \forall x.$$

Such $\omega(x)$ exists for all x (and is cont.)

Definition

Denote by $\mathcal{D}^k(\mathbb{R}^n)$ the space (endowed with the struct. of locally convex top. vector space) of smooth k-forms with compact support on $\mathbb{R}^n$

The space of k-currents on $\mathbb{R}^n$, $\mathcal{D}_k(\mathbb{R}^n)$, is defined as the dual of $\mathcal{D}^k(\mathbb{R}^n)$ (space of cont. linear funct.)

"generalized oriented k-surfaces"

$\forall T \in \mathcal{D}_k(\mathbb{R}^n)$, ∂T (boundary of T)

is defined as $\partial T(\omega) := T(d\omega)$

The mass (k-dim. volume) of T

is
$$\mathbb{M}(T) := \sup \left\{ T(\omega) : \begin{array}{l} \omega \in \mathcal{D}^k(\mathbb{R}^n) \\ |\omega(x)| \leq 1 \ \forall x \end{array} \right\}$$