GHT 19/20 beture 20 22/5/20

Nou we construct a vector Space (space of k-vectors on V) whose duel is $\Lambda^{K}(V)$ so that simple k-vectors are naturally embedded in this space. We vely on the canonical identification of V and V** (Vis finite dim.) Def. $(\forall k = 0, 1, 2, \dots)$ The space of k-vectors on V is $\bigwedge_{k}(V) := space of k-covectors on V*$ $= \bigwedge^{\mathcal{K}}(\vee^*)$ Then $\wedge_1(\vee) = (\vee^*)^* = \vee$ And we have the wedge product $\wedge: \ \wedge_{\varrho}(\mathsf{V}) \times \wedge_{\kappa}(\mathsf{V}) \longrightarrow \wedge_{\varrho+\kappa}(\mathsf{V})$

Cliven a basis en,..., en of V we contre $e_{\underline{i}} := e_{i_1} \wedge \dots \wedge e_{i_k} \quad \forall \underline{i} \in \mathcal{I}_{n,k}$ $\in \Lambda_{k}(V)$ We already proved that $\{e_{\underline{v}}: \underline{v} \in J_{n,k}\}$ is a basis of $\Lambda_k(V)$ We define the duality pairing <;> of $\Lambda^{k}(V)$ and $\Lambda_{k}(V)$ by setting (\pm) $\langle e_{\underline{v}}^{*}; e_{\underline{j}} \rangle := S_{\underline{v}}$ (Thurs {et } is the dual basis assaulted to $\{e_{\underline{i}}\}$ that is, the \underline{i} coord. of $W \in \Lambda_{\underline{k}}(\underline{k})$ wrt $\{e_{\underline{i}}\}$ is given by $\langle e^*, w \rangle !$ Everything works as it should

 $(lume 7 \quad \forall x \in \Lambda^{k}(V), \quad V_{1}, \dots, V_{k} \in V$ there helds $\langle \chi; V_1 \land \dots \land V_k \rangle = \chi(V_1, \dots, V_k) (\star \star)$ Mook (*) says that (**) holds for $\alpha \in \{e_{\underline{i}}^{*}\}, \forall_{\underline{j}} \in \{e_{\underline{i}}\}$ using linearity in a we get that (+*) holds for $\alpha \in \Lambda^{\kappa}(V)$, $V_{j} \in \{e_{i}\}$ using lineanty in each v; we get that (++) holds for $x \in \Lambda^{k}(V), \quad V_{j} \in V$ Corollary 8 The duality pairing <; > does NOT depend on the choice of the basis {e; }

Corollary 9 $\begin{pmatrix} (\mathcal{V}_1, \ldots, \mathcal{V}_k) \\ \downarrow \\ \mathcal{V}_1 \land \cdots \land \mathcal{V}_k \end{pmatrix} \sim \begin{pmatrix} \widetilde{\mathcal{V}}_1, \ldots, \widetilde{\mathcal{V}}_k \end{pmatrix}$ Thus use identify [V1, ..., vie] with V, A... AVie this notation disappears from now on !! Proof $\begin{pmatrix} \mathcal{V}_1, \ldots, \mathcal{V}_k \end{pmatrix} \sim \begin{pmatrix} \widetilde{\mathcal{V}}_1, \ldots, \widetilde{\mathcal{V}}_k \end{pmatrix}$ $\alpha(v_1, \dots, v_k) = \alpha(\widetilde{v}_1, \dots, \widetilde{v}_k) \quad \forall k \in \Lambda^k(V)$ $\langle \alpha, \forall, \Lambda \dots \Lambda \forall k \rangle = \langle \alpha, \forall, \Lambda \dots \Lambda \forall k \rangle \quad \forall \alpha \in \Lambda^{k}(V)$ (use that Λ^k is dual to Λ_k) $V_1 \wedge \cdots \wedge V_k = V_1 \wedge \cdots \wedge V_k$

Assume now that V is endowed with a scalar product (•) and $e_{1,...,e_n}$ is an orthonormal basis. Then eve can endow $\Lambda^{\kappa}(V)$ and $\Lambda_{\kappa}(V)$ with scalar products s.t. $\{e_{i}^{*}\}$ and $\{e_{i}\}$ are orthonormal basis. I never use these sealar products, Only the associated norm (on $\Lambda_{k}(V)$) $W = \sum_{\underline{i}} W_{\underline{i}} \cdot e_{\underline{i}}$ Given $\wedge_{k}(V)$ $\langle e^{*}_{\underline{i}}, w \rangle$ then $|W| := \sqrt{\sum_{i=1}^{2} W_{\underline{i}}^{2}}$ Proposition 10 $|V_1 \wedge \cdots \wedge V_k| = \mathcal{H}^k(\mathcal{R}(V_{1, \cdots, N}, V_k))$ $[v_1, \dots, v_{k_2}]$ $\operatorname{Ree}^{\mathbb{V}}\left[\mathbb{V}_{1}\wedge\cdots\wedge\mathbb{V}_{k}\right] \stackrel{1}{=} \sqrt{\sum_{i}\left(\langle e_{\underline{i}}^{*}; \mathcal{V}_{1}\wedge\cdots\wedge\mathcal{V}_{k}\rangle\right)^{2}}$ lema 7 $\sqrt{\sum_{\underline{v}} \left(\mathcal{C}_{\underline{v}}^{*} \left(\mathcal{V}_{1}, \dots, \mathcal{V}_{k} \right) \right)^{2}}$ Two lectures ago: = matrix of $= \sqrt{\sum_{i} (\det(W_{i}))^2}$ W := matrix of Wi := minor of W corresp. = V det (WtW) to rows ei,..., ik Binet @ Binet formale

$$T \operatorname{enerse}_{\operatorname{preservap}} = |\det T| \quad \operatorname{unit} \operatorname{cube} \operatorname{unit} R^{k}$$

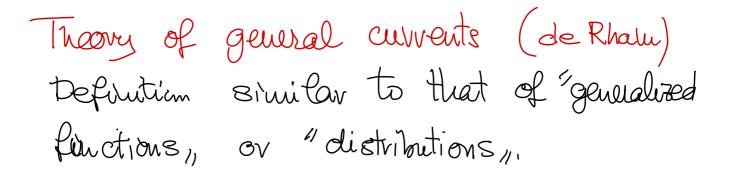
$$g \operatorname{ven}_{\operatorname{by}} = |\det T| \cdot \operatorname{vel}_{k}(R(\widehat{e}_{1}, \dots, \widehat{e}_{k}))$$

$$T : \widehat{e}_{i} \to v_{i}$$

$$R^{k} \quad V = \mathcal{H}^{k}(T(R(\widehat{e}_{i}, \dots, \widehat{e}_{k}))$$

$$= \mathcal{H}^{k}(R(v_{1}, \dots, v_{k})) \cdot$$

Concluding remarks (1) There are two natural choices of norm on $\Lambda_{k}(V)$: a) Euclidean norm, défined above: [.] b) "mass norm, ϕ , namely the largest norm of Mk(V) st. $\Phi(V_1 \land \dots \land V_k) = |V_1 \land \dots \land V_k| \quad \forall V_j \in V$ that is, convex envelope of the restriction of 1.1 to simple k-rectors $\phi(W) := \inf \{ \sum_{i \in W_i} W = \sum_{i \in W_i} \}$ Convex Combiliation of simple vest.



BASIC OBSERVATION(S) let I be a closed, oriented k-dim. Surface of closs CI in Rth. Then $T_{\Sigma}:\omega \longmapsto \int_{\Sigma} \omega$ is a linear functional on the space of R-forms (continuous 4 compact support) Kovenver 1) I is uniquely determined by Tz $(\Sigma \neq \widetilde{\Sigma} \Rightarrow T_{\Sigma} \neq T_{\widetilde{\Sigma}})$ 2) Stokes theorem becomes $T_{\partial \Sigma}(\omega) = T_{\Sigma}(d\omega)$ 3) $\operatorname{Vol}_{k}(\Sigma) = \sup \{T_{\Sigma}(\omega) : | [\omega(k)] \leq | \forall k \}$

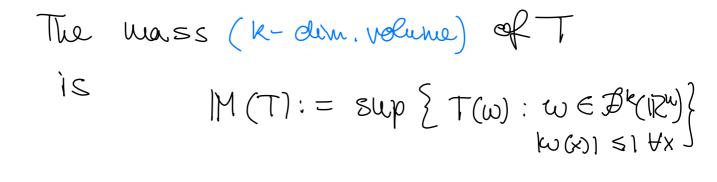
$$T_{\Sigma}(\omega) = \int \omega := \int \langle \omega(x); \tau(x) \rangle d\mathcal{H}^{k}(x)$$

$$\leq \int |\omega(x)| \cdot |\tau(x)| d\mathcal{H}^{k}(x)$$

$$if |\omega(x)| \leq 1 \qquad \leq \int 1 d\mathcal{H}^{k}(x) = vol_{k}(\Sigma)$$

$$\forall x \qquad \leq \int 1 d\mathcal{H}^{k}(x) = vol_{k}(\Sigma)$$
and you get = if $\omega(x)$ is such that
$$\langle \omega(x); \tau(x) \rangle = |\tau(x)| = 1 \quad \forall x.$$
Such $\omega(x) = xists$ for all X (and is cent.)

Denote by $\mathcal{D}^{k}(\mathbb{R}^{h})$ the space of every Defilition Smooth R-forms with compact support on R The space of K-currents on IR", D_k (R^h), is defend as the dual of $\mathcal{D}^{k}(\mathbb{R}^{n})$ (space of cont. boundary of enear find.) "generalized boundary of enear find.) oriented k - surfaces //is defend as $\partial T(\omega) := T(d\omega)$



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