GMT 19/20 lecture 19 21/5/20

Simple k-vectors (continued)

Prop. 5

(i) $(v_1, \dots, v_k) \sim (0, \dots, 0)$ iff $v_1, \dots, v_k$ are linearly dependent;

(ii) if $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k) \not\sim 0$ then $\mathrm{Span}(v_1, \dots, v_k) = \mathrm{span}(\tilde{v}_1, \dots, \tilde{v}_k)$ and the matrix of change of basis M satisfies $\det(M) = 1$

$$\tilde{v}_i = \sum_j M_{ij} v_j \quad \forall j$$

(iii) Conversely, if $(v_1, \dots, v_k)$ and $(\tilde{v}_1, \dots, \tilde{v}_k)$ span the same subspace W and the matrix M satisfies $\det M = 1$, then $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k)$

Proof

Step 1 : if $v_1, \dots, v_k$ are lin. dep. then $\alpha(v_1, \dots, v_k) = 0 \quad \forall \alpha \in \Lambda^k(V)$

$\Rightarrow \quad (v_1, \dots, v_k) \sim (0, \dots, 0)$.

Step 2 : if $(v_1, \dots, v_k)$ are lin. indep. then $\exists\ \alpha$ s.t. $\alpha(v_1, \dots, v_k) \neq 0$

$\Rightarrow \quad (v_1, \dots, v_k) \not\sim (0, \dots, 0)$ (Statem. (i) is proved.)

Indeed, complete $v_1, \dots, v_k$ to a basis $v_1, \dots, v_n$ of V.

Let $v_1^*, \dots, v_n^*$ be the dual basis.

Let $\bar{\alpha} := v_1^* \wedge \dots \wedge v_k^*$.

Then $\bar{\alpha}(v_1, \dots, v_k) = \det(I) = 1 \neq 0$

prev. lecture

Step 3 Take $(v_1, \dots, v_k) \sim (\tilde{v}_1, \dots, \tilde{v}_k)$ and assume (by contrad.) that

$W := \mathrm{span}\{v_1, \dots, v_k\} \neq \mathrm{span}\{\tilde{v}_1, \dots, \tilde{v}_k\}$

Then in the constr. of $v_{k+1}, \dots, v_n$ above, I can assume that $v_{k+1} = \tilde{v}_j$ $\overset{\parallel}{\tilde{v}_1}$

for j (let's say $j=1$).

Then $\underset{\overset{\parallel}{v_1^* \wedge \dots \wedge v_k^*}}{\bar{\alpha}} (\tilde{v}_1, \dots, \tilde{v}_k) = \det (M_{(1,\dots,k)})$

$= \det((0, \dots)) = 0$

where M = matrix of the coeff. of $\tilde{v}_1, \dots, \tilde{v}_k$ wrt $v_1, \dots, v_n$.

But recall that $\bar{\alpha}(v_1, \dots, v_k) = 1 \neq 0$

Contradiction!

We have proved $\mathrm{Span}(\{v_j\}) = \mathrm{span}(\{\tilde{v}_j\})$.

as in step 2

Step 4 $\quad \bar{\alpha}(v_1, \dots, v_k) = 1$

by ass. $\rightarrow$ $\parallel$ $\qquad \Longrightarrow \det M = 1$

$\bar{\alpha}(\tilde{v}_1, \dots, \tilde{v}_k) = \det M$ $\quad$ ((ii) is proved)

by last lecture

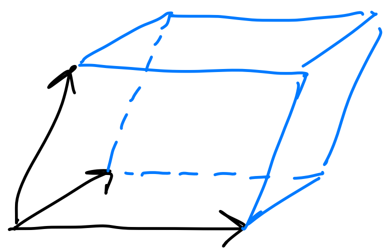
Step 5 By ass. $\bar{\alpha}(v_1,\dots,v_k) = \bar{\alpha}(\tilde{v}_1,\dots,\tilde{v}_k)$ and the same holds for all $\alpha \in \Lambda^k(V)$ because the restriction of α to W is a multiple of (the restr. of) $\bar{\alpha}$ (to W). (because $\Lambda^k(W)$ has dimension 1).

Hence $(v_1,\dots,v_k) \sim (\tilde{v}_1,\dots,\tilde{v}_k)$!

$\square$

Given $v_1,\dots,v_k \in V$, let

$R(v_1,\dots,v_k) :=$ " ~~rectangle~~ parallelepiped (?) spanned by $v_1,\dots,v_k$ "



$$= \left\{ \sum_{j=1}^{k} \lambda_j v_j \;\middle|\; \lambda_j \in [0,1] \;\forall j \right\}$$

Take $\tilde{v}_1,\dots,\tilde{v}_k$ s.t. $(\tilde{v}_1,\dots,\tilde{v}_k) \sim (v_1,\dots,v_k) \neq \phi$. Then

$$R(\tilde{v}_1,\dots,\tilde{v}_k) = T(R(v_1,\dots,v_k))$$

where $T : W \longrightarrow W$ is given, $W = \mathrm{Span}\{v_1,\dots,v_k\} = \mathrm{Span}\{\tilde{v}_1,\dots,\tilde{v}_k\}$,

by $T : v_j \longrightarrow \tilde{v}_j \;\forall j$.

Note that the matrix associated to T w.r.t. the basis $v_1, \dots, v_k$ is the usual M and $\det M = 1$

Assume now that V is endowed with a scalar product $(\cdot)$

Then

$$\mathcal{H}^k(R(\tilde{v}_1, \dots, \tilde{v}_k)) = \mathcal{H}^k(R(v_1, \dots, v_k)) \qquad (*)$$

$\Big($ choose $e_1, \dots, e_k$ orthon. basis of W

$\dots\dots$ $\Big)$

We call "norm" of $[v_1, \dots, v_k]$ the value

$$|[v_1, \dots, v_k]| := \mathcal{H}^k(R(v_1, \dots, v_k))$$

This def. is well-posed because of $(*)$. The reason to call it a norm will be clear later.

Recall: an orientation of V is an equivalence class of basis w.r.t. the equiv. relation $\approx$ where $(v_1,\dots,v_k) \approx (\tilde v_1,\dots,\tilde v_k)$ means that the change-of-basis matrix M has $\det M > 0$. (not that $\sim$ if finer that $\approx$)

We have defined

$$\begin{array}{ccc} [v_1,\dots,v_k] & \longmapsto & \mathrm{Span}(v_1,\dots,v_k) \\ \cap & & \cap \\ \{\text{unitary simple vectors}\} & & Gr_{or}(V,k) \end{array}$$

that is $\mathcal{H}^k(R(v_1,\dots,v_k)) = 1$

Grassmannian of k dimensional oriented subspaces of V

Prop 6 This map is bijective

Proof: trivial ....

Differential forms

A k-form on an open set $\Omega \subset \mathbb{R}^n$ is a "map"

$$\omega : \underset{\Omega}{\underset{\cap}{x}} \longmapsto \omega(x) \in \Lambda^k(\mathbb{R}^n)$$

In coordinates

$$\omega(x) := \sum_{\underline{i} \in I_{n,k}} \omega_{\underline{i}}(x)\, dx_{\underline{i}}$$

<u>Note</u> ω is of class C^k means that all $\omega_{\underline{i}}$ are of class C^k....

Exterior derivative (or differential)

If ω is of class C^1, $d\omega$ is the $(k+1)$-form given by

$$d\omega(x) := \sum_{j=1}^{n} dx_j \wedge \frac{\partial \omega}{\partial x_j}(x)$$

$$= \sum_{\underline{i} \in I_{n,k}} \boxed{\sum_{j=1}^{n} \frac{\partial \omega_{\underline{i}}}{\partial x_j}(x)\, dx_j} \wedge dx_{\underline{i}}$$

$$= \sum_{\underline{i} \in I_{n,k}} d\omega_{\underline{i}}(x) \wedge dx_{\underline{i}}$$

(there is also an intrinsic def. that does not depend on the choice of the basis).

Orientation of a surface

Let Σ be a k-dim. surface of class C^1 of $\mathbb{R}^n$.

submanifold with bdry

An orientation of Σ is

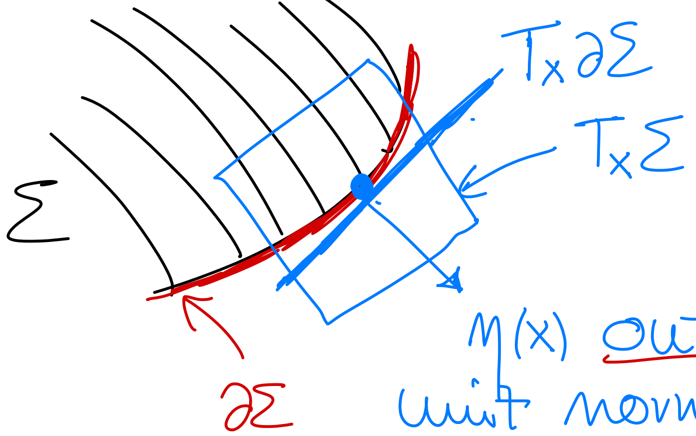
~~" a continuous choice of a orientation for each $T_x\Sigma$ "~~

topological space

a continuous map $\tau : \Sigma \longrightarrow \{\text{simple } k\text{-vect}\}$

s.t. $\tau(x)$ is unitary and spans $T_x\Sigma$

Orientation of the boundary



If Σ is oriented by τ, we orient $\partial\Sigma$ by τ' s.t. $\forall x \in \partial\Sigma$

$$[\eta(x), \tau_1'(x), \dots, \tau_{k-1}'(x)] = [\tau_1(x), \dots, \tau_k(x)]$$

Integration of forms on ORIENTED surf.

Let Σ be a k-dim. surface of class C^1 in $\mathbb{R}^n$, let ω be a k-form defined on (an open neighbourhood of) Σ. Then define

with orientation τ

$$\int_\Sigma \omega := \int_\Sigma \underbrace{\langle \omega(x); \tau(x) \rangle}_{\text{action of } \omega(x)\in\Lambda^k(\mathbb{R}^n) \text{ on } \tau_1(x),\dots,\tau_k(x)} d\mathcal{H}^k(x)$$

$$\| $$

$$\int_D \left\langle \omega(\phi(s)); \frac{\partial\phi}{\partial s_1}, \dots, \frac{\partial\phi}{\partial s_k} \right\rangle ds$$

> if $\phi : D \to \Sigma$ is a parametrization of Σ that preserves orientation ($D \subset \mathbb{R}^k$)

compare with the def. of $\int_\gamma \omega$

(this definition makes sense if, for example, $\omega \in L^1(\mathcal{H}^k \llcorner \Sigma)$)

Stokes Theorem

Let Σ be a compact oriented k-surface of class C^1 in $\mathbb{R}^n$, let ω be a $(k-1)$-form of class C^1 (defined in a neighb. of Σ)

Then
$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega$$

(It is enough Σ is closed and ω has compact support)