GMT 19/20, lecture 11, 23/4/20

Self-similar fractals (continued)

<u>Setting:</u>

<u>Given:</u> $\phi_1, \dots, \phi_N$ contractive similarities of $\mathbb{R}^n$,

$$\phi_i : x \longmapsto x_i + \lambda_i R_i x$$

with $x_i \in \mathbb{R}^n$, $R_i \in O(n)$, $\lambda_i \in (0,1)$

<u>Then</u> d is the sol. of

$$\sum_{i=1}^{N} \lambda_i^d = 1$$

<u>Theorem</u>

(i) $\exists!$ compact set $C \subset \mathbb{R}^n$ s.t.

$$C = \bigcup_{i=1}^{N} \phi_i(C)$$

(ii) $\mathcal{H}^d(C) < +\infty$

open set cond.

(iii) Assume $\exists$ V open s.t. $V \supset \phi_i(V)$ $\forall i$ & $\phi_i(V)$ are disjoint

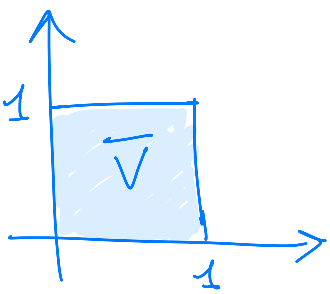
Then $\mathcal{H}^d(C) > 0$.

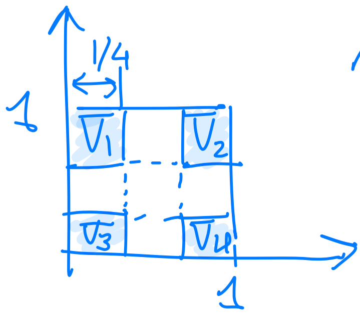
Proof of (i)

C is the (unique) fixed point of the contraction

$$\Phi : E \longmapsto \bigcup_{i=1}^{N} \phi_i(E)$$

Examples 1 $(n=2)$

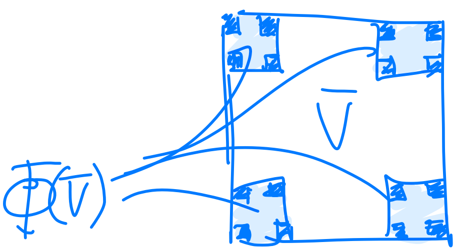




$\phi_1, \dots, \phi_4$ map $\bar V$ onto $\bar V_i$

thus $\lambda_i = \frac{1}{4}$

Then d solves $4 \cdot \frac{1}{4^d} = 1 \Longrightarrow d = 1$



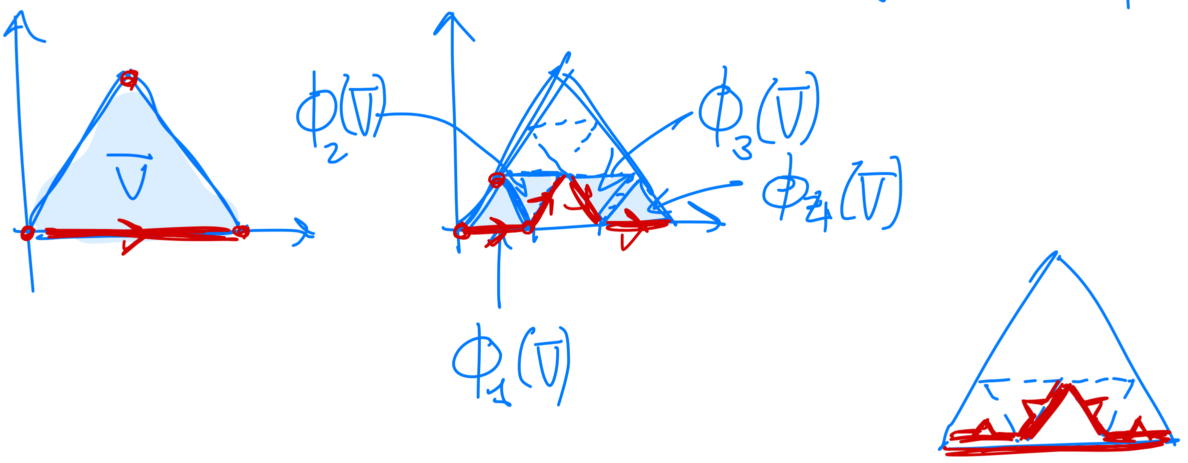
$\Phi^2(V)$

$\Phi^m(V)$ = union of 4^m squares with side $\frac{1}{4^m}$

$$C = \bigcap_{m=0}^{\infty} \Phi^m(V)$$

Example 2 $(n=2)$

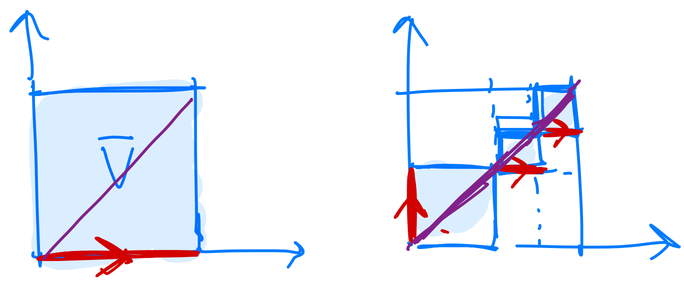
$\Phi_1, \dots, \Phi_4$



C is the von Koch curve!

d solves $\quad 4 \cdot \frac{1}{3^d} = 1 \Rightarrow d = \frac{\log 4}{\log 3}$

Example 3



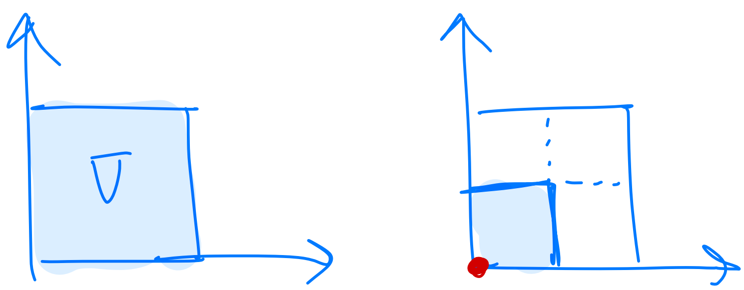
Φ_1, Φ_2, Φ_3

scaling factors

$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{4}$

d solves $\quad 1 = \frac{1}{2^d} + \frac{1}{4^d} + \frac{1}{4^d} \Rightarrow d = 1$

Example 4



$\Phi_1 = \Phi_2$ scaling factor $\frac{1}{2}$

$d = 1$

proof of (ii)

$$C = \bigcup_{i_1=1}^{N} \phi_{i_1}(C)$$

$$= \bigcup_{i_1=1}^{N} \phi_{i_1}\Big(\bigcup_{i_2=1}^{N} \phi_{i_2}(C)\Big)$$

$$= \bigcup_{1\le i_1, i_2\le N} \phi_{i_1}\circ\phi_{i_2}(C)$$

$$\vdots$$

$$= \bigcup_{\substack{\cancel{1\le i_1,\dots,i_m\le N}\\ \underline{i}\in I^m}} \underbrace{\overbrace{\phi_{i_1}\circ\phi_{i_2}\circ\cdots\circ\phi_{i_m}}^{\phi_{\underline{i}}}(C)}_{C_{\underline{i}}}$$

similarity with scal. fact. $\lambda_{\underline{i}} := \lambda_{i_1}\cdot\ldots\cdot\lambda_{i_m}$

$I := \{1,\dots,N\}$

$\underline{i} = (i_1,\dots,i_m)$

Note that $\mathrm{diam}(C_{\underline{i}}) = \lambda_{\underline{i}}\cdot \underbrace{\mathrm{diam}(C)}_{=R}$

$$\le \lambda_{max}^m R$$

$1 > \lambda_{max} := \max_{1\le i\le N} \lambda_i$

$$\mathcal{H}^d_{\delta}(C) \le \cancel{\frac{\alpha_d}{2^d}} \Big(\sum_{\underline{i}\in I^m} \lambda_{\underline{i}}^d\Big) R^d = \underbrace{\cancel{\Big(\sum_{i=1}^{N} \lambda_i^d\Big)^m}}_{=1} R^d$$

$\delta \ge \lambda_{MAX}^m R$

$$\Longrightarrow \mathcal{H}^d(C) \le R^d.$$

Proof of (iii) $(\mathcal{H}^d(C) > 0!)$

I construct a probability measure μ on C s.t. $\Theta^*_d(\mu, x) < +\infty \quad \forall x \in C$.

Instead of open set cond, I assume a stronger condition: $\phi_i(\overline{V})$ disjoint!!

this implies that $\phi_i(C)$ are disjoint! (Ex.). Take μ so that

$$(*)\begin{cases} \mu(C) = 1 \\ \mu(\phi_i(C)) = \lambda_i^d \\ \dots\dots \\ \mu(C_{\underline{i}}) = \lambda_{\underline{i}}^d \quad \forall m \ \forall \underline{i} \in I^m \\ \quad \| \\ \phi_{\underline{i}}(C) \end{cases}$$

How do I construct μ?

$\mu = \lim \mu_k$ where

$$\mu_0 = \delta_{\bar{x}} \quad (\bar{x} \in C)$$

$$\mu_1 = \sum_{i=1}^N \lambda_i^d \, \delta_{\phi_i(\bar{x})}$$

$$\vdots$$

$$\mu_m = \sum_{\underline{i} \in I^m} \lambda_{\underline{i}}^d \, \delta_{\phi_{\underline{i}}(\bar{x})}$$

NOTE THAT $\boxed{\mu_m(C_{\underline{i}}) = \lambda_{\underline{i}}^d}$ $\forall m' \leq m \quad \forall \underline{i} \in I^{m'}$

Ex: prove that μ_m converge (in the sense of measures on C) to a prob. measure μ with (*)

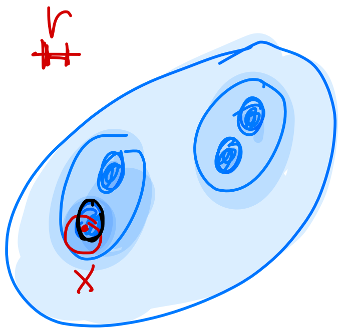
(use that $\mathbb{1}_{C_i}$ is continuous because C_i is open and closed in C).

Upper bound on $\Theta^*_d(\mu, x)$.

Fix $x \in C$.

$\exists \ \underline{i} \in I^{\mathbb{N}}$ s.t. $\forall m$ $x \in C_{\underline{i}_m}$ with

$\underline{i} = (i_1, i_2, \ldots\ldots)$ $\qquad$ $\underline{i}_m$ = truncation of $\underline{i}$ $= (i_1, \ldots, i_m)$



Fix $0 < \delta \leq \operatorname{diam} \phi_i(\overline{V})$, $\forall i$

& $\delta \leq \operatorname{dist}(\phi_i(\overline{V}), \phi_j(\overline{V}))$ $\forall i \neq j$

Fix $r > 0$ choose m s.t.

$$\underbrace{\delta \lambda_{\underline{i}_{m+1}}}_{= \delta \cdot \lambda_{i_1} \cdots \lambda_{i_{m+1}}} < r \leq \underbrace{\delta \lambda_{\underline{i}_m}}_{= \delta \cdot \lambda_{i_1} \cdots \lambda_{i_m}} \qquad \overline{B(x,r) \cap C} \subset C_{\underline{i}_m}$$

$$\mu\left(\overline{B(x,r)}\right) \leq \mu(C_{\underline{i}_m}) = (\lambda_{\underline{i}_m})^d$$

$$r \geq \delta \lambda_{\underline{i}_{m+1}} = \delta \lambda_{\underline{i}_m} \cdot \lambda_{i'_{m+1}}$$

$$\frac{\mu\left(\overline{B(x,r)}\right)}{r^d} \leq \frac{\cancel{(\lambda_{\underline{i}_m})^d}}{(\delta \cancel{\lambda_{\underline{i}_m}} \cdot \lambda_{i'_{m+1}})^d} \leq \frac{1}{(\delta \lambda_{min})^d}$$

$$\Rightarrow \quad \Theta^*_d(\mu, x) \leq \frac{1}{(\delta \lambda_{min})^d} \qquad \square$$

<u>Corollary</u> upper and lower bound on $\Theta^*_d(\mu, x)$, $\Theta_{*d}(\mu, x)$

$\Theta^*_d(C, x)$, $\Theta_{*d}(C, x)$

Haar measures

G topological group

$\forall y \in G$ let $\tau^{\ell}_y : x \longmapsto yx$

$\tau^{r}_y : x \longmapsto xy$

a meas. μ on G is left/right invariant if

$$\mu(E) = \mu(\tau^{\ell/r}_y(E)) \quad \forall \underset{\mathcal{B}(G)}{E \in} \ \forall y \in G$$

Examples

- Lebesgue measure on $\mathbb{R}^n$
- $\mathcal{H}^1$ on $S^1 \subset \mathbb{C}^*$
- $\mathcal{H}^3$ on $S^3 \subset \mathbb{H}^*$

Main Theorem

If G is compact then it admits a (unique!) invariant prob. measure μ

(If G is locally compact + ... then there exists a μ-invariant measure which is locally finite & unique up to a constant factor)