GMT 19/20 Lecture 9, 9/4/20

$\boxed{\text{Th 2 of prev. lect.}}$ metric space

Given E Borel in X, $\mathcal{H}^d(E) < +\infty$,
then

(a) $\Theta_d(E,x) = \Theta^*_d(E,x) = 0$ for $\mathcal{H}^d$-a.e. $x \notin E$

(b) $2^{-d} \leq \Theta^*_d(E,x)$ for $\mathcal{H}^d$-a.e. $x \in E$

(c) $\Theta^*_d(E,x) \leq \begin{cases} 1 & \text{if } X = \mathbb{R}^n \\ 5^d & \end{cases}$ for $\mathcal{H}^d$-a.e. $x \in E$

<u>Proof of (a)</u>

let $\mu := \mathcal{H}^d \llcorner E$ (finite measure!)

Fix $\delta > 0$ and let

$$E_\delta := \{ x \in E \text{ s.t. } \Theta^*_d(E,x) > \delta \}$$

Claim $\mathcal{H}^d(E_\delta) = 0$.

Fix A open $\supset E_\delta$. Let

$$\mathcal{F} := \left\{ B = \overline{B(x,r)} \;\middle|\; \begin{array}{l} x \in E_\delta,\ B \subset A,\ \& \\ \mathcal{H}^d(E \cap B) \geq \delta \alpha_d r^d \end{array} \right\}$$

$\mathcal{F}$ is a Besicovitch cover of E_δ

Take $\mathcal{G} \subset \mathcal{F}$ disjoint s.t. $\widehat{\mathcal{G}}$ covers E_δ

$$\mathcal{H}^d_\infty(E_\delta) \le \frac{\alpha_d}{2^d} \sum_{B\in\mathcal{G}} (\operatorname{diam}\widehat{B})^d$$

$$= \frac{5^d}{\delta} \sum_{B\in\mathcal{G}} \delta \alpha_d r^d$$

$$\le \frac{5^d}{\delta} \sum_{B\in\mathcal{G}} \mu(B) \le \frac{5^d}{\delta}\mu(A)$$

Take the inf. over all $A \supset E_\delta$

$$\mathcal{H}^d_\infty(E_\delta) \le \mu(E_\delta) = \mathcal{H}^d(\cancel{E_\delta \cap E}) = 0 \qquad \overset{\emptyset}{\shortparallel}$$

$\square$

Proof of (b) $\qquad \Theta^*_d(E,x) \ge \frac{1}{2^d}$ for $\mathcal{H}^d$-a.e. $x \in E$

$$\Uparrow$$

$\forall \lambda < \frac{1}{2^d}$ let $E_\lambda := \{x\in E \text{ s.t. } \Theta^*(E,x) < \lambda\}$

then $\mathcal{H}^d(E_\lambda) = 0$ $\qquad \mu(\overline{B(x,r)}) \le \lambda \alpha_d r^d$

$$\Uparrow$$

$\forall \lambda < \frac{1}{2^d}$ $\forall r_0 > 0$ let $\cancel{E_{\lambda,r_0}}$ $\tilde{E}$ $:= \left\{ x\in E : \cancel{\dfrac{\mathcal{H}^d(E\cap \overline{B(x,r)})}{\alpha_d r^d}} \le \lambda \right\}$

then $\mathcal{H}^d(E_{\lambda,r_0}) = 0$ $\qquad \forall r \le r_0$

Fix $\delta > 0$, $\delta \le r_0$,

For every $\varepsilon > 0$ $\exists$ $\{E_i\}$ count. cover of $\tilde{E}$

$$\mathcal{H}^d_\delta(\tilde{E}) + \varepsilon \geq \frac{\alpha_d}{2^d} \sum_i \big(\underbrace{\mathrm{diam}(E_i)}_{\overset{!!}{r_i}}\big)^d$$

$\forall i$ choose $x_i \in E_i \cap \tilde{E}$

then $E_i \subset \overline{B(x_i, r_i)}$

$$= \frac{\alpha_d}{2^d} \sum_i r_i^d$$

$$\geq \frac{1}{\lambda 2^d} \sum_i \mu\big(\overline{B(x_i, r_i)}\big)$$

$$\geq \frac{1}{\lambda 2^d} \mu(\tilde{E}) = \frac{1}{\lambda 2^d} \mathcal{H}^d(\tilde{E})$$

Take inf over $\varepsilon > 0$ and sup over $\delta > 0$

$$\mathcal{H}^d(\tilde{E}) \geq \underbrace{\frac{1}{\lambda 2^d}}_{\neq 1} \mathcal{H}^d(\tilde{E})$$

Then $\mathcal{H}^d(\tilde{E}) = 0$ $\square$

Proof of (c) first case, $X=\mathbb{R}^d$

$$\Theta^{*}_{d}(E,x)\leq 1 \quad \text{for } \mathcal{H}^d\text{-a.e. } x\in E$$

$$\Uparrow$$

let $m>1$, let $E_m:=\{x\in E \mid \Theta^*_d(E,x)>m\}$

then $\mathcal{H}^d(E_m)=0$.

Fix $\delta>0$, let $\mathcal{Y}:=\left\{ \overline{B(x,r)} \;\middle|\; x\in E_m,\ r\leq\delta \ \& \ \cancel{\frac{\mathcal{H}^d(\overline{B(x,r)}\cap E)}{\alpha_d r^d}} \geq m \right\}$

$\mathcal{Y}$ Besicov. cover of E_m $\qquad \mu(\overline{B(x,r)}) \geq m\alpha_d r^d$.

then $\forall \varepsilon>0$ $\exists$ $\mathcal{G}\subset\mathcal{Y}$ cover of E_m s.t.

$$\begin{aligned}
\mathcal{H}^d(E_m)+\varepsilon \geq \mu(E_m)+\varepsilon &\geq \sum_{B\in\mathcal{G}}\mu(B)\\
&\geq \sum_{B\in\mathcal{G}} m\alpha_d r^d\\
&= m\left[\frac{\alpha_d}{2^d}\sum_{B\in\mathcal{G}}(\operatorname{diam} B)^d\right]\\
&\geq m\,\mathcal{H}^d_{2\delta}(E_m)
\end{aligned}$$

Then $\mathcal{H}^d(E_m)\geq \underset{\vee\atop 1}{m}\,\mathcal{H}^d(E_m) \Rightarrow \mathcal{H}^d(E_m)=0.$

Proof of (c) Case 2

$\Theta^*_d(E,x) \leq 5^d$ for $\mathcal{H}^d$-a.e. $x \in E$.

$$\Uparrow$$

$\forall m > 5^d$ let $E_m := \{x \in E \mid \Theta^*_d(E,x) > m\}$

then $\mathcal{H}^d(E_m) = 0$

Fix A open, $A \supset E$, fix $\delta > 0$, let

$$\mathcal{F} := \left\{ \underset{\underset{B}{\shortparallel}}{\overline{B(x,r)}} \;\middle|\; \begin{array}{l} x \in E_m,\ r \leq \delta,\ \cancel{\dfrac{\mathcal{H}^d(E \cap \overline{B(x,r)})}{\alpha_d r^d}} \geq m \\ B \subset A \end{array} \right\}$$

$\mu(\overline{B(x,r)}) \geq m \alpha_d r^d$

Take $\mathcal{G} \subset \mathcal{F}$ disjoint s.t. $\hat{\mathcal{G}}$ covers E_m

Then

$$\mu(A) \geq \sum_{B \in \mathcal{G}} \mu(B) \geq \frac{m \alpha_d}{2^d} \sum_{B \in \mathcal{G}} \cancel{r^d} (\operatorname{diam} B)^d$$

$$\geq \frac{m}{5^d} \left[\frac{\alpha_d}{2^d} \sum_{B \in \mathcal{G}} (\operatorname{diam} \hat{B})^d \right]$$

$$\geq \frac{m}{5^d} \mathcal{H}^d_{10\delta}(E_m)$$

Then

$$\underset{\underset{\mathcal{H}^d(E_m)}{\shortparallel}}{\mu(A)} \geq \underset{\underset{1}{\vee}}{\frac{m}{5^d}} \mathcal{H}^d(E_m)$$

$\square$

Further applications of covering theorems

Lemma 1 μ, λ finite measures on X and assume μ doubling or $X = \mathbb{R}^n$

If $\lambda \perp \mu$ then $\frac{d\lambda}{d\mu}(x) := \lim_{r \to 0} \frac{\lambda(\overline{B(x,r)})}{\mu(\overline{B(x,r)})} = 0$ for μ-a.e. x

Corollary 2 If X, μ, λ are as above then $\frac{d\lambda}{d\mu}(x) = 0$ μ.a.e. $\frac{d\lambda}{d\mu}(x) = +\infty$ λ-a.e.

Theorem 3 Assume μ doubling or $X = \mathbb{R}^n$

Take $f \in L^p(\mu)$, $1 \leq p < +\infty$, then

$$\fint_{B(\bar{x},r)} |f(x) - f(\bar{x})|^p \, d\mu(x) \xrightarrow[r \to 0]{} 0 \quad \text{for } \mu\text{-a.e. } \bar{x}$$

Corollary 4 Take X, μ, f as above.

Then

$$\fint_{\overline{B(\bar{x},r)}} f(x) \, d\mu(x) \xrightarrow[r \to 0]{} f(\bar{x}) \quad \text{for } \mu\text{-a.e. } \bar{x}$$

Corollary 5 Take λ, μ s.t. μ doubling or $X = \mathbb{R}^n$

Let $\lambda = f\mu + \lambda_s$ then $\frac{d\lambda_a}{d\mu} = \frac{d\lambda}{d\mu}(x) = f(x)$ for μ-a.e. x

Proof of lemma 1 $\quad \frac{d\lambda}{d\mu} = 0 \quad \mu\text{-a.e.}$

Since $\lambda \perp \mu$ $\exists$ F s.t. $\mu(X \setminus F) = 0$ & $\lambda(F) = 0$

let $m > 0$ and let

$$E_m := \left\{ x \in F \;\middle|\; \limsup_{r \to 0} \frac{\lambda(\overline{B(x,r)})}{\mu(\overline{B(x,r)})} > m \right\}$$

$\boxed{\text{Claim } \mu(E_m).}$

Fix A open $A \supset E_m$ and let

$$\mathcal{F} = \left\{ \underbrace{\overline{B(x,r)}}_{= B} \;\middle|\; \cancel{x \in E_m},\ B \subset A,\ \& \ \boxed{\lambda(\overline{B(x,r)}) \geq m\, \mu(\overline{B(x,r)})} \right\}$$

$\mathcal{F}$ is a Besicovitch cover of E_m

Then $\forall \varepsilon > 0$ $\exists$ $\mathcal{G} \subset \mathcal{F}$ disjoint that covers μ-a.e. of E_m. Then

$$\mu(E_m) \leq \sum_{B \in \mathcal{G}} \mu(B) \leq \frac{1}{m} \sum_{B \in \mathcal{G}} \lambda(B) \leq \frac{1}{m} \lambda(A)$$

Then $\quad \mu(E_m) \leq \frac{1}{m} \lambda(E_m) \leq \frac{1}{m} \lambda(F) = 0$

$\square$

Proof of Th. 3

$$\fint_{B(\bar{x},r)} |f(x)-f(\bar{x})|^p \, d\mu(x) \xrightarrow[r\to 0]{} 0 \quad (*)$$

for μ-a.e. $\bar{x}$.

Fix $\varepsilon > 0$. By Lusin theorem $\exists \tilde{f}$ continuous and E s.t. $\mu(E^c) \le \varepsilon$ s.t. $f = \tilde{f}$ on E.

Then $(*)$ holds for μ a.e. $\bar{x} \in E$.

Fix $\bar{x} \in E$

$$\fint_{\underset{B}{\overset{\shortparallel}{B(\bar{x},r)}}} |f(x)-f(\bar{x})|^p \, d\mu(x) =$$

$$= \frac{1}{\mu(B)} \int_{B\cap E} |\underset{\tilde{f}(x)}{\underbrace{f(x)}} - \underset{\tilde{f}(\bar{x})}{\underbrace{f(\bar{x})}}|^p d\mu + \frac{1}{\mu(B)} \int_{B\setminus E} |\cdots|^p d\mu)$$

$|a-b|^p \le 2^{p-1}(|a|^p+|b|^p)$

$$\le \left(\operatorname*{osc}_B \tilde{f}\right)^p + \frac{2^{p-1}}{\mu(B)} \int_{B\setminus E} |f(\bar{x})|^p + |f(x)|^p \, d\mu(x)$$

$\lambda := |f|^p \mu \llcorner E^c$

$$\le \left(\operatorname*{osc}_B \tilde{f}\right)^p + 2^{p-1}|f(\bar{x})|^p \cdot \frac{\mu(B\setminus E)}{\mu(B)} + 2^{p-1}\frac{\lambda(B)}{\mu(B)}$$

$\tilde{\mu} := \mu \llcorner E$

$$= \left(\operatorname*{osc}_B \tilde{f}\right)^p + 2^{p-1}|f(\bar{x})|^p \frac{\mu(B\setminus E)}{\mu(B)} + 2^{p-1}\frac{\lambda(B)}{\tilde{\mu}(B)} \cdot \frac{\mu(B\cap E)}{\mu(B)}$$

$r \to 0$: $\left(\operatorname*{osc}_B \tilde{f}\right)^p \to 0$, $\quad \frac{\mu(B\setminus E)}{\mu(B)} \to \Theta_\mu(E^c,\bar{x})$, $\quad \frac{\lambda(B)}{\tilde{\mu}(B)} \to 0$, $\quad \frac{\mu(B\cap E)}{\mu(B)} \to \Theta(E,\bar{x}) = 1$

for μ a.e. $\bar{x} \in E$

$\square$