GMT 19/20, lecture 7, 2/4/20

Covering theorems a la Besicovitch

We are in $\mathbb{R}^n$!!

μ is (locally finite) measure on $\mathbb{R}^n$

Th. 1 Let $E \subset \mathbb{R}^n$, $\mu(E) < +\infty$.

Let $\mathcal{F}$ be a family of closed balls s.t.

$\mathcal{F}$ is a Besicovitch cover of E | $\forall x \in E \quad \inf \{ r \mid \overline{B(x,r)} \in \mathcal{F} \} = 0$

Then $\forall \varepsilon > 0 \ \exists \mathcal{F}' \subset \mathcal{F}$ s.t.

- $\mathcal{F}'$ is disjoint and covers μ-a.a. of E;
- $\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$

Th 2 Let E, $\mathcal{F}$ as in Th. 1.

Then $\forall \varepsilon > 0 \ \exists \mathcal{F}' \subset \mathcal{F}$ s.t.

- $\mathcal{F}'$ covers E;
- $\sum_{B \in \mathcal{F}'} \mu(B) \leq \mu(E) + \varepsilon$.

Lemma 1 (Besicovitch's covering theorem)

Let $\mathcal{F}$ be a family of balls in $\mathbb{R}^d$ with bounded radii.

Let E be the set of centers of balls in $\mathcal{F}$.

Then $\exists\ \mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}$ s.t.

- each $\mathcal{G}_i$ is disjoint
- $\mathcal{G} := \bigcup_{i=1}^{N} \mathcal{G}_i$ covers E.

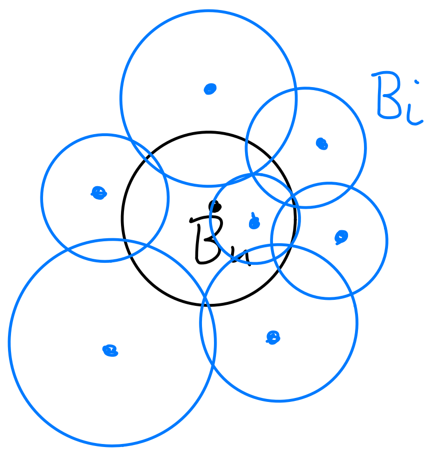
Where N depends ONLY on d.

Lemma 0 (not proved)

$B_i = B(x_i, r_i)$

Let $B_1, \dots, B_n$ balls in $\mathbb{R}^d$ s.t.

- $B_i \cap B_n \neq \emptyset \quad \forall i < n$
- $r_i \geq \frac{r_n}{2} \quad \forall i < n$
- $x_i \notin B_j \quad \forall i \neq j < n$



Then $n \leq N(d)$

depends only on d.

Proof of lemma 1

$\boxed{\text{Case 1}}$ $\forall B, B' \in \mathcal{F}$ $\quad rad(B') \leq 2\, rad(B)$

Take $(\mathcal{G}_1, \dots, \mathcal{G}_N)$ from lemma 0

- each $\mathcal{G}_i \subset \mathcal{F}$ and disjoint
- $\forall B = B(x,r) \in \mathcal{G}_i$ then
  $x \notin B' \quad \forall B' \in \mathcal{G}_j,\ j \neq i$
- $(\mathcal{G}_1, \dots, \mathcal{G}_N)$ is maximal w.r.t. the partial order
  $$(\mathcal{G}_1', \dots, \mathcal{G}_N') \leqslant (\mathcal{G}_1, \dots, \mathcal{G}_N)$$
  defined by
  $$\mathcal{G}_1' \subseteq \mathcal{G}_1, \dots, \mathcal{G}_N' \subseteq \mathcal{G}_N.$$

(existence follows by Zorn's lemma)

Take now $B = B(x,r) \in \mathcal{F}$.

Assume that $x \notin \bigcup_{i=1}^{N} \left(\bigcup_{B \in \mathcal{G}_i} B \right)$.

by contrad.

Then by maximality $\forall i\ \exists B_i \in \mathcal{G}_i$ s.t. $B \cap B_i \neq \emptyset$. But then $\{B_1, \dots, B_N, B\}$ contradicts lemma 0.

Case 2 General case. (Let $R := \sup_{B \in \mathcal{F}} \mathrm{rad}(B)$)

Divide $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots$

where $\mathcal{F}_k := \left\{ B(x,r) \in \mathcal{F} \,\middle|\, \frac{R}{2^{k+1}} < r \le \frac{R}{2^k} \right\}$

Now, extract $\mathcal{G}_{0,1}, \dots, \mathcal{G}_{0,N}$ $\boxed{\text{as in case 1}}$ from $\mathcal{F}_0$

Let $\mathcal{F}_1' := \{$ balls in $\mathcal{F}_1$ whose centers are not contained in $\bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{G}_{0,i}} B$ $\}$

Extract $\mathcal{G}_{1,1}, \dots, \mathcal{G}_{1,N}$ from $\mathcal{F}_1'$ proceed as in case 1 (to be fixed!!)

and so on....

Let $\mathcal{G}_i := \mathcal{G}_{0,i} \cup \mathcal{G}_{1,i} \cup \dots \dots \quad \forall i.$

$\square$

Lemma 2 Let E be set with $\mu(E)<+\infty$. Let $\mathcal{F}$ be a family of balls with bounded radii whose centers cover E.
Then $\exists\, \mathcal{G} \subset \mathcal{F}$, $\mathcal{G}$ disjoint, s.t.

$$\mu\Big(E \cap \big(\bigcup_{B\in\mathcal{G}} B\big)\Big) \geq \frac{1}{N}\mu(E)$$

as in Lemma 1

Proof Apply Lemma 1 to $\mathcal{F}$ and obtain $\mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}$, $\mathcal{G}_i$ disjoint s.t.

$$\bigcup_{i=1}^{N} \bigcup_{B\in\mathcal{G}_i} B \supset E$$

then

$$\mu(E) = \mu\Big(\bigcup_{i=1}^{N}\big(E \cap \bigcup_{B\in\mathcal{G}_i} B\big)\Big) \leq \sum_{i=1}^{N} \mu\Big(E\cap \bigcup_{B\in\mathcal{G}_i} B\Big)$$

$\Rightarrow \exists\, \underline{i}$ s.t. $\frac{\mu(E)}{N} \leq \mu\Big(E\cap \bigcup_{B\in\mathcal{G}_{\underline{i}}} B\Big)$

Let $\mathcal{G} := \mathcal{G}_{\underline{i}}$. $\square$

Proof of Th. 1

Fix A open s.t. $A \supset E$, $\mu(E) \le \mu(E) + \varepsilon$.

Step 1. Let $\mathcal{F}_0 := \{ B \in \mathcal{F} \mid B \subset A_0,\ \mathrm{rad}(B) \le 1 \}$

Choose $\mathcal{G}_0 \subset \mathcal{F}_0$ as in Lemma 2.

Then
$$\mu\Big(E \cap \bigcup_{B \in \mathcal{G}_0} B\Big) \geq \frac{\mu(E)}{N}$$

Take $\mathcal{G}_0' \subset \mathcal{G}_0$ finite s.t.
$$\mu\Big(E \cap \bigcup_{B \in \mathcal{G}_0'} B\Big) \geq \frac{\mu(E)}{2N}$$

then
$$\mu\Big(\underbrace{E \setminus \bigcup_{B \in \mathcal{G}_0'} B}_{E_1}\Big) \leq \Big(1 - \frac{1}{2N}\Big)\mu(E)$$

Step 2. Let $\mathcal{F}_1 := \{ B \in \mathcal{F} \mid B \subset A,\ B \cap \Big(\bigcup_{B' \in \mathcal{G}_0'} B'\Big) = \emptyset \}$ closed set

Then $\mathcal{F}_1$ is a Besicovitch cover of E_1

Take $\mathcal{G}_1' \subset \mathcal{F}_1$ finite s.t.
$$\mu\Big(E_1 \setminus \bigcup_{B \in \mathcal{G}_1'} B\Big) \leq \Big(1 - \frac{1}{2N}\Big)^2 \mu(E).$$

and so on ....

Finally let $\mathcal{F}' := \mathcal{G}_0' \cup \mathcal{G}_1' \cup \mathcal{G}_2' \dots$ $\square$

<u>Lemma 3</u> Let E_0 be s.t. $\mu(E_0) = 0$.
Let $\mathcal{F}$ be a Besicovitch cover of E_0
$\forall \varepsilon > 0$ Then $\exists\, \mathcal{G} \subset \mathcal{F}$ s.t. $\mathcal{G}$ covers E_0
& $\sum_{B \in \mathcal{G}} \mu(B) \leq \varepsilon$.

<u>Proof</u> Choose A open s.t. $A \supset E_0$
& $\mu(A) \leq \frac{\varepsilon}{N}$. as in lemma 1

Let $\mathcal{F}' := \{ B \in \mathcal{F} \mid B \subset A \}$
Use lemma 1 to extract from $\mathcal{F}'$
$\mathcal{G}_1, \dots, \mathcal{G}_N$ s.t. $\mathcal{G}_i$ disjoint
& $\mathcal{G} := \bigcup_i \mathcal{G}_i$ covers E_0.

<u>Then</u> $\forall i$ $\sum_{B \in \mathcal{G}_i} \mu(B) = \mu\left(\bigcup_{B \in \mathcal{G}_i} B \right) \leq \mu(A) \leq \frac{\varepsilon}{N}$

$$\Rightarrow \sum_{B \in \mathcal{G}} \mu(B) \leq N \cdot \frac{\varepsilon}{N} = \varepsilon. \qquad \square$$

Applications of covering theorems

1 $\mathcal{H}^d = \mathcal{H}^d_\delta = \mathcal{L}^d$ on $\mathbb{R}^d$ $\forall \delta \in [0,+\infty]$

Step 1 $\boxed{\mathcal{L}^d(E) \geq \mathcal{H}^d_\delta(E)}$ $\forall \delta > 0.$

Step 2 $\mathcal{H}^d(E) \geq \mathcal{H}^d_\delta(E) \geq \boxed{\mathcal{H}^d_\infty(E) \geq \mathcal{L}^d(E)}$

Proof of Step 1

Use Theorem 2 (of any of the last two lectures) to find balls closed B_i

s.t. $\bigcup B_i \supset E$ & $\sum_i \mathcal{L}^d(B_i) \leq \mathcal{L}^d(E) + \varepsilon$

diam $(B_i) \leq \delta$

then $\varepsilon + \mathcal{L}^d(E) \geq \sum_i \mathcal{L}^d(B_i)$

$$= \sum_i \frac{\alpha_d}{2^d} (\operatorname{diam} B_i)^d$$

$$= c_d \sum_i (\operatorname{diam} B_i)^d$$

$$\geq \mathcal{H}^d_\delta(E)$$