GMT 19/20, lecture 3, 19/3/20 [version: 28/4/2020]

Review of Basic Measure Theory (continued).

2.2 Vector-valued (Borel) measures

Let be given:

- a positive (Borel) measure μ on X,
- a finite dimensional normed space F,
- a map $\rho : X \to F$ in $L^1(\mu)$.

We denote by $\rho\mu$ the F-valued measure on X defined by

$$[\rho\mu](E) := \int_E \rho \, d\mu \, .$$

<u>Theorem 1</u> Every F-valued Borel measure λ on X can be represented as $\lambda = \rho\mu$ for suitable ρ, μ as above.

<u>Notation</u> Given $\lambda = \rho\mu$ as above, we define

(a) the variation $|\lambda| := |\rho|_F \cdot \mu$ (a positive measure on X);

(b) the mass $\mathbb{M}(\lambda) = \|\lambda\| := |\lambda|(X) = \int_X |\rho|_F \, d\mu = \|\rho\|_{L^1(\mu)}$.

(c) $\mathcal{M}(X,F)$ the space of all F-valued measures on X.

We write $\mathcal{M}(X)$ for $\mathcal{M}(X,\mathbb{R})$.

The representation $\lambda = \rho\mu$ is not unique, but $|\lambda|$ and $\mathbb{M}(\lambda)$ do NOT depend on the choice of the representation.

Theorem 1 does not hold if F is infinite dimensional.

<u>Proposition 2</u> The space $\mathcal{M}(X,F)$ endowed with the norm $\mathbb{M}(\lambda) = \|\lambda\|$ is a Banach space.

Riesz Theorem

Given $\lambda = \rho\mu \in \mathcal{M}(X,F)$ and a bounded (Borel) function $g : X \to F^*$ set

dual of F

$$T_\lambda(g) := \int_X g\, d\lambda := \int_X \underbrace{\langle g(x) ; \rho(x) \rangle}_{\text{action of } g(x) \in F^* \text{ on } \rho(x) \in F} d\mu(x)$$

Assume now that X is compact:

Proposition 3 T_λ is a bounded linear functional on $\mathcal{C}(X,F^*)$ and $\|T_\lambda\| = \mathbb{M}(\lambda)$.

this means $\sup\limits_{\|g\|_\infty \leq 1} \int_X \langle g ; \rho \rangle \, d\mu = \|\rho\|_{L^1(\mu)}$

space of continuous functions from X to F^* with supremum norm $\|\cdot\|_\infty$

Theorem 4 (Riesz Th.) The operator $\lambda \mapsto T_\lambda$ is a linear isometry from $\mathcal{M}(X,F)$ onto the dual of $\mathcal{C}(X,F^*)$.

The nontrivial (and fundamental) part is the surjectivity. The rest of the statement is contained in Proposition 3.

If X is locally compact we only consider g in $\mathcal{C}_0(X,F^*)$. That is, the space of all continuous $g : X \to F^*$ s.t. $\boxed{\lim\limits_{x\to\infty} g(x) = 0}$ endowed with the supremum norm.

This limit is understood in the sense of the one-point (or Alexandrov) compactification of X; it means that $\forall \varepsilon > 0$ $\exists K$ compact in X s.t. $|g(x)|_{F^*} \leq \varepsilon$ $\forall x \in X \setminus K$.

Theorem 5 (Riesz Th.) The operator $\lambda \mapsto T_\lambda$ is a linear isometry from $\mathcal{M}(X,F)$ onto the dual of $\mathcal{C}_0(X,F^*)$.

Weak * convergence of measures

$\boxed{2.2.1}$ Weak * topology and convergence

The identification of $\mathcal{M}(X,F)$ and the dual of $\mathcal{C}(X,F^*)$ (or $\mathcal{C}_0(X,F^*)$) induces a weak* topology on $\mathcal{M}(X,F)$, and a sequence of measures λ_n converge to λ iff

$$\int_X g \, d\lambda_n \longrightarrow \int_X g \, d\lambda \quad \text{for every } g \in \mathcal{C}(X,F^*).$$

We say that "λ_n converge to λ in the sense of measures" and write $\lambda_n \xrightarrow{*} \lambda$.

This topology is not metrizable, but its restriction to any any bounded subset of $\mathcal{M}(X,F)$ is metrizable.

In practice, this is the most relevant topology on $\mathcal{M}(X,F)$, much more than the norm topology.

$\boxed{2.2.2}$ Compactness

If λ_n is a bounded sequence in $\mathcal{M}(X,F)$, that is, $\mathbb{M}(\lambda_n) \leqslant C < +\infty$, then, up to subsequence, λ_n converge weakly* to some λ.

This is an immediate consequence of Banach-Alaoglu theorem.

$\boxed{2.2.3}$ Lower semicontinuity of the mass

If $\lambda_n \xrightarrow{*} \lambda$ in $\mathcal{M}(X,F)$ then $\liminf_{n\to\infty} \mathbb{M}(\lambda_n) \geqslant \mathbb{M}(\lambda)$.

This is an immediate consequence of the weak* semicontinuity of the norm of the dual. Or (equivalently) of the fact that the mass can be written as sup of weakly* continuous linear functionals:

$$\mathbb{M}(\lambda) = \sup\left\{ T_\lambda g \; : \; g \in \mathcal{C}(X,F^*), \, \|g\|_\infty \leqslant 1 \right\}.$$

2.2.4 Weak* convergence and non-continuous test functions. I

Assume that X is compact, μ_n are positive finite measures, $\mu_n \overset{*}{\rightharpoonup} \mu$, and $g : X \to \mathbb{R}$ is Borel but NOT continuous, e.g. $g = 1_E$.

What can we say about $\int_X g \, d\mu_n$?

(a) g bounded below and l.s.c. $\Rightarrow$ $\liminf\limits_{n\to\infty} \int_X g \, d\mu_n \geqslant \int_X g \, d\mu$;

in particular A open in X $\Rightarrow$ $\liminf\limits_{n\to\infty} \mu_n(A) \geqslant \mu(A)$;

(b) g bounded above and u.s.c. $\Rightarrow$ $\liminf\limits_{n\to\infty} \int_X g \, d\mu_n \geqslant \int_X g \, d\mu$;

in particular C closed in X $\Rightarrow$ $\limsup\limits_{n\to\infty} \mu_n(C) \leqslant \mu(C)$;

(c) g bounded and $\mu(\underbrace{\mathrm{Sing}(g)}_{\text{discontinuity set of } g}) = 0$ $\Rightarrow$ $\lim\limits_{n\to\infty} \int_X g \, d\mu_n = \int_X g \, d\mu$;

in particular E Borel and $\mu(\partial E) = 0$ $\Rightarrow$ $\lim\limits_{n\to\infty} \mu_n(E) = \mu(E)$.

Sketch of proof

(a) Take $g_n \in \mathcal{E}(X)$ s.t. $g_n(x) \uparrow g(x)$ $\forall x \in X$.
(Such g_n exist because g is bounded below and l.s.c.)

Let $\Lambda_n(\mu) := \int_X g_n \, d\mu$ and $\Lambda(\mu) := \int_X g \, d\mu$ $\forall \mu \in \mathcal{M}^+(X)$.

class of positive finite meas. on X

Then Λ_n is (weak*) continuous on $\mathcal{M}^+$ and $\Lambda_n \uparrow \Lambda$.

Hence Λ is (weak*) l.s.c. on $\mathcal{M}^+$.

(b) Apply (a) to $-g$.

(c) $\forall x \in X$ set

g^* and g_* are the upper and lower semicontinuous envelopes of g

$$g^*(x) := \limsup_{y \to x} g(y) , \quad g_*(x) := \liminf_{y\to x} g(y) .$$

Then

(1) g^* is bounded and u.s.c.;

(2) g_* is bounded and l.s.c.;

(3) $g_* \leqslant g \leqslant g^*$ on X;

(4) $g_* = g = g^*$ on $X \setminus \mathrm{Sing}(g)$.

Then

$$\limsup_{n\to\infty} \int_X g(x)\, d\mu \overset{(3)}{\leqslant} \limsup_{n\to\infty} \int_X g^*(x)\, d\mu$$

$$\xrightarrow{(1)+(a)} \leqslant \int_X g^* d\mu$$

$$(4) + \mu(\mathrm{Sing}(g))=0 \quad = \int_X g\, d\mu$$

$$= \int_X g_* d\mu \underset{(2)+(b)}{\leqslant} \liminf_{n\to\infty} \int_X g_*(x)\, d\mu \underset{(3)}{\leqslant} \liminf_{n\to\infty} \int_X g(x)\, d\mu \qquad \square$$

$\boxed{2.2.5}$ Weak* convergence and non-continuous test functions. II

In the setting of § 2.2.4, assume that X is locally compact. Then statements (a)-(c) hold under the additional assumption

$$(*) \qquad \lim_{n\to\infty} M(\mu_n) = M(\mu).$$

If (*) does not hold the assumptions in statements (a)-(c) should be modified as follows:

(a) g is l.s.c. and $\liminf_{x\to\infty} g(x) \geqslant 0$; A is open;

(b) g is u.s.c. and $\limsup_{x\to\infty} g(x) \leqslant 0$; C is compact;

(c) $\mu(\mathrm{Sing}(g)) = 0$ and $\lim_{x\to\infty} g(x) = 0$; E is compact and $\mu(\partial E) = 0$.

$\boxed{2.2.6}$ Weak* convergence and non-continuous test functions. III

Let λ_n be $\mathbb{F}$-valued measures on X such that $\lambda_n \xrightarrow{*} \lambda$ and $|\lambda_n| \xrightarrow{*} \mu$ (note that $|\lambda| \leq \mu$ but $=$ may not hold).
Let g be a function on X with values in $\mathbb{R}$ or $\mathbb{F}^*$.
Then

$$\lim_{n\to\infty} \int_X g \, d\lambda_n = \int_X g \, d\lambda$$

if ONE of the following sets of assumptions holds:

(a) X compact, g bounded, $\mu(\mathrm{Sing}(g)) = 0$;

(b) X locally compact, $M(\lambda_n) \to \mathbb{M}(\mu)$, g bounded, $\mu(\mathrm{Sing}(g)) = 0$;

(c) X locally compact, $\lim_{x\to\infty} |g(x)| = 0$, $\mu(\mathrm{Sing}(g)) = 0$.

Remarks

- Let $\mu_n \xrightarrow{*} \mu$ on X. If the measure μ_n are positive and X is compact then $\mathbb{M}(\mu_n) \to \mathbb{M}(\mu)$.
  Indeed $\mu \geq 0$ implies

$$\mathbb{M}(\mu) = \int_X 1 \, d\mu$$

and the RHS is a weak* continuous functional on $\mathcal{M}(X)$ because 1 is a function in $\mathcal{C}(X)$.

In general the conclusion may not hold if either the μ_n are not positive or X is not compact.

$\delta_x :=$ Dirac delta at $x \in X$

Example (a): $X := [0,1]$, $\mu_n := \delta_0 - \delta_{1/n}$; then $\mu_n \xrightarrow{*} 0$ and $\mathbb{M}(\mu_n) = 2$ $\forall n$.

Example (b): $X = \mathbb{R}$, $\mu_n := \delta_n$; then $\mu_n \xrightarrow{*} 0$ and $\mathbb{M}(\mu_n) = 1$ $\forall n$.

- Let $\lambda_n \xrightarrow{*} \lambda$ on X. If the measures λ_n are not positive then it may happen that $|\lambda_n| \cancel{\xrightarrow{*}} |\lambda|$.

  Example: $X := [0,1]$, $\lambda_n := \delta_0 - \delta_{1/n}$; then $\lambda_n \xrightarrow{*} 0$ and $|\lambda_n| = \delta_0 + \delta_{1/n} \xrightarrow{*} 2\delta_0$.

  However, if $\mathbb{M}(\lambda_n) \longrightarrow \mathbb{M}(\lambda)$ then $|\lambda_n| \xrightarrow{*} |\lambda|$.

2.3 Outer measures

Given a set X, an outer measure on X is a set function $\mu : \mathcal{P}(X) \longrightarrow [0,+\infty]$ (where $\mathcal{P}(X)$ is the power set of X) s.t. :

(a) $\mu(\emptyset) = 0$;

(b) $E \subset E' \Rightarrow \mu(E) \leqslant \mu(E')$ (monotonicity) ;

(c) $E \subset \bigcup_{n=1}^{\infty} E_n \Rightarrow \mu(E) \leqslant \sum_{1}^{\infty} \mu(E_n)$ (σ-subadditivity).

Note that (b) is implied by (c).

Examples

(1) counting measure: $\mu(E) := \# E$;

(2) Dirac delta at $x \in X$: $\delta_x(E) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$;

(3) $\mu(E) := \begin{cases} 1 & \text{if } E \neq \emptyset \\ 0 & \text{if } E = \emptyset \end{cases}$.

Definition

A set $E \subset X$ is μ-measurable (according to Caratheodory) if

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) \quad \forall F \subset X .$$

Equivalently: $\mu = \mu \llcorner E + \mu \llcorner (X \setminus E)$.

Note that $\mu \leqslant \mu \llcorner E + \mu \llcorner (X \setminus E)$ for every $E \subset X$ by subadditivity.

$\mathcal{M}_\mu$ denotes the class of all μ-measurable sets.

Proposition 1

The class $\mathcal{M}_\mu$ is a σ-algebra and the restriction of μ to $\mathcal{M}_\mu$ is σ-additive.

The proof is an exercise.

It is easy to see that $\mathcal{M}_\mu = \mathcal{P}(X)$ is μ is the counting measure on X or a Dirac delta.

What is $\mathcal{M}_\mu$ for the third example of outer measure given above?

Theorem 2 (Caratheodory)

If X is a metric space and μ is additve on distant sets, that is,

$$\mu(E \cup E') = \mu(E) + \mu(E') \quad \text{if} \quad \overbrace{\mathrm{dist}(E,E')}^{\inf\{d(x,x') : x \in E, x' \in E'\}} > 0,$$

then $\mathcal{M}_\mu$ contains the Borel σ-algebra $\mathcal{B}(X)$.

In particular the restriction of μ to $\mathcal{B}(X)$ is a measure.

The proof of this result is nontrivial.