GMT 19/20, Exercise Sheet 2 [version: 4/5/2020]

Reference: lectures 2 & 3

In the following measures are always positive Borel measures on a locally compact, separable metric space X.

Ex. 1

Let μ be a measure on X.
Prove that the smallest closed set C which supports μ exists.
Thus the definition of support of μ is well posed.

Ex. 2

Give examples of finite measures μ, λ on $[0,1]$ s.t. $\mu \perp \lambda$ and $\mathrm{supp}(\mu) = \mathrm{supp}(\lambda) = [0,1]$.

Ex. 3

Prove that it is not possible to associate to every measure μ on X a Borel set $S(\mu) \subset X$ s.t. $S(\mu)$ supports μ and $S(\mu) \cap S(\lambda) = \emptyset$ whenever $\mu \perp \lambda$.

An atom of a measure μ is a point $x \in X$ s.t. $\mu(\{x\}) > 0$; μ is non-atomic if it has not atoms; μ is atomic if every set E with $\mu(E) > 0$ contains an atom.
Note that these are not quite the definitions you may find in textbooks on general measure theory, but are equivalent for Borel measures.

A measure μ is called σ-finite if there exist countably many (Borel) sets X_i s.t. $\mu(X_i) < +\infty$ $\forall i$ and $\bigcup X_i = X$.

Ex. 4

Given a measure μ on X, let $\rho(x) := \mu(\{x\})$ for every $x \in X$, and let

$$\tilde{\mu}(E) := \sum_{x\in E} \rho(x) := \sup\left\{ \sum_{x\in E'} \rho(x) : \begin{array}{l} E' \subset E \\ E' \text{ finite} \end{array} \right\}.$$

Prove that

(a) $\tilde{\mu}$ is an atomic measure on X;

(b) $\mu = \tilde{\mu} + \lambda$ where λ is a non-atomic measure;

(c) ρ can be any function from X to $[0,+\infty]$, even not Borel.

Ex. 5

Let μ be a σ-finite measure. Then the following are equivalent:

(a) μ is atomic;

(b) there exists the smallest Borel set E that supports μ.

Ex. 6

Let μ be a non-atomic locally finite measure. Then for every $0 \leq m \leq M(\mu)$ there exists E s.t. $\mu(E) = m$.

Hint for $X = \mathbb{R}$: use that the function $f : [0,+\infty] \to [0,+\infty]$ defined by $f(x) := \mu((-x,x))$ is increasing and continuous.

We denote by $\mathcal{H}^0$ the measure that "counts points" on every space X, namely $\mathcal{H}^0(E) := \# E$ = the number of points of E if E is finite and $+\infty$ otherwise.

Ex. 7

Let μ be a finite, non-atomic measure on X. Then $\mu \ll \mathcal{H}^0$ but Radon-Nikodym Theorem does not apply, that is, μ cannot be written as $\mu = \rho\,\mathcal{H}^0$ for any $\rho \in L^1(\mathcal{H}^0)$.

Ex. 8

Let μ be a (Borel) measure on X, $f: X \to X'$ a Borel map, and let $f_\# \mu$ be the push-forward of μ according to f, that is,

$$[f_\# \mu](E) := \mu(f^{-1}(E)) \qquad \forall E \in \mathcal{B}(X')$$

Prove that $f_\# \mu$ is indeed a (Borel) measure on X'.

Ex. 9

Let μ be a measure on X. Then for every $x \in \operatorname{supp}(\mu)$ (and in particular for μ-a.e. x) there holds

$$\mu(B(x,r)) > 0 \qquad \forall r > 0.$$

This proves that the ratios in the limits and the limsup in Theorem 2, § 2.1, Lect. 2, are well-defined for μ-a.e. x.

Ex. 10

Let μ be a σ-finite measure on X and fix $x \in X$.
Then $\mu(\partial B(x,r)) = 0$ for all $r > 0$ except countably many.
In particular $\mu(\overline{B(x,r)}) = \mu(B(x,r))$ for all such r.

Ex. 11

Let μ, λ be locally finite measures on X, let $x \in \operatorname{supp}(\mu)$, and for $r > 0$ set

$$g(r) := \frac{\lambda(\overline{B(x,r)})}{\mu(\overline{B(x,r)})}; \qquad h(r) := \frac{\lambda(B(x,r))}{\mu(B(x,r))}.$$

Then

(a) $g(r)$ and $h(r)$ are well-defined for $0 < r \leq r_0$ where r_0 is such that $\overline{B(x,r_0)}$ is compact;

(b) g, h are resp. right- and left-continuous on $(0, r_0)$;

(c) $g(r) = h(r)$ iff $\lambda(\partial B(x,r)) = \mu(\partial B(x,r)) = 0$ (that is, all $r>0$ except countably many);

(d) g, r are continuous at all $r \in (0, r_0)$ s.t. $g(r) = h(r)$ (that is, all $r>0$ except countably many);

(e) $\limsup_{r\to 0} g(r) = \limsup_{r\to 0} h(r)$; $\liminf_{r\to 0} g(r) = \liminf_{r\to 0} h(r)$.

This exercise shows that all limits and limsups that appear in Theorems 2 and 3 and in Corollary 4 in § 2.1, Lecture 2, do not change if we replace the closed balls $\overline{B(x,r)}$ with the open balls $B(x,r)$.

A locally finite measure μ on X has the doubling property if there exists $M < +\infty$ s.t.

$$\mu(\overline{B(x,2r)}) \leqslant M\, \mu(\overline{B(x,r)}) \quad \forall \text{ ball } \overline{B(x,r)} ; \qquad (*)$$

μ has the asymptotic doubling property if

$$\limsup_{r\to 0} \frac{\mu(\overline{B(x,2r)})}{\mu(\overline{B(x,r)})} < +\infty \quad \text{for } \mu\text{-a.e. } x . \qquad (**)$$

Ex. 12

Prove that the definitions above are NOT affected if one replaces

(a) the closed balls in $(*)$ and $(**)$ with the corresponding open ones;

(b) the number 2 in $\overline{B(x,2r)}$ in $(*)$ and $(**)$ with any $m>1$ (and in $(**)$ m may depend on x).

▷ In the next exercises F is a finite dimensional normed space, and $\lambda = \rho\mu$ an F-valued measure.

Ex. 13

Data $\rho \in L^1(\mu, F)$, dimostrare che effettivamente

$$\lambda(E) := \int_E \rho \, d\mu$$

è una misura di Borel a valori in F.

Cosa succede se invece $\rho \in L^1_{loc}(\mu, F)$?

Ex. 14

Let $\lambda = \rho\mu$ and $\lambda = \tilde{\rho}\tilde{\mu}$ be two representations of λ. Prove that $|\rho|_F \mu = |\tilde{\rho}|_F \tilde{\mu}$ and $\|\rho\|_{L^1(\mu)} = \|\tilde{\rho}\|_{L^1(\tilde{\mu})}$.

Thus the definition of $|\lambda|$ and $\mathbb{M}(\lambda)$ do not depend on the representation of λ.

Ex. 15

For every set $E \subset \mathbb{N}$ let $\lambda(E) \in \ell^2$ be defined by

$$\big(\lambda(E)\big)_n := \begin{cases} \frac{1}{n+1} & \text{if } n \in E, \\ 0 & \text{if } n \in \mathbb{N} \setminus E. \end{cases}$$

Prove that

(a) λ is a σ-finite measure on $\mathbb{N}$;

(b) λ cannot be written as $\lambda = \rho \mathscr{H}^0$ for any $\rho \in L^1(\mathscr{H}^0)$;

(c) Theorem 1 in § 2.2, Lecture 3, does not apply to λ.

Given a locally compact topological space X, $\tilde{X} := X \cup \{\infty\}$ denotes the one-point (or Alexandrov) compactification of X.
Thus a base of neighbourhoods of ∞ is given by the sets $X \setminus K$ with K compact in X; in particular ∞ is an isolated point iff X is compact. Moreover if X is metrizable so is $\tilde{X}$.

Ex. 16

Consider the following classes of maps $g : X \to F^*$:

$$\mathcal{B}_1 := \left\{ g : g \text{ Borel}, \ |g|_{F} \leq 1 \text{ on } X \right\},$$

$$\mathcal{B}_2 := \left\{ g \in \mathcal{B}_1 : g \text{ continuous} \right\},$$

$$\mathcal{B}_3 := \left\{ g \in \mathcal{B}_2 : \lim_{x \to \infty} g(x) = 0 \right\}.$$

this make sense only if f is not compact

Given $\lambda \in \mathcal{M}(X, F)$, prove that

$$M(\lambda) = \inf_{g \in \mathcal{B}_1} \int_X g \, d\lambda = \inf_{g \in \mathcal{B}_2} \int_X g \, d\lambda = \inf_{g \in \mathcal{B}_3} \int_X g \, d\lambda .$$

and the first infimum is a minimum.

This exercise proves (most of) Proposition 3 of §2.2, Lect. 3.

The key point is that for every $v \in F$ there holds

$$\langle w, v \rangle \leq |w|_{F^*} |v|_F \quad \forall w \in F^*$$

(by the definition of the dual norm $|\cdot|_{F^*}$) and equality holds for some $w \in F^*$ (this is a corollary of Hahn-Banach Theorem).

Note that the proof does not really uses that F is finite dimensional.

Ex. 17

Show that Riesz Theorem (Theorem 4 of § 2.2, Lecture 3) for an arbitrary (finite dimensional) normed space F follows from the case $F = \mathbb{R}$.

Ex. 18

Show that Riesz Theorem for X locally compact (Theorem 5...) follows from Riesz Theorem for the one-point compactification $\tilde{X}$ (Theorem 4).

Hint: Let X be locally compact. Then $\mathcal{C}_0(X, F^*)$ is naturally identified with a subspace of $\mathcal{C}(\tilde{X}, F^*)$; then a bounded linear functional T on $\mathcal{C}_0(X, F^*)$ can be extended to a bounded linear functional $\tilde{T}$ on $\mathcal{C}(\tilde{X}, F^*)$ and the is represented by a measure $\tilde{\lambda} \in \mathcal{M}(\tilde{X}, F)$. Let then λ be the restriction of $\tilde{\lambda}$ to X...

Ex. 19

Let $\lambda_n \in \mathcal{M}(X, F)$ be of the form $\lambda_n = \alpha_n \delta_{x_n}$ with $x_n \in X$ and $\alpha_n \in F$.

(a) if $x_n \to x$ in X, $\alpha_n \to \alpha$ then $\lambda_n \xrightarrow{*} \alpha . \delta_x$;

(b) if $\lambda_n \xrightarrow{*} \lambda$ and $\lambda \neq 0$ then x_n and α_n converge;

(c) if $\lambda_n \xrightarrow{*} 0$, what can we say about α_n and x_n?

Ex. 20

For every $N = 1, 2, \ldots$ let D_N be the set of all $\lambda \in \mathcal{M}(X, F)$ of the form $\lambda = \alpha_1 \delta_{x_1} + \cdots + \alpha_N \delta_{x_N}$ with $\alpha_i \in F$, $x_i \in X$, and let $D := \bigcup_{N=1}^{\infty} D_N$.

Prove that each D_N is weak* closed and D is dense in $\mathcal{M}(X, F)$.

Ex. 21

Let λ_n, λ be measures in $\mathcal{M}(X,F)$ such that:

(a) $\mathbb{M}(\lambda_n) \leqslant C$ for some $C<+\infty$ independent of n;

(b) $\int_X g \, d\lambda_n \xrightarrow[n\to\infty]{} \int_X g \, d\lambda$ for every g in X dense in $\mathscr{C}(X)$. ← not $\mathscr{C}(X,F^*)$!

Then $\lambda_n \xrightarrow{*} \lambda$. Moreover (a) is automatically verified if $X=\mathscr{C}(X)$.

Ex. 22

Let $X := [0,1]$ and let $\lambda_n := n(\delta_{1/n} - \delta_0) \in \mathcal{M}(X)$. Then:

(a) $\int_X g \, d\lambda_n$ converges for every $g \in C^1(X)$;

(b) for every $\alpha \in [0,1)$ there exists $g \in \boxed{C^{0,\alpha}(X)}$ s.t. $\int_X g \, d\lambda_n \longrightarrow +\infty$.

space of α-Hölder functions $g: X \to \mathbb{R}$

Ex. 23

Let C be the standard Cantor set, that is, $C := \bigcap_{n=0}^{\infty} C_n$ where $C_0 := [0,1]$, $C_1 := [0,1/3] \cup [2/3,1]$ and so on ... Thus each C_n is the union of closed intervals $I_{n,1}, \dots, I_{n,2^n}$ with length 3^{-n}.

For every n let $\mu_n := 2^{-n} \sum_{i=1}^{2^n} \delta_{x_{ni}}$ with $x_{ni} \in I_{ni}$.

Then $\mu_n \xrightarrow{*} \mu$ in $\mathcal{M}^+(C)$ where μ is uniquely determined by $\mu(I_{n,i}) = 2^{-n}$ for every n, i.

Ex. 24

Let $\mathcal{F}$ be a family of functions $g: X \to [-\infty,+\infty]$, and let f^+, f^- be the upper and lower envelopes of $\mathcal{F}$, that is

$$f^-(x) := \inf\{g(x) : g \in \mathcal{F}\}; \qquad f^+(x) := \sup\{g(x) : g \in \mathcal{F}\}.$$

Prove that:

(a) if every $g \in \mathcal{F}$ is lower semicont. then f^+ is l.s.c.;

(b) if every $g \in \mathcal{F}$ is upper semicont. then f^- is u.s.c.

Note that this exercise works with X any topological space.

Ex. 25

Let $\mathcal{F}$ be a family of functions $g : X \to \mathbb{R}$ with $L := \sup_{g \in \mathcal{F}} \underbrace{\mathrm{Lip}(g)}_{\boxed{\text{Lipschitz constant of } g := \sup\left\{\frac{|g(x)-g(x')|}{d(x,x')} : \begin{array}{l} x,x' \in X \\ x \neq x' \end{array}\right\}}} < +\infty$, and let f^+, f^- be as above.

Then either $f^+ = +\infty$ on X or f^+ takes values in $\mathbb{R}$ and $\mathrm{Lip}(f^+) \leq L$.

And a similar statement holds for f^-.

Ex. 26

Let $E \subset X$, let $f : E \to (-\infty, +\infty]$ be finite at some point of E, and for every $x \in X$, $m \geq 0$ set

$$f_m(x) := \inf \left\{ f(y) + m \cdot d(x,y) \ : \ y \in E \right\}$$

Then

(a) f_m is Lipschitz and $\mathrm{Lip}(f) \leq m$ for every $m \geq 0$;

(b) $f_m(x)$ is increasing in m for every $x \in X$;

(c) $\lim_{m \to +\infty} f_m(x) = \sup_{m \geq 0} f_m(x) \leq f(x) \quad \forall x \in E$ and $=$ holds iff f is l.s.c. at x.

(d) if f is l.s.c., $f_m(x) \uparrow f(x) \quad \forall x \in E$.

In Exercises 25 and 26 it is not needed that X is locally compact.

Ex. 27

Using Exercises 24-26 fill the missing details in the proof of § 2.2.4, Lecture 3.

Ex. 28

space of positive finite measures on X

Given $\mu \in \underbrace{\mathcal{M}^+(X)}$, let $\tilde{\mu} \in \mathcal{M}^+(\tilde{X})$ be the natural extension of μ to the compactification $\tilde{X}$, that is,

$$\tilde{\mu}(E) := \mu(E \cap X) = \mu(E \setminus \{\infty\}) \quad \forall E \in \mathcal{B}(\tilde{X}).$$

Let $\mu_n \xrightarrow{*} \mu$ in $\mathcal{M}^+(X)$ and assume that there exists

$$m := \lim_{n\to\infty} M(\mu_n).$$

Prove that $\tilde{\mu}_n \xrightarrow{*} \tilde{\mu} + c \cdot \delta_\infty$ with $c := m - M(\mu)$.

Ex. 29

Prove the statements contained in § 2.2.5, Lecture 3.

Hint: Use Exercise 28 to reduce to the statements in § 2.2.4.

Ex. 30

Let $\lambda_n \xrightarrow{*} \lambda$ in $\mathcal{M}(X,F)$ and assume that $|\lambda_n| \xrightarrow{*} \mu$. Then

(a) $|\lambda| \leq \mu$ (that is, $|\lambda|(E) \leq \mu(E)$ $\forall E$);

(b) if $M(\lambda_n) \longrightarrow M(\lambda)$ then $|\lambda| = \mu$.

Hint for (a): prove first that for every A open in X

$$|\lambda|(A) = \sup\left\{ \int_X g\, d\lambda : \ g \in \mathcal{C}_0(A, F^*),\ |g|_F \leq 1 \right\}$$

and deduce that $|\lambda|(A) \leq \mu(A)$.

Ex. 31

Prove the statements contained in § 2.2.6, Lecture 3.