Currents
13/14

Lecture 1
12/3/14

lectures on the
Theory of Currents
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An introduction

The notion of currents generalizes that of (oriented) surface (i.e. submanifold of $\mathbb{R}^n$), and was developped in present form by Federer and Fleming to provide a solution of the (homological) Plateau problem.

This is not the only use of currents and indeed they were originally introduced by de Rham to serve other purposes.

Plateau problem:
"Find the surfaces S of minimal area that spans a given curve Γ (in the space)"

Nowadays "find" means "prove the existence of", S is supposed to be a d-dimensional surface in some ambient space, by "area" we mean the d-dimensional volume, and the requirement is that ∂S is a given $(d-1)$-dimensional surface.

"Proving existence" is already a challenging task (contrary to other variational problems); computing effectively minimal surfaces is even more challenging.

We keep the "existence issue" as a guideline for the

One possible approach to prove existence is by the "direct method", that is, semicontinuity and compactness. $\rightsquigarrow$ analogy with sol. of $\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$

To this end one needs to "construct" a class of generalized surface, extend to this class the notion of "area of surface" and that of "boundary"; moreover the class has to be <u>large enough</u> to have good compactness properties (with respect to a suitable topology such that the area is lower semicontinuous).

On the other hand, this class should be kept as small as possible, in order to ensure at least <u>density</u> or regular surfaces, and possibly some regularity theory.

We do not dwelve in regularity theory but we will do all the rest, including some "pre-regularity".

The regularity theory is particularly hard because in the end minimal surfaces may be not regular.

Let me discuss this point in deeper detail.

Let S_0 be a 2d-dimensional compact surface in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ whose tangent space is a complex subspace of $\mathbb{C}^n$ at every point.

Then it is not difficult to show (we will do it during this course) that S_0 minimizes the area among all surfaces S with boundary ∂S_0, and it's the only minimizer. Now, the same is true even if S_0 is "a piece" of a complex subvariety of $\mathbb{C}^n$, e.g., the set of solutions of some polinomial equation.

In particular $S_0 := \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2^3 ; |z_1| \leq 1\}$ minimizes the area among all surfaces S with boundary
$$\partial S_0 = \{(z_1, z_2) : |z_1| = 1 \text{ and } z_2^3 = z_1^2\} = \{(e^{3it}, e^{2it}) : t \in [0, 2\pi]\},$$
and is the <u>only</u> minimizers (which actually means that every minimizing sequence of surfaces will converge to S_0). But S_0 is NOT REGULAR in $(0,0)$!

In general minimal surfaces of dimension d may exhibit a singular set of dimension $d-2$.

The situation is slightly better for hypersurfaces, that is, when $d = n-1$; in that case the singular set may have dimension at most $d-7$. In particular minimal hypersurfaces in $\mathbb{R}^n$ with $n < 8$ (that is, $d < 7$) are regular (and more precisely analytic, the "classical" example of minimal surface with a singularity point being the Simons cone in $\mathbb{R}^8$.

We will not discuss regularity theory in this course.

I conclude this introduction with a brief overview of other approaches to the Plateau problem.

Purely set-theoretic approach.

This is particularly effective for $d=1$. In this case the Plateau problem reduces to (a particular case of) the Steiner problem: finding the set S minimal length which contains a given finite set and is connected (possibly in a general metric space).

A solution is provided by the fact that length (or more precisely, the 1-dimensional Hausdorff measure) is lower semicontinuous on the class of connected compact subsets of a given metric space, endowed with Hausdorff distance.

The Plateau problem can be formulated in a purely set theoretic setting also for higher d: for instance, one may look for the set S with minimal area (2-dim. Hausdorff measure) such that a given curve Γ can be retracted to a point within S. The problem, however, is that lack of any lower semicont. result for the Hausdorff measure with dimension $d>1$. Still this approach was carried out by Reifenberg.

The parametric approach.

One may decide to view surfaces as images of parametrizations $\phi: D \longrightarrow \mathbb{R}^n$.

reference domain in $\mathbb{R}^2$ (or $\mathbb{R}^d$)

In this case the area of $S := \Phi(D)$ is given by $\quad F(\phi) = \int_D \left| \frac{\partial \phi}{\partial t_1} \times \frac{\partial \phi}{\partial t_2} \right| dt$.

So one might consider finding a minimizer of F using the direct method in some suitable Sobolev class of parametrizations. Semicontinuity is not an issue, since F is l.s.c. in the weak* topology of $W^{1,p}$ (at least for some p), but compactness is: given a minimizing sequence (ϕ_n), in general it is not compact in any relevant topology.

This is due to the fact that F is invariant under the action of the group of diffeomorphisms of the reference domain D, and this group is far too large.

There is, however, a way around this.

Let

$$G(\phi) := \int_D \frac{1}{2} |\nabla u|^2 \, dt = \int_D \frac{1}{2} \left(\left|\frac{\partial \phi}{\partial t_1}\right|^2 + \left|\frac{\partial \phi}{\partial t_2}\right|^2 \right) dt.$$

Thus $G(\phi) \geq F(\phi)$ for every ϕ and equality holds if

$$\frac{\partial \phi}{\partial t_1} \perp \frac{\partial \phi}{\partial t_2} \quad \& \quad \left|\frac{\partial \phi}{\partial t_1}\right| = \left|\frac{\partial \phi}{\partial t_2}\right|$$

that is, if

ϕ is a conformal

Now, the key point is that (more or less) every surface that admit a parametrization also admit a conformal parametrization.

This means that minimizing G (instead of F) gives the conformal parametrization ϕ of the required minimal surface $S = \phi(D)$.

This is the base of the approach of Douglas to Plateau problem.

Remarks

- One tricky point is that in minimizing $G(\phi) = \frac{1}{2}\int_D |\nabla\phi|^2$ one cannot prescribe the boundary values of ϕ, but has to consider the weaker contraint $\phi(\partial D) = \Gamma$ where Γ is the curve to be spanned by S.
- This approach is seen also in dimension $d=1$, where geodesics are obtained by the minimization of $\int |\dot\gamma|^2$ insted of $\int|\dot\gamma|$ (which would be the length). Indeed the minimization of $\int|\dot\gamma|^2$ yields not just a parametrization γ of a certain geodesic, but a parametrization with constant speed...
- This approach does not work in dimension $d \geq 2$ because of lack of conformal parametrizations.

Finite perimeter sets

There is finally the approach to minimal surfaces via finite perimeter sets, pioneered by Caccioppoli and DeGiorgi. This is very similar to currents, and (to a certain extent) finite perimeter sets can be viewed as examples of n-dimensional currents in $\mathbb{R}^n$....

Structure of this Course

1. Basic notion from Geometric Measure Theory
(Hausdorff measures and dimension, rectifiable sets)

2. Review of the basic notions of multilinear algebra

3. Currents in the Euclidean setting

4. Additional topics

Textbooks / Reference Books

F. Morgan : Introduction to GMT ← very introductory!

L. Simon : Lectures on GMT ← covers much more than this course

S. Krantz & H. Parks : Geometric Integration Theory
↑ More introductory and elementary than Simon

H. Federer : Geometric Measure Theory.
↑ Most complete and detailed account of the theory of currents still today.
It's not a textbook!

Currents 13/14 — Lecture 2 13/3/14

In this lecture we review some basic notions from measure theory and then give the definition of Hausdorff measure.

In this course we will only need two notion of (positive) measure:

outer measures are defined for all subsets of a given ambient space, and are σ-subadditive, that is $\mu(E) \leq \sum_i \mu(E_i)$ whenever the countable family $\{E_i\}$ covers E (that is, $\bigcup E_i \supseteq E$).
| well, it is also required that $\mu(\emptyset)=0$

measures are defined on the appropriate Borel σ-algebra (thus the ambient space must be endowed with a topology) and are σ-additive.

The former are there because they are easy to construct.
The latter are there because are the tools we need most. Note that confining ourselves to the Borel σ-algebra is not restrictive for our purposes, and actually simplifies many details.
The connection between these two concepts is made by Caratheodory theorem, which we state after a preliminary definition.

<u>Definition</u> Let μ be an outer measure on X.
A set $E \subset X$ is measurable in the sense of Caratheodory if

$$\mu(F) = \mu(F \cap E) + \mu(F \cap E^c) \quad \forall F \subset X.$$

<u>Proposition</u> The class $\mathcal{S}$ of all such sets is a σ-algebra and the restriction of μ to $\mathcal{S}$ is σ-additive.

<u>Theorem</u> (Caratheodory). Let X be a metric space and assume that μ is additive on <u>distant</u> sets, that is

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$$

if

$$0 < \mathrm{dist}(E_1; E_2) := \inf_{\substack{x_1 \in E_1 \\ x_2 \in E_2}} d(x_1; x_2) .$$

Then $\mathcal{S}$ contains all Borel sets.

We recall now some basic notation and terminology about <u>measures</u> (so, σ-additive meas. on Borel σ-algebra of some space X).

$\lambda \ll \mu$ (λ is abs. cont. wrt μ) if $\lambda(E) \Rightarrow \mu(E) > 0$;

$\lambda \perp \mu$ (λ and μ are mutually sing.) if they are supported on two disjoint sets, i.e., $\exists$ E and F (Borel) s.t. $E \cap F = \emptyset$ and $\lambda(E^c) = 0$; $\mu(F^c) = 0$;

$f\mu$ measure given by $[f\mu](E) := \int_E f \, d\mu$;

f is a positive Borel function

$\|\mu\|$ mass of $\mu := \mu(X)$;

Hence $\|f\mu\| = \int_X f \, d\mu = \|f\|_{L^1(\mu)}$.

$\mu \llcorner E := 1_E \cdot \mu$ restriction of μ to the set E.

Theorem 1 (Hahn - Lebesgue - Radon - Nikodym)

- If λ and μ are finite measures then λ can be decomposed as
$$\lambda = \lambda_a + \lambda_s$$
with $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$.

a.c. part of λ w.r.t. μ singular part of λ w.r.t. μ.

- This decomposition is unique.
- λ_a is of the form $\lambda_a = f\mu$ with $f \in L^1(\mu)$.

Radon-Nikodym density of λ_s wrt μ

If in addition $X = \mathbb{R}^n$ (or X metric space and μ has the so called doubling property) then
$$f(x) = \lim_{r \to 0} \frac{\lambda(B(x,r))}{\mu(B(x,r))} \quad \text{for } \mu\text{-a.e. } x.$$

Theorem 2 (points of L^p- approximate continuity).

If $X = \mathbb{R}^n$ (or X is a metric space and μ has the doubling property), μ is a finite measure and $f \in L^p$ with $p < +\infty$ then f is L^p-approx. continuous at μ-a.e. x, that is
$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|^p \, d\mu(y) \xrightarrow[r \to 0]{} 0 \quad \text{for } \mu\text{-a.e. } x.$$

The second part of Theorem 1 and Theorem 2 are consequences of well-known covering theorems. Hopefully, these two statements are all we need in the rest of this course, and therefore I will not discuss covering theorems (not now, at least).

Vector valued measures

Let μ be a positive measure on X, E a normed space with finite dimension, and $f : X \to E$ a Borel map such that $\int_X |f| \, d\mu < +\infty$, that is, $f \in L^1(\mu; E)$.
Let then $\lambda = f\mu$ be the measure defined as usual by

$$\lambda(F) = [f\mu](F) = \int_F f \, d\mu \quad \forall F \text{ Borel set in } X.$$

Then λ is an example of E-valued measure on X.
It can be proved (using the Lebesgue-Radon-Nik. theorem) that every E-valued measure on X can be represented as $\lambda = f\mu$ as above.
One can additionally require that $|f|_E = 1$ μ-a.e., and under this additional assumption f and μ are uniquely determined.
The total variation of λ is the (finite) positive measure $|\lambda| := |f| \mu$.
The mass of λ is $\|\lambda\| := |\lambda|(X) = \int_X |f|_E \, d\mu = \|f\|_1$.

The mass is a norm on the space $\mathcal{M}(X, E)$ of all E-valued measures on X, which turns out to be a Banach space.

If in addition X is a compact metric space and $\mathcal{C}(X, E^*)$ denotes the space of continuous maps $g : X \to E^*$ endowed with the supremum norm, then

<u>Proposition</u> $\mathcal{M}(X,E)$ is isometric to the dual of $\mathcal{C}(X,E^*)$ via the duality pairing

$$\underset{\mathcal{M}}{\langle \lambda}, \underset{\mathcal{C}}{g\rangle} := \int_X \langle g \,; d\lambda\rangle = \int_X \langle g(x)\,;\, f(x)\rangle \, d\mu(x)$$

<u>Corollary</u> (compactness of measures)
Given a sequence of measures λ_n with uniformly bounded masses, there exists a subsequence (still denoted by λ_n) which converges to some measure λ in <u>the sense of measures</u>, that is

$$\langle \lambda_n \,; g\rangle \longrightarrow \langle \lambda \,; g\rangle \quad \forall g \in \mathcal{C}(X,E^*).$$

<u>Remarks</u> • If X is a separable, locally compact metric space, then the proposition above holds with $\mathcal{C}(X,E^*)$ replaced by $\mathcal{C}_0(X,E^*)$, namely the space of continuous functions $g : X \to E^*$ such that

$$\lim_{x\to\infty} |g(x)|_{E^*} = 0$$

(where "∞" is intended in the sense of Alexandrov compactification of X, and $\mathcal{C}_0(X,E^*)$ is still endowed with the supremum norm).

• The statements above rely in an essential way on the fact that E is finite dimensional, and therefore E^{**} is canonically isomorphic to E.

Definition of d-dimensional Hausdorff measure.

Let X be a metric space, $d \in [0,+\infty)$, $\delta \in (0,+\infty)$.
For every set $E \subset X$ we define the Hausdorff pre-measure $\mathcal{H}^d_\delta(E)$ by

$$\mathcal{H}^d_\delta(E) := \underset{\substack{\uparrow \\ \text{renormalization} \\ \text{factor}}}{c_d} \cdot \inf \sum_i (\operatorname{diam} E_i)^d$$

where the infimum is taken over all countable families $\{E_i\}$ of subsets of X that cover E and satisfy $\operatorname{diam}(E_i) \le \delta \ \forall i$; the factor c_d is given by

$$c_d := \frac{\text{volume of unit ball in } \mathbb{R}^d}{2^d} \qquad \text{for every } d \text{ integer}$$
(see remarks below)

Finally, the d-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}^d(E) := \sup_{\delta>0} \mathcal{H}^d_\delta(E) = \lim_{\delta \downarrow 0} \mathcal{H}^d_\delta(E).$$

$\delta \mapsto \mathcal{H}^d_\delta(E)$ is clearly decreasing in δ.

Fundamental properties and remarks

- It is easy to check that that each $\mathcal{H}^d_\delta$ is an outer measure, and is additive on couple of sets E_1, E_2 s.t. $\operatorname{dist}(E_1,E_2) > \delta$.

- From the previous fact it follows <u>easily</u> that $\mathcal{H}^d$ is an outer measure which is additive on distant sets. In particular <u>the restriction of $\mathcal{H}^d$ to the Borel σ-algebra is σ-additive</u>!

- It follows easily from the definition that given a lipschitz map $f : E \subset X \to X'$, then

$$\mathcal{H}^d(f(E)) \leq L^d \, \mathcal{H}^d(E)$$

metric space

where L is the lipschitz constant of f.
It follows that $\mathcal{H}^d$ is preserved under <u>isometries</u>, and if $f : \mathbb{R}^n \to \mathbb{R}^n$ is an homothety with scaling factor λ then

$$\mathcal{H}^d(f(E)) = \lambda^d \, \mathcal{H}^d(E) \, .$$

This justifies calling $\mathcal{H}^d$ "d-dimensional".

- The choice of the renormalization factor C_d has the purpose of making $\mathcal{H}^d = \mathcal{L}^d$ on $\mathbb{R}^d$.
Note that it is relatively easy to see that $\mathcal{H}^d$, being translation invariant, must agree with the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$ up to some constant factor. That the right factor is the number C_d given above is less obvious.
For us is essential that $\mathcal{H}^d = \mathcal{L}^d$ on $\mathbb{R}^d$, but in other contexts this is not so important, and often the renormalization factor is simply omitted (or set $=1$).

- Note that the sets E_i in the covering of E can be freely assumed to be subsets of E without changing the value of $\mathcal{H}^d_s(E)$ and $\mathcal{H}^d(E)$. Therefore those quantities depend only on the restriction of the metric of X to E, and not on the ambient space X.

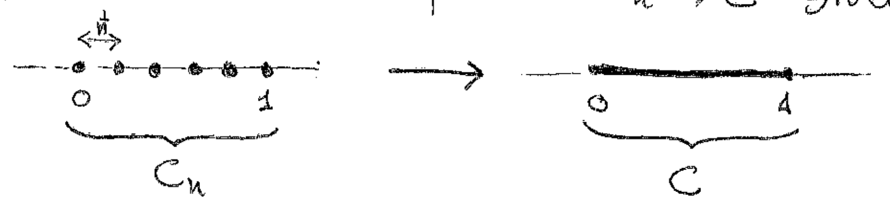
- One can add different restrictions to the set E_i in the covering of E without changing the values of $\mathcal{H}^d_s(E)$ and $\mathcal{H}^d(E)$. For example one can require that the sets E_i are
    - closed (because $\operatorname{diam}(\overline{E}) = \operatorname{diam}(E)$);
    - open (because $\forall\, \varepsilon > 0\ \exists\, A$ open $\supset E$ s.t. $\operatorname{diam}(A) \leq \operatorname{diam}(E) + \varepsilon$);
    - convex (if X is a normed space, because $\operatorname{diam}(\operatorname{Co}(E)) = \operatorname{diam}(E)$);
    - etc. etc.

Note that assuming that the sets E_i are <u>balls</u> is <u>too</u> restrictive (in general it is not true that a set E is contained in a ball with same diameter, consider for instance $E =$  in $\mathbb{R}^2$).

Using balls instead of arbitrary sets yields a different measure, known as Hausdorff spherical measure. There are indeed many different notions of d-dimensional measures, but for our purposes they are ultimately equivalent (and H. measure is the notion most widely used).

On the (lack of) semicontinuity of Hausdorff measure w.r.t. Hausdorff convergence of closed sets.

Consider for instance the sequence $C_n \to C$ given by



Then $\mathcal{H}^1(C_n) = 0 \ \forall n$, but $\mathcal{H}^1(C) = 1$.

(More generally, for every compact set C in a metric space X there exists a sequence of finite sets C_n which converge to C, and therefore for every $d>0$ $\mathcal{H}^d(C_n) = 0$, no matter what $\mathcal{H}^d(C)$ is....).

There is however a significant fact:

Gołąb theorem

Let C_n be a sequence of connected compact sets that converge to C in the Hausdorff distance. Then

$$\liminf_{n\to+\infty} \mathcal{H}^1(C_n) \geq \mathcal{H}^1(C).$$

The proof of this theorem is far from obvious. It can be obtained (for example) using the following characterization of the 1-dim. H. measure for (compact) connected sets:

If C is a compact connected sets in X metric space, then

$$\mathcal{H}^1(C) = \sup \sum_i \operatorname{diam}(C_i)$$

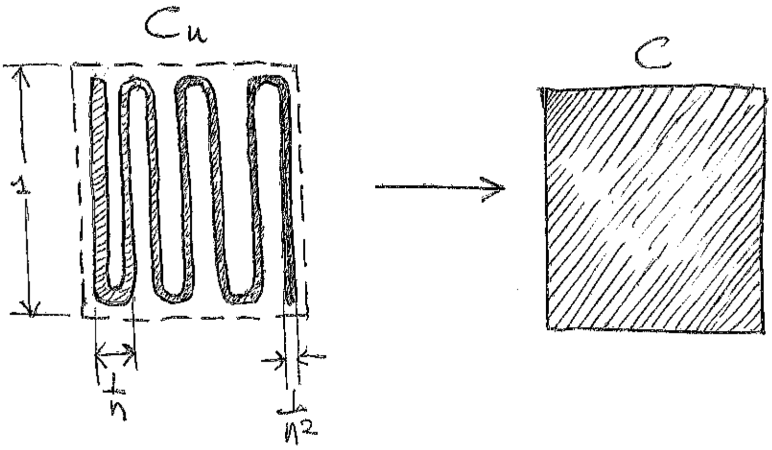
where the supremum is taken over all finite,

<u>disjoint</u> families $\{C_i\}$ of compact, connected subsets of C.

Note that no variant of this result is known for d-dimensional H. measures with $d>1$.

The key obstruction to keep in mind is that any reasonable d-dimensional set C can be approximated by d-dimensional surfaces C_n with $\mathcal{H}^d(C_n) \to 0$, and satisfying every reasonable topological constraint.

For instance, let us approximate a square C in $\mathbb{R}^2$ with a sequence of C_n which are homeomorphic to a disk



<u>Hausdorff dimension</u>

As one might expect, a set with finite length has zero area, and a set with positive area must have infinite length. More generally given $\alpha<\beta$ there holds

$$\mathcal{H}^\alpha(E) < +\infty \Rightarrow \mathcal{H}^\beta(E) = 0,$$

and

$$\mathcal{H}^\beta(E) > 0 \Rightarrow \mathcal{H}^\alpha(E) = +\infty.$$

(This is easy to prove!)

This implies in particular that given d such that $0<\mathcal{H}^d(E)<+\infty$, then $\mathcal{H}^\alpha(E)=+\infty \ \forall \alpha<d$ and $\mathcal{H}^\beta(E)=0 \ \forall \beta>d$.

It thus makes sense to define such special d as the Hausdorff dimension of E.

However such a d may not exists, but we can still define the H. dimension of E as

$$\dim_H(E) := \sup\{\alpha\geq 0 : \mathcal{H}^\alpha(E)=+\infty\}$$
$$= \inf\{\beta\geq 0 : \mathcal{H}^\beta(E)=0\}$$

(with the convention that $\sup\emptyset=0$, $\inf\emptyset=+\infty$).

<u>Remarks</u>

- As mentioned above, $d=\dim_H(E)$ does not imply that $\mathcal{H}^d(E)>0$ or $\mathcal{H}^d(E)<+\infty$.
- $\dim_H(E)$ is invariant under bi-Lipschitz transformation but <u>not</u> under homeomorphisms: it is a metric invariant, not a topological one.
- $\dim_d(E)$ may be not integer, see below.

<u>Example</u> : <u>the Cantor set</u>.

Let $C=\bigcap_{n=0}^{\infty} C_n$ be the standard Cantor set; thus C_n is the union of 2^n closed intervals with length 3^{-n}.

We claim that $\mathcal{H}^d(C)=1$ where $d:=\frac{\log 2}{\log 3}\left(=\dim_H(C)\right)$.

Note first that the only possible d such that $0<\mathcal{H}^d(C)<+\infty$ is $d=\frac{\log 2}{\log 3}$, and this easy explained. Indeed C can be seen as the union of 2 copies of C scaled by a factor $\frac{1}{3}$ (namely $\underbrace{C\cap[0,\frac{1}{3}]}_{C_1}$ and $\underbrace{C\cap[\frac{2}{3},1]}_{C_2}$).
Hence

$$\mathcal{H}^d(C) = \mathcal{H}^d(C_1)+\mathcal{H}^d(C_2)$$

$$= \frac{2}{3^d}\,\mathcal{H}^d(C)$$

Hence d satisfies the equation $1=\frac{2}{3^d}$ (but this only if $0<\mathcal{H}^d(E)<+\infty$), that is $d=\frac{\log 2}{\log 3}$.

Now we show that $\mathcal{H}^d_\delta(C)\leq 1$ for every $\delta>0$, and in particular $\mathcal{H}^d(C)\leq 1$.
To this end it suffices to cover C with the 2^n intervals with length 3^{-n} that form C_n (note that these intervals have diameter $\leq\delta$ if $3^{-n}\leq\delta$, that is, $n\geq-\frac{\log\delta}{\log 3}$). Indeed we get

$$\mathcal{H}^d_\delta(C)\leq 2^n\cdot(3^{-n})^d=\left(\frac{2}{3^d}\right)^n \overset{\text{by the defin. of } d}{=} 1^n=1.$$

Here we have no renormalization factor!!

Finally, we would like to prove the opposite inequality. This is the difficult part (finding lower bounds for the Hausdorff measure is always more complicated than upper bounds).
The claim is that for every cover $\{E_i\}$ of C there holds $\sum_i(\mathrm{diam}(E_i))^{1/3}\geq 1$. Since we can assume that the sets E_i are open and convex, that is, open

intervals, and since C is compact, we can reduce to the case that $\{E_i\}$ is a finite collection of open intervals. Next, for every i one finds a (disjoint, finite) collection $\{I_{i,j} : j\}$ of closed intervals such that a) each $I_{i,j}$ is contained in E_i and is a connected component of one of the set C_n whose intersection gives C; b) $\{I_{i,j} : i,j\}$ covers C.

The final key step consists in showing that it is "convenient" to replace the cover $\{E_i\}$ with $\{I_{i,j}\}$, that is

$$\sum_j (\mathrm{diam}(I_{i,j}))^d \leq \mathrm{diam}(E_i) \quad \forall i.$$

To prove this, the key inequality is that

$$a^d + b^d \leq (a+b+3^{-n})^d \quad \forall n,a,b \text{ with } 0 < a,b \leq 3^{-n}.$$

We just remark here that sharp lower bounds for Hausdorff measure are, in general, quite hard to prove, and one is usually happy with non-sharp ones. Those are usually obtained in a different way, namely by constructing measures supported on the set under considerations with suitable density properties. We will not discuss this issue any further.

The area formula

The area formula is used to compute the d-dimensional H. measure of a subset of a d-dimensional surface of class C^1 in $\mathbb{R}^n$.

There are many variants of this formula.

We shall start from the simplest one, which hopefully is almost self evident, and then give further generalizations, specifying for each one what are the new ingredients, but without giving complete proofs ....

Let us begin by recalling a well-known fact.
Let T be a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$, and let M be the associated matrix. The for every (Borel) set $E \subset \mathbb{R}^d$ there holds

$$\mathscr{L}^d(T(E)) = |\det M| \, \mathscr{L}^d(E),$$

in particular $|\det M| = \mathscr{L}^d(R)$ where R is the rectangle spanned by the columns $m_1, \dots, m_d$ of M, that is, $R := \{ \sum_{i=1}^d t_i m_i : 0 \le t_i \le 1 \ \forall i \}$, that is, the image according to T of the unit cube $[0,1]^d$.

Let now T be a linear map from V to W, d-dimensional vector spaces endowed with a scalar product. Then we can define $|\det T|$ as $|\det M|$ where M is the matrix associated to T by a choice of orthonormal bases on V and W.

Note that $|\det M|$ does not depend on the choice of the bases (while $\det M$ does) and therefore the definition of $|\det T|$ is well-posed. Moreover for every Borel set $E \subset V$ there holds

$$\mathcal{H}^d(T(E)) = |\det T| \, \mathcal{H}^d(E).$$

(Here we use that $\mathcal{H}^d = \mathcal{L}^d$ on $\mathbb{R}^d$.)

Area formula (first version)

Let $\phi : D$ (open set $\subset \mathbb{R}^d$) $\longrightarrow S$ (d-dimensional surface of class $\mathscr{C}^1$ in $\mathbb{R}^n$) be a parametrization of class $\mathscr{C}^1$, that is $\phi(D)$ is open in S, $\phi : D \to \phi(D)$ is an homeomorphism, and $d\phi(t)$ has rank d at every $t \in D$.

We use $\nabla\phi(t)$ for the gradient matrix ($n \times d$) and $d\phi(t)$ for the linear map (differential) from $\mathbb{R}^d$ to $\mathrm{Tan}(S, x)$, $x = \phi(t)$

Then for every $E \subset D$ there holds

Lebesgue measure on $\mathbb{R}^d$

$$(*) \qquad \mathcal{H}^d(\phi(E)) = \int_E J\phi(t) \, dt$$

Jacobian (determ.) of ϕ at t

where we have set

$$J\phi(t) := |\det(d\phi(t))| \quad \forall t \in D.$$

There are two issues here: 1) prove the area formula (*) 2) find alternative formulas to compute the Jacobian $J\phi(t)$.

Sketch of proof of the area formula.

The idea is that around each point $t_0 \in D$, $\phi(t)$ can be written as the composition of an affine map $\phi_0(t) := \phi(t_0) + d\phi(t_0)(t-t_0)$ composed by a C^1 map $\phi_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is "almost" an isometry, that is

$$(1-\varepsilon)|x-x'| \le |\phi_2(x)-\phi_2(x')| \le (1+\varepsilon)|x-x'|$$

where ε can be taken as close to 0 as one wishes provided that x, x' are sufficiently close to $x_0 := \phi(t_0)$.

(Actually ϕ_2 is "almost" the identity, in the sense that $|\phi_2(x) - x| \le \varepsilon |x - x_0| \ldots$)

Let now F_0 be a subset of D "sufficiently close to t_0". Then we know from the previous remarks that the area formula holds for the affine map ϕ_1, that is,

$$\begin{aligned} \mathcal{H}^d(\underbrace{\phi_1(F_0)}_{F_1}) &= |\det d\phi(t_0)|\, \mathscr{L}^d(F_0) \\ &= J\phi(t_0) \cdot \mathscr{L}^d(F_0); \end{aligned}$$

moreover ϕ_2 is "almost" an isometry, and therefore it "almost" preserves $\mathcal{H}^d$, that is

$$(1-\varepsilon)^d \mathcal{H}^d(F_1) \le \mathcal{H}^d(\underbrace{\phi_2(F_1)}_{F_2}) \le (1+\varepsilon)^d\, \mathcal{H}^d(F_1);$$

and therefore, taking into account that $F_2 := \phi_2(F_1) = \phi_2(\phi_1(F_0)) = \phi(F_0)$, we obtain that

$$\mathcal{H}^d(\phi(F_0)) = (1+O(\varepsilon))\,\mathcal{H}^d(F_1) = (1+O(\varepsilon))\; J\phi(t_0)\,\mathcal{L}^d(F_0)\,.$$

Now the idea would be to cover E with a (finite) family of "small sets" F_0^i as above, with corresponding points t_0^i, and obtain

$$\mathcal{H}^d(\phi(E)) = (1+O(\varepsilon))\left[\sum_i J\phi(t_0^i)\,\mathcal{L}^d(F_0^i)\right]$$

and then notice that the sums approximate $\int_E J\phi(t)\,dt$ ....

$\square$

Which properties of Hausdorff measure did we use?

The proof essentially relies on two properties of $\mathcal{H}^d$:

1) $\mathcal{H}^d = \mathcal{L}^d$ on $\mathbb{R}^d$;

2) $\mathcal{H}^d(f(E)) \leq (\mathrm{lip}(f))^d\, \mathcal{H}^d(E)$ for every set E and every Lipschitz map f.

This means that the area formula holds as well for any other notion of d-dimensional measure that shares these properties, such as (for example) the spherical Hausdorff measure $\mathcal{H}^d_S$ and the integralgeometric measure $\mathcal{I}^d$.

It follows that all these measures agree on surfaces of class $\mathcal{C}^1$ and dimension d.

This means that for our purposes these measures turns out to be essentially equivalent, the preference for Hausdorff measure being mostly a tradition....

Alternative formulas to compute the Jacobian

- Given a linear map $T : V \to W$ we defined $|\det T| := |\det M|$ where M is the $(d\times d)$-matrix associated to T and a choice of orthonormal bases on V and W.

  Let now W be a subspace of $\tilde{W}$ (n dim. linear space, e.g. $\mathbb{R}^n$) and that $\tilde{M}$ is the $(m\times d)$-matrix associated to T, the previous base on V, and some orthonormal base on $\tilde{W}$ (possibly irrelated to that on W).

  We claim that

  $$M^t M = \tilde{M}^t \tilde{M}^t .$$

  Indeed, M^t is the matrix associated to the adjoint $T^* : W \to V$ (here the adjoint is defined using the scalar product, that is by the identity $\langle Tv, w\rangle_W = \langle v, T^* w\rangle_W$) while $\tilde{M}^t$ is the matrix associated to $T^* : \tilde{W} \to V$. But the latter adjoint is just an extension of the former and in particular there is only one map $T^*T : V \to V$, which is represented both by M^tM and $\tilde{M}^t\tilde{M}$, therefore these two matrices agree.

  Now, as a consequence of the formula above we obtain that

  $$\begin{aligned} |\det T| &:= |\det M| \\ &= \left(\det M^* \cdot \det M\right)^{\frac{1}{2}} \\ &= \left(\det (M^* M)\right)^{\frac{1}{2}} = \left(\det (\tilde{M}^* \tilde{M})\right)^{\frac{1}{2}} . \end{aligned}$$

- Let now $\phi : D$ open set in $\mathbb{R}^d \to S$ be a map of class C^1, with S d-dimensional surface in $\mathbb{R}^n$. Then, for every $t \in D$, $d\phi(t)$ is a linear map from $\mathbb{R}^d$ in $\mathrm{Tan}(S, \phi(t)) \subset \mathbb{R}^n$, and if on $\mathbb{R}^d$ and $\mathbb{R}^n$

We choose the standard bases, the $(m\times d)$-matrix associated to $d\phi(t)$, as a linear map from $\mathbb{R}^d$ to $\mathbb{R}^n$, is the gradient $\nabla\phi(t)$. Therefore the last formula in the previous paragraph yields

$$J\phi(t) := |\det(d\phi(t))| = \left(\det(\nabla^t\phi(t)\cdot\nabla\phi(t))\right)^{1/2}.$$

Note that this formula is often given as <u>definition</u> of $J\phi$.

- Let N be an $(n\times d)$-matrix.

  Then the (generalized) Binet formula, which we'll state and prove later, gives
  $$\det(N^tN) = \sum_M (\det M)^2$$
  where the sum is taken over all $d\times d$ minors M of the matrix N.

  By applying this formula to the formula for the Jacobian given above we get

  $$J\phi(t) = \left(\det(\nabla^t\phi(t)\,\nabla\phi(t))\right)^{\frac{1}{2}}$$
  $$= \left(\sum_M (\det M)^2\right)^{\frac{1}{2}}$$
  where the sum is taken over all $(d\times d)$-minor M of the gradient $\nabla\phi(t)$.

<u>Generalized Pythagorean theorem</u>

Let $m_1,\dots,m_d$ be vectors in $\mathbb{R}^n$, and let R be the "rectangle" spanned by these vectors, that is,
$$R := \left\{\sum_{i=1}^d t_i m_i \;:\; t_i\in[0,1]\ \forall i\right\}.$$

Then R is the image of the unit cube $[0,1]^d$ according to the linear map $T: \mathbb{R}^d \to \mathbb{R}^n$ defined by $Te_i = n_i$ for $i=1,\dots,d$. Thus the matrix associated to T is $N = (n_1,\dots,n_d)$.

Using the previous formulas we have that

$$\mathrm{vol}_d(R) = \mathcal{H}^d(R) = |\det T| = \sqrt{\det(N^t N)} = \sqrt{\sum_M (\det M)^2}$$

where M ranges among all $(d\times d)$-minors of N.
Note that is M is the minor associated to the rows with indexes $i_1,\dots,i_d$ of M, then the corresponding columns are the projections of $n_1,\dots,n_d$ on the coordinate plane $x_{i_1},\dots,x_{i_d}$, and therefore $|\det M|$ is the volume of the rectangle R' which is spanned by these projections, that is, the projection of R on the coordinate plane above.
The formula above yields therefore

$$\mathrm{vol}_d(R) = \sqrt{\sum_{R'} (\mathrm{vol}(R'))^2}$$

where R' ranges over all projections of R on the d-dimensional coordinate planes of $\mathbb{R}^n$.
This is know as Generalized Pythagorean Theorem.

<u>Area formula (second version)</u>

Let Φ : D open set in $\mathbb{R}^d \to \mathbb{R}^n$ be a map of class C^1 (not necessarily a parametrization, or even an injective map), and let E be a (Borel!) subset of D. Then the d-dimensional measure of $\phi(E)$, <u>counted with multiplicity</u>, is given by

$$(*) \qquad \int_{x\in\mathbb{R}^n} \#\left(\phi^{-1}(x)\cap E\right) d\mathcal{H}^d(x) = \int_{t\in E} J\phi(t)\,dt$$

($\#A$ stands for the number of points of A)

One can further strengthen formula $(*)$: for every (Borel!) function $h : D \to [0,+\infty]$ there holds

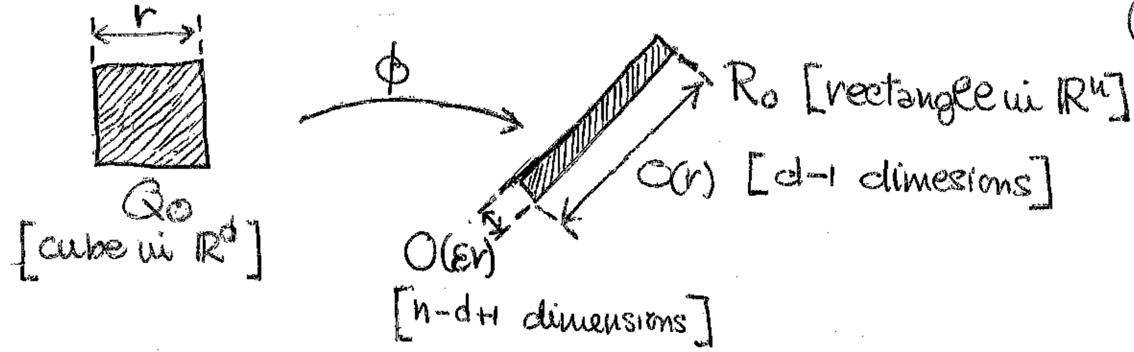
$$(**) \qquad \int_{x\in\mathbb{R}^n} \Big(\sum_{t\in\phi^{-1}(x)} h(t)\Big)\, d\mathcal{H}^d(x) = \int_{t\in E} h(t)\, J\phi(t)\,dt\,.$$

<u>Remarks</u> $(**)$ includes $(*)$ as a particular case. Moreover $(**)$ can be derived from $(*)$ by standard techniques of integration theory. Note that there is a measurability issue (which we happily ignore) concerning the left-hand sides of $(*)$ and $(**)$.

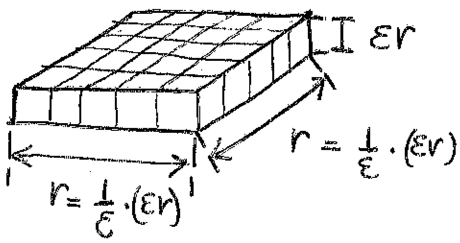
<u>Proof</u> Let S be the set of all $t\in D$ s.t. <u>$\mathrm{rank}(d\phi(t))<d$</u>. Thus $D\setminus S$ is open and ϕ agrees, locally in $D\setminus S$, with a parametrization of a C^1 surface. Thus we can derive $(*)$ with $E\setminus S$ in place of E from the <u>first version</u> of the area formula.

We easily verify that the proof of $(*)$ is completed by showing that $\mathcal{H}^d(\phi(E\cap S))=0$. And indeed $\mathcal{H}^d(\phi(S))=0$.

The key observation for the proof of this equality is the following: let $t_0\in S$, and let Q_0 be a cube containing t_0 with side length r. Since $d\phi(t_0)$ has rank $\le d-1$, given $\varepsilon>0$, for r sufficiently small $\phi(Q_0)$ is contained in a rectangle R_0 with $d-1$ sides of length $O(r)$ and $n+d-1$ sides of length $O(\varepsilon r)$.



In particular we can cover R_0 with $O(\varepsilon^{1-d})$ n-dim. cubes with sides of length $O(\varepsilon r)$, denoted by Q_1^j



Now we cover S with N cubes Q_0^i with side length r as above, and $N = O(r^{-d})$ (S is closed, and we can actually assume it is compact).

Then we cover $\phi(S)$ with the cubes Q_1^{ij} as above, and use this cover to estimate from above the Hausdorff measure of $\phi(S)$: then (if $\delta > \sqrt{n}\, O(\varepsilon r)$)

$$\mathcal{H}^d_\delta(S) \leq \sum_{ij} (\operatorname{diam} Q_1^{ij})^d$$

$$= \underset{\substack{\uparrow \\ \text{number} \\ \text{of } i}}{O(r^{-d})} \cdot \underset{\substack{\uparrow \\ \text{numer} \\ \text{of } j}}{O(\varepsilon^{1-d})} \cdot \underset{\substack{\uparrow \\ \operatorname{diam}(Q_1^{ij})}}{\left(O(\varepsilon r)\right)^d}$$

$$= O(\varepsilon) \, .$$

$\square$

Lecture 4

20/3/14

Recall the definition of Lipschitz map $f : X \to Y$ (metric spaces): there exists $L < +\infty$ s.t.

$$d_Y(f(x_1); f(x_2)) \leq L \, d_X(x_1; x_2) \qquad \forall x_1, x_2 \in X$$

and the best (smallest) of all these L is called the Lipschitz constant of f, denoted by $\mathrm{Lip}(f)$.

Under many regards, Lipschitz maps are the "right" class of maps when dealing with Hausdorff dimension and measures (which are preserved by bi-Lipschitz maps, but not by homeomorphisms) and even in the Euclidean context (the one of currents) they are somewhat preferable to $\mathcal{C}^1$ maps because, even though they are less regular (but not much less, as we will see presently), because of the following properties.

Useful (elementary) properties of Lipschitz maps

- Compactness: If X is separable, Y is compact, and (f_n) is a sequence of Lipschitz maps from X to Y with $\mathrm{Lip}(f_n) \leq L < +\infty$, then, up to a subsequence, f_n converge uniformly to some $f : X \to Y$ with $\mathrm{Lip}(f) \leq L$. (This is a particular case of Arzelà-Ascoli theorem.)

- <u>McShane extension Lemma</u>. Let $E\subset X$ and $f:E\to\mathbb{R}$ be a Lipschitz function. Then f admits an <u>extension</u> $F:X\to\mathbb{R}$ with $\mathrm{lip}(F)=\mathrm{lip}(f)$.

(that is, Lipschitz with constant $\leq L$)

<u>Proof</u> Just set $L:=\mathrm{lip}(f)$ and

$$F(x):=\inf_{t\in X}\{f(t)+L\,d(x,t)\}.$$

It is straightforward to check that F is L-Lipschitz (recall that $x\mapsto d(x,t)$ is 1-Lipschitz $\forall t$) and that $F=f$ on E. $\square$

<u>Remark</u> The F above is actually the <u>largest</u> Lipschitz extension of f, in the sense that given $g:X\to\mathbb{R}$ such that $g=f$ on E and $\mathrm{lip}(g)\leq\mathrm{lip}(f)$, then $g(x)\leq F(x)$ $\forall x\in X$. The smallest Lipschitz extension of f is, clearly,

$$\tilde{F}(x):=\sup_{t\in X}\{f(t)-L\,d(x,t)\}.$$

<u>Remark</u> Using McShane extension Lemma one can extend maps $f:E\subset X\to\mathbb{R}^n$ (just by extending each component of f) but in this case the Lipschitz constant of the extension may be larger than that of f.
We have, however,

- <u>Kirszbraun Theorem</u>. If X and Y are Hilbert spaces and $f:E\subset X\to Y$ is Lipschitz, then f admits an extension $F:X\to Y$ with $\mathrm{lip}(F)=\mathrm{lip}(f)$.

This result is, however, much more delicate than McShane extension lemma.

- <u>Rademacher theorem</u>. Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz map. Then f is differentiable at (Lebesgue-) a.e. point of $\mathbb{R}^d$.

  <u>Remarks</u>. It suffices to prove this result for $m=1$. There are many proofs of this fact. One consists in noticing that f belongs (locally) to the Sobolev class $W^{1,\infty}$, and then use that every <u>continuous</u> function in the Sobolev class $W^{1,p}$ with $p > d$ is differentiable a.e. (the proof of this fact is rather simple).

- <u>Lusin property with $\mathscr{C}^1$ maps</u>. Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a Lipschitz map. Then for every $\varepsilon > 0$ there exists an (open) set A with $\mathscr{L}^d(A) < \varepsilon$ and a map $g : \mathbb{R}^d \to \mathbb{R}^n$ of class $\mathscr{C}^1$ such that $f = g$ in $\mathbb{R}^d \setminus A$. Moreover we can assume that f is differentiable at every point $x \in \mathbb{R}^d \setminus A$ and $df(x) = dg(x)$.

  <u>Remarks</u>. As before, it suffices to prove the result for $n=1$. Also, one can assume that the domain of f is a bounded open subset of $\mathbb{R}^d$ (a ball) instead of the whole $\mathbb{R}^d$. There are many ways to prove this property.

One, which we sketch below, relies on Whitney extension theorem.
Other proofs rely on the Sobolev version of Whitney extension theorem (see the book by Ziemer: "Weakly differentiable functions") or on local smoothing techinques. Both approaches give stronger versions of the Lusin property stated above.

Sketch of proof. When the order of differentiability is 1 (as in our case) Whitney extension lemma can be stated as follows: let C be a closed set in $\mathbb{R}^d$, $f : C \to \mathbb{R}$ a function, and assume that for every $x \in C$, f admists a 1st order Taylor expansion at x as follows:

$$f(x+h) = f(x) + L_x(h) + R_x(h)$$

for every h s.t. $x+h \in C$, where $L_x : \mathbb{R}^d \to \mathbb{R}$ is a linear function for every x and $x \mapsto L_x$ is continuous, and $R_x(h) = o(|h|)$ uniformly locally in x.
Then f admits an extension $g : \mathbb{R}^d \to \mathbb{R}$ of class C^1.
In order to apply this result to our Lipschitz function f, we notice that the existence of a Taylor expansion of order 1 is equivalent to differentiability and holds for a.e. $x \in \mathbb{R}^d$.
In order to have that $x \mapsto L_x = df(x)$ is continuous we must restrict our x to a subset of $\mathbb{R}^d$ (the complement of an open set with small measure)

and this can be done using the classical Lusin theorem (applied to the map $x \longmapsto df(x)$).
Finally, the uniformity in the remainder $R_x(h)$ can be achieved by further restricting the set of "admissible" x, using the Severini-Egorov theorem for this purpose..... $\square$

We can now give one of the fundamental definitions of this course

<u>Rectifiable Sets</u>

Let $d = 1, 2, \dots$ A (Borel) set E contained in $\mathbb{R}^n$ (or in any other metric space X) is called d-rectifiable, or rectifiable of dimension d, if it can be decompose as $E = \bigcup_{i=0}^{\infty} E_i$ where (Borel set!)

- $\mathcal{H}^d(E_0) = 0$
- E_i is contained in the image of a Lipschitz map f_i from (a Borel subset of) $\mathbb{R}^d$ to $\mathbb{R}^n$ for every $i \geq 1$.

<u>Remarks</u> • The class of d-rectifiable sets is the largest class of d-dimensional sets for which there is still a (very weak) notion of tangent bundle. They are the building blocks in the construction of d-dimensional integral currents.

- The name for what we called rectifiable sets varies (slightly) from author to author.
  Simon and Krantz & Parks call these sets "countably d-rectifiable". Federer uses an even more complicated term (due to the fact that he defines many different classes of rectifiable sets).
- Lipschitz images of d-rectifiable sets are d-rectifiable. This statement is almost obvious, but there is an issue with the measurability of the image (in general is not Borel).
- If E is d-rectifiable it may happen that $\mathcal{H}^d(E) = +\infty$ (if is a d-plane, for example) but in any case $\dim_{\mathcal{H}}(E) \leq d$ (one easily checks that indeed $\mathcal{H}^\alpha(E) = 0 \quad \forall \alpha > d$).
- Countable unions of d-rectifiable sets are d-rectifiable. Every (Borel) subset of a d-rectifiable set is d-rectif. In this sense we may say that d-rectifiable sets form a σ-ideal.
- If E is a (Borel) subset of a d-dimensional surfaces S of class $\mathcal{C}^1$ in $\mathbb{R}^d$, then E is d-rectifiable (because S can be parametrized by (countably many) maps of class $\mathcal{C}^1$, which in particular are locally lipschitz).
- An example of "ugly" 1-rectifiable set in $\mathbb{R}^2$ to keep in mind (for counterexamples):

$$E := \bigcup_{a,b \in \mathbb{Q}} E_{a,b}$$

where $E_{a,b}$ is the straight line with equation $y = ax + b$.

Since each $E_{a,b}$ is the image of a lipschitz (actually, affine) map from $\mathbb{R}$ to $\mathbb{R}^2$, E is clearly 1-rectifiable. Note that given any disk $B \subset \mathbb{R}^2$, $E \cap B$ contains segments with a dense set of directions ....

- The set E_0 in the definition of d-rectifiable set plays an important role from the technical point of view. Note that it cannot be dispensed with, in the sense that there are $\mathcal{H}^d$-null sets that cannot be covered by countably many lipschitz images of $\mathbb{R}^d$ (indeed, these sets can even be taken compact and with Hausdorff dimension equal to 0).

this is a nice and feasible exercise

A useful characterization of rectifiable sets

A (Borel) set $E \subset \mathbb{R}^n$ is d-rectifiable if and only if it can be decomposed as $E = \bigcup_{j=0}^{\infty} E_j$ where

- $\mathcal{H}^d(E_0) = 0$
- E_i is contained in a d-dimensional surface of class $\mathcal{C}^1$ for every $i \geq 1$.

Proof The "if" part is obvious since we already mentioned that (a subset of) a surface of class $\mathcal{C}^1$ is d-rectifiable (and that the class of rectifiable sets is closed under countable union).

To prove the "only if" part it suffices to show that a lipschitz image of $\mathbb{R}^d$ can be decomposed as in the statement above.

Let then $E = f(\mathbb{R}^d)$ with $f : \mathbb{R}^d \to \mathbb{R}^n$ Lipschitz.

Now we use the Lusin type property of lipschitz functions with $\mathscr{C}^1$ functions to find a sequence of (open) sets A_j and $\mathscr{C}^1$ maps $f_j : \mathbb{R}^d \to \mathbb{R}^n$ such that

- $\mathscr{L}^d(A_j) \to 0$
- $f = f_j$ on $\mathbb{R}^d \setminus A_j$.

Set now $A := \bigcap_j A_j$. Then $\mathscr{L}^d(A) = 0$ and

$$E := f(\mathbb{R}^d) \subset \overbrace{f(A)}^{E_0} \cup \Big(\bigcup_j \overbrace{f_j(\mathbb{R}^d)}^{E_j} \Big).$$

Since $\mathscr{L}^d(A) = 0$, we have that $\mathscr{H}^d(E_0 := f(A)) = 0$.

Therefore, in order to show that $f(E)$ can be decomposed as in the statement above it suffices to show that this is true for each $E_j := f_j(\mathbb{R}^d)$.

For every j, let Ω_j be the set of all $t \in \mathbb{R}^d$ such that $\operatorname{rank}(df_j(t)) = d$. Then Ω_j is open and f_j agrees <u>locally</u> on Ω_j with a parametrization of a $\mathscr{C}^1$ surfaces. This means that $f_j(\Omega_j)$ can be covered by countably many $\mathscr{C}^1$ surfaces of dimen. d in $\mathbb{R}^n$.

To conclude the proof it suffices to show that $\mathscr{H}^d(f_j(\mathbb{R}^d \setminus \Omega_j)) = 0$, and this follows from the area formula and the fact that $Jf_j(t) = 0$ for every $t \notin \Omega_j$ (because $\operatorname{rank}(df_j(t)) < d$).

$\square$

We conclude with a definition and a few remarks.

<u>d-purely unrectifiable sets</u>

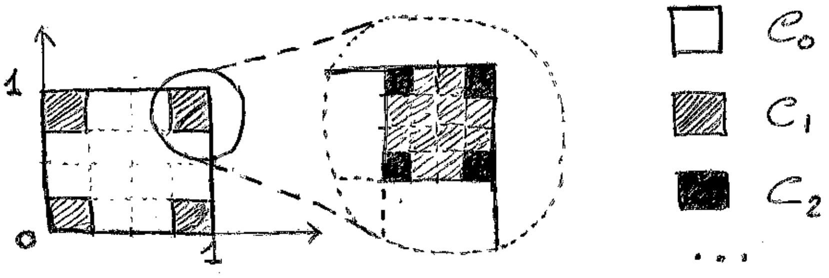
A Borel set E in $\mathbb{R}^n$ (or in any metric space) is d-purely unrectifiable if there holds

$$\mathcal{H}^d(E \cap F) = 0$$

for every set F in $\mathbb{R}^d$ which is d-rectifiable (or, which is clearly equivalent, every F which is a Lipschitz image of $\mathbb{R}^d$, or even every F which is a $\mathcal{C}^1$-surface of class $\mathcal{C}^1$ and dimension d).

$\boxed{\text{Additional remarks}}$ • Note first of all that there are d-unrectifiable sets E with $\mathcal{H}^d(E) > 0$. Indeed there are d-unrectifiable sets in $\mathbb{R}^n$ with dimension even equal to n.

A simple example is the following Cantor type set C in the plane: $C = \bigcap_{n=0}^{\infty} C_n$ where C_n consists of 4^n squares with side length 4^{-n} as in the picture



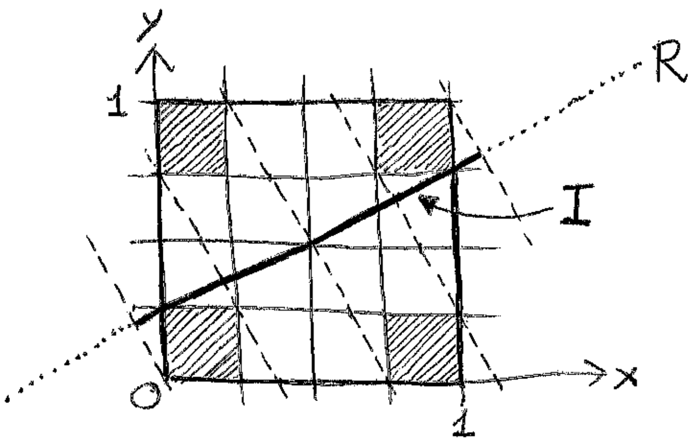
We claim that:

(i) $\mathcal{H}^1(C) \leq \sqrt{2}$.

To prove this it suffices to the coverings of C given by the 4^n squares with side-length 4^{-n} (and diameter $\sqrt{2}\cdot 4^{-n}$) that form C_n.

(ii) $\mathcal{H}^1(C) > 0$.

We use the fact that there exists a lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(C)$ has positive length. More precisely f is the orthogonal projection of $\mathbb{R}^2$ onto the dotted line R in the picture (that we identify with $\mathbb{R}$, if you like):



Indeed one easily checks that $f(C_n)$ agrees with the segment I in the picture for every n, and so does $f(C)$. Since f is 1-lipschitz and I has length $3/\sqrt{5}$, we infer that $\mathcal{H}^1(C) \geq 3/\sqrt{5}$.

(iii) C is purely unrectifiable.

We must show that $\mathcal{H}^1(C \cap F) = 0$ for every 1-rectifiable set F in $\mathbb{R}^2$. But taking into account the characterization of rectifiable sets given above, we may restrict to the case F is a $\mathcal{C}^1$ curve (intended as a submanifold). And since $\mathcal{C}^1$ curves can be locally written as graphs of $\mathcal{C}^1$ functions $y = y(x)$ or $x = x(y)$, we can further assume that F is one of these. Say the graph of $y = y(x)$, the other case being the same by simmetry.

Now, let C' be the projection of C on the x-axis. Then $C \cap F \subset (C' \times \mathbb{R}) \cap F$ and therefore

$$\mathcal{H}^1(C \cap F) \leq \mathcal{H}^1((C' \times \mathbb{R}) \cap F)$$

Lebesgue measure

$$= \int_{C'} \sqrt{1 + (y')^2}\, dx = 0$$

because of the formula for the length of the graph of a function $y(x)$ (recall elementary calculus), or the area formula, if you really need!

because the Cantor type set $C' \subset \mathbb{R}$ has Lebesgue measure equal to 0.

Let us proceed with further remarks.

- The construction above can be generalized by showing that every product set $E = E_1 \times E_2 \subset \mathbb{R}^2$ with E_1, E_2 Lebesgue negligible set is $\mathbb{R}$ is 1- purely unrectifiable.
  By suitably choosing E_1 and E_2 one can have that E has even Hausdorff dimension $= 2$ (in our example it has dimension $= 1$).

- Every set $E \subset \mathbb{R}^n$ with $\mathcal{H}^d(E) < +\infty$ can be decomposed as $E = E_r \cup E_{pu}$ with E_r d-rectifiable and E_{pu} d-purely unrectifiable. (Consider the class of all subsets of E that are d-rectifiable and show that this class is closed under countable union and therefore admits an element which maximizes $\mathcal{H}^d$; let E_r be such element.)

  This decomposition is unique up to $\mathcal{H}^d$-null subsets. This means that given another such decomposition $E'_r \cup E'_{pu}$ there holds

$$\mathcal{H}^d\big(\underbrace{E_r \Delta E'_r}_{\substack{\text{symmetric}\\ \text{difference of}\\ E_r \text{ and } E'_r}}\big) = \mathcal{H}^d(E_{pu} \Delta E'_{pu}) = 0$$

- The existence of sets E which are not d-rectifiable raises the question of characterizing the sets E (with $\mathcal{H}^d(E) < +\infty$) which are d-rectifiable.

  We collect here two of the few existing results. One of them played a key role in the original theory of integral currents (Federer & Fleming) but it no longer does. As far as I know, none of these results admits (at the moment) a simple proof.

- Characterization of rectifiability by projection

Let us begin with a simple remark.
We denote by $G(n,d)$ the Grassmannian of d-planes (d-dimensional subspaces) in $\mathbb{R}^n$, and for every $V \in G(n,d)$ denote by P_V the projection of $\mathbb{R}^n$ onto V.

Let now E be a $\mathcal{C}^1$ surface with dimension d in $\mathbb{R}^n$ and let $x_0 \in E$. If $V \in G(n,d)$ is such that the restriction of P_V to $\mathrm{Tan}(E, x_0)$ has rank d, then $P_V(x_0)$ is an interior point of $P_V(E)$, which therefore has positive $\mathcal{H}^d$-measure.

We have thus proved that $\mathcal{H}^d(P_V(E)) > 0$ for all $V \in G(n,d)$ except a subset of codimension at least 1. This implies that

$$(*) \qquad \mathcal{H}^d(P_V(E)) > 0 \quad \text{for a.e. } V \in G(n,d)$$

["a.e." with respect to the "natural measure on $G(n,d)$", namely the probability measure which is invariant under the action of isometries — note that every isometry on $\mathbb{R}^n$ defines an homeomorphism of $G(n,d)$ into itself, and actually the group of isometries acts on $G(n,d)$; moreover the group of isometries is endowed with a (unique!) invariant probability measure — the Haar measure — and this allows to define a (unique!) probability measure on $G(n,d)$, invariant under the action of this group....]

It is not difficult to see that $(*)$ holds even if E is a subset of a $\mathcal{C}^1$ surface S with $\mathcal{H}^d(E) > 0$. (Take as x_0 a point of density 1 of E, namely a point of approximate continuity of 1_E w.r.t. the restriction of $\mathcal{H}^d$ to S, and then apply the area formula to the map $P_V : S \to V$ and the set E.)

As a consequence we obtain immediately that (*) holds as well if E is a d-rectifiable set with $\mathcal{H}^d(E) > 0$.

It turns out, and this is a highly non trivial fact, that purely unrectifiable sets behave in the opposite way: if E is d-purely unrectifiable and $\mathcal{H}^d(E) < +\infty$ (or, at least, E is $\mathcal{H}^d$ σ-finite) then

(**) $$\mathcal{H}^d(P_V(E)) = 0 \quad \text{for a.e. } V \in G(n,d).$$

Thus every set E with $0 < \mathcal{H}^d(E) < +\infty$ such that (**) does not hold must contain a non-trivial rectifiable subset, that is, $\mathcal{H}^d(E_r) > 0$.

rectifiable part of E, see above

This result is due to Besicovitch for $n=2$, $d=1$, and to Federer for the general case.

- It is a nice exercise to check (**) directly for the 1-unrectifiable set C at page 9.

- <u>Characterization of rectifiability by density</u>

Let us start again by a simple remark. If E is surface of class C^1 and dimension d then E has d-dimensional density equal to 1 at every point (except boundary point), that is

$$(*) \qquad \lim_{r\to 0} \frac{\mathcal{H}^d(E \cap \overbrace{B(x,r)}^{\text{ball with center } x \text{ and radius } r \text{ in } \mathbb{R}^n})}{\underbrace{\omega_d}_{\substack{\text{volume of} \\ \text{unit ball in } \mathbb{R}^d}} r^d} = 1 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in E.$$

We will see in the next lecture that $(*)$ holds for every d-rectifiable set E with $\mathcal{H}^d(E) < +\infty$ as well.

A highly non trivial fact is that the existence of the density implies rectifiability: if $\mathcal{H}^d(E) < +\infty$ and the limit in $(*)$ exists for $\mathcal{H}^d$-a.e. x (not necessarily equal to 1) then d is <u>integer</u> and E is d-rectifiable.
(This statement summarizes two separate results by Marstrand and Preiss.)

- <u>About the definition of rectifiability</u>

We have seen that in the decomposition $E = \bigcup_{i=0}^{\infty} E_i$ that defines d-rectifiability, the sets E_i with $i \geq 1$ can be Lipschitz images of $\mathbb{R}^d$ or, equivalently, C^1 surfaces of dimension d. One might wonder if C^2 surfaces might work as well. The answer is <u>NO</u>, because there exist C^1 surfaces S such that the intersection of S with <u>any</u> C^2 surface is $\mathcal{H}^d$-null, and therefore S cannot be covered by any countable family of C^2 surfaces.
What we are saying can be rephrased as follows: there exist C^1 functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$\mathcal{L}^d(\{t : f(t) = g(t)\}) = 0$ for every $g : \mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}^2$.

Or, stated differently, $\mathcal{C}^1$ functions don't have the Lusin property with functions of class $\mathcal{C}^2$ (nor $\mathcal{C}^{1,\alpha}$ for any $\alpha > 0$, for that matter).

- Rectifiable sets in metric spaces.

The definition of d-rectifiable sets makes sense in every metric spaces.
However, particularly for $d > 1$, it may happen that very "reasonable" metric spaces X contain no nontrivial d-rectifiable set, even if $\dim_H(X) > d$.

A relevant example is the Heisenberg group H^1, which is the simplest example of sub-Riemannian geometry. As a set it is $\mathbb{R}^3$, and the group structure carries over a (left) invariant metric which induces the usual topology of $\mathbb{R}^3$.
However H^1 is not bi-Lipschitz equivalent of $\mathbb{R}^3$, and indeed its Hausdorff dimension is 4.
It turns out that every 2-rectifiable set $E \subset H^1$ is trivial, in the sense that $\mathcal{H}^2(E) = 0$ (this is not easy to prove from scratch).

In the same spirit (but the example is not equally relevant) one might show that if X is $\mathbb{R}$ endowed with the distance $d(x_1, x_2) := \sqrt{|x_1 - x_2|}$, then $\dim_H(X) = 2$ (use that the Peano curve $\gamma : I \to Q$ turns out to be Lipschitz if I is endowed with the metric d) but for every Lipschitz map

γ from a subset A of $\mathbb{R}$ into X there holds $\mathcal{H}^1_X(\gamma(A)) = 0$. Hence every 1-rectifiable set $E \subset X$ must satisfy $\mathcal{H}^1_X(E) = 0$.

One might think that even though the definition of rectifiable sets makes sense in metric space, still these set are "modeled" , much like manifolds, on the Euclidean space, and therefore they may not fit truly non-Euclidean metrics....

Lecture 5
26/3/14

We begin by giving a (very weak) notion of tangent bundle to a rectifiable set.
The definition is based on the following elementary observation:

Lemma 1
Let S_1, S_2 be C^1 surfaces with dimension d in $\mathbb{R}^n$. Then

(*) $\qquad \mathrm{Tan}(S_1,x) = \mathrm{Tan}(S_2,x) \quad$ for $\mathcal{H}^d$-a.e. $x \in S_1 \cap S_2$.

In fact there holds more: the equality in (*) holds for all $x \in S_1 \cap S_2$ except a subset of (Hausdorff) dimension at most $d-1$ (which could be the whole $S_1 \cap S_2$, of course ....)

Since d-dimensional surfaces can be locally written as graphs of maps from $\mathbb{R}^d$ to $\mathbb{R}^{n-d}$, with some care we can derive Lemma 1 from the following statement on functions:

Lemma 2 Let $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ be functions of class C^1 and let $I := \{x : f_1(x) = f_2(x)\}$. Then

(**) $\qquad df_1(x) = df_2(x) \quad$ for $\mathcal{L}^d$-a.e. $x \in I$,

and actually the equality in (**) holds for all $x \in I$ except a subset with dimension at most $d-1$.

Proof Let $I' := \{x : f_1(x) = f_2(x) \text{ and } df_1(x) \neq df_2(x)\}$.
Then I' is defined by the equation $f(x) = 0$ with $f := f_1 - f_2$, and $df(x) \neq 0 \ \forall x \in I'$, which implies

that Γ' is a $\mathcal{C}^1$ surface of codimension 1 in $\mathbb{R}^d$, and in particular it has dimension $d-1$ (or it is empty). $\square$

<u>Proposition 3</u> (Definition and existence of the weak tangent bundle to a rectifiable set).

Let E be a d-rectifiable set in $\mathbb{R}^n$.
Then there exists a (Borel!) map $\tau : E \to G(n,d)$ such that for every surface S of class C^1 and dimension d in $\mathbb{R}^n$ there holds

$G(n,d)$: Grassmannian of d-planes in $\mathbb{R}^n$

$$(*) \qquad \mathrm{Tan}(S,x) = \tau(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in S \cap E.$$

Moreover such τ is unique up to an $\mathcal{H}^d$-null subset of E, meaning that given another τ' satisfying $(*)$, there holds $\tau(x) = \tau'(x)$ for $\mathcal{H}^d$-a.e. $x \in E$.

Thus τ is called weak tangent bundle of E, and sometime denoted by τ_E.

<u>Remarks</u>
- The weak tangent bundle is not defined in any pointwise way, in particular it does not make sense to specify $\tau(x)$ at some given point x (as it does not make sense to specify the value of an L^p function at a given point).
- If E is (a subset of) a d-dimensional surface of class $\mathcal{C}^1$ the tangent bundle τ satisfies
$$\tau(x) = \mathrm{Tan}(E,x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x.$$

Indeed, that this bundle satisfies (*) in Proposition 3 is an immediate consequence of Lemma 1.
Thus the notion of tangent bundle given above is <u>compatible</u> with the classical one from geometry.

<u>Proof</u> (<u>of Proposition 3</u>).
EXISTENCE: since E is rectifiable, it can be written as $E = \bigcup_{i=0}^{\infty} E_i$ where $\mathcal{H}^d(E_0) = 0$ and each E_i is contained in a C^1 surface S_i for $i \geq 1$.
Then we set

$$\tau(x) := \begin{cases} \mathrm{Tan}(S_i, x) & \text{if } x \in \overbrace{E_i \setminus \Big(\bigcup_{j<i} E_j\Big)}^{E_i'} \\ & \text{with } i = 1, 2, \dots \\ \text{whatever} & \text{if } x \in \overbrace{E_0 \setminus \Big(\bigcup_{j=1}^{\infty} E_j\Big)}^{E_0'} . \end{cases}$$

To check that such τ satisfies (*) in Prop. 3 is easy using Lemma 1: indeed this lemma yields

$$\mathrm{Tan}(S, x) = \mathrm{Tan}(S_i, x) = \tau(x)$$

for $\mathcal{H}^d$-a.e. $x \in S \cap E_i'$ for every $i \geq 1$. We then get (*) by observing that the sets E_i' with $i = 1, 2, \dots$ cover $\mathcal{H}^d$-a.e. of E and therefore $S \cap E_i'$ cover $\mathcal{H}^d$-a.e. of $S \cap E$.

UNIQUENESS: Let τ' be a bundle satisfying (*), we show that τ' agrees $\mathcal{H}^d$-a.e. with the bundle τ defined above in this proof. Indeed applying (*) to τ'

yields that $\tau'(x) = \mathrm{Tan}(S_i, x)$ for $\mathcal{H}^d$-a.e. $x \in E \cap S_i$ and in particular $\tau'(x) = \tau(x)$ for $\mathcal{H}^d$-a.e. $x \in E \cap E_i'$, and as before we deduce that $\tau'(x) = \tau(x)$ for $\mathcal{H}^d$-a.e. $x \in E$.

$\square$

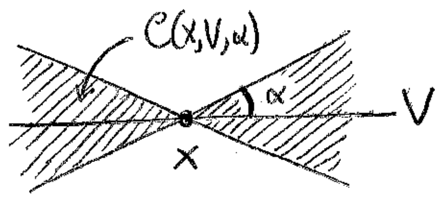
In the next statements we assume that E is a d-rectifiable set satisfying the additional assumption

$$\mathcal{H}^d(E) < +\infty$$

(it actually suffices that E has locally finite $\mathcal{H}^d$-measure).

These staments express the fact that close to a generic point $x \in E$ ("generic" meaning "$\mathcal{H}^d$-a.e.") the set E looks similar (in some sense or another) to an (affine) d-dimensional plane given by $\tau(x)$.

We need first some notation: given a d-plane $V \in G(n,d)$, a point $x \in \mathbb{R}^n$, $a > 0$ we denote by $C(x, V, a)$ the cone with axis V centered at x and with angle α s.t. $\sin\alpha = a$



that is,

$$\mathcal{C}(x,V,a) := x + \mathcal{C}(V,a)$$

$$:= x + \{h \in \mathbb{R}^n : \mathrm{dist}(h,V) \le a|h|\}$$

We express the fact that $\tau(x)$ is tangent to E at x by showing that "close to x" most of E is to be found inside the cone $\mathcal{C}(x, \tau(x), \varepsilon)$ for every $\varepsilon > 0$.

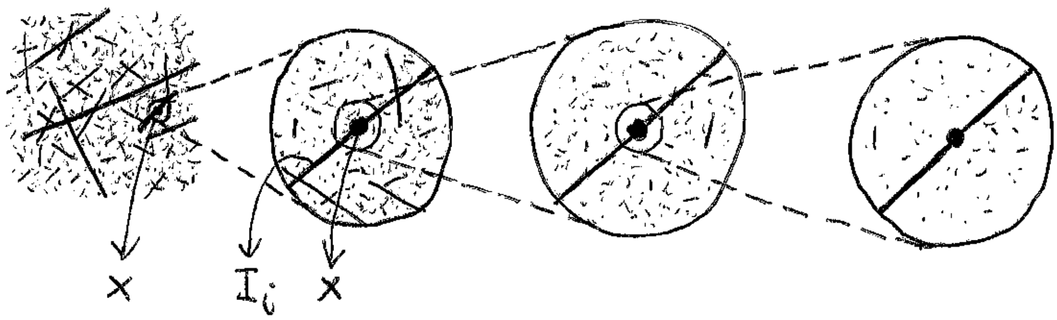
More precisely:

<u>Theorem 4</u> Let E be a d-rectifiable set with $\mathcal{H}^d(E)<+\infty$, and let τ be the tangent bundle to E, as given in Proposition 3.
Then, for $\mathcal{H}^d$-a.e. x and every $\varepsilon>0$, there holds

$$(*)\qquad \begin{aligned} &\mathcal{H}^d(E\cap B(x,r)\cap \mathcal{C}(x,\tau(x),\varepsilon)) \sim \omega_d r^d \text{ as } r\to 0,\\ &\mathcal{H}^d(E\cap (B(x,r)\setminus \mathcal{C}(x,\tau(x),\varepsilon))) \ll r^d \text{ as } r\to 0. \end{aligned}$$

where ω_d is, as usual, the volume of the unit ball in $\mathbb{R}^d$, and therefore $\omega_d r^d$ is the $\mathcal{H}^d$-measure of the (flat) d-dimensional disk with radius r (in $\mathbb{R}^n$).

<u>Remark</u> This statement is not at all obvious. To understand its subtlety, consider the following example: E is the 1-rectifiable set in $\mathbb{R}^2$ given by the union of a sequence of segments I_i with length ℓ_i so that $\sum \ell_i<+\infty$. Note that these segments can be chosen so that for every disk $B\subset\mathbb{R}^2$, the directions of the segments I_i contained in D are <u>dense</u> (in the space of directions).
What happens is the following: if x is a "generic" (or "random") point of I_i, and we look at $E\cap B(x,r)$ with r smaller and smaller, this sets will always include (parts of) other segments besides I_i, but I_i becomes "predominant" in term of measure.
I try to give an idea of this in the picture below



Now, it is instructive to try and give a direct proof of this claim (without using the Radon-Nikodym theorem, that is!).

<u>Proof (of Theorem 4)</u>

Let $\{S_i\}$ be a countable family of $\mathcal{C}^1$ surfaces that cover $\mathcal{H}^d$-almost all of E.

By property (*) in Proposition 3 we have that for every i

$$\tau(x) = \mathrm{Tan}(S_i, x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \underbrace{E \cap S_i}_{E_i}.$$

Therefore the claim in Theorem 4 reduces to the following: $\forall i$, $\forall \varepsilon > 0$ and for $\mathcal{H}^d$-a.e. $x \in E_i := E \cap S_i$ there holds

$$(1) \qquad \begin{aligned} &\mathcal{H}^d(S \cap B(x,r) \cap \mathcal{C}(x, \mathrm{Tan}(S_i,x), \varepsilon)) \sim \omega_d r^d \quad \text{as } r \to 0, \\ &\mathcal{H}^d(S \cap (B(x,r) \setminus \mathcal{C}(x, \mathrm{Tan}(S_i,x), \varepsilon))) \ll r^d \quad \text{as } r \to 0. \end{aligned}$$

We define the following auxiliary measures:

$$\lambda := 1_{S_i} \cdot \mathcal{H}^d = \text{restriction of } \mathcal{H}^d \text{ to } S_i$$

characteristic function of S_i

$$\mu' := 1_{E_i} \cdot \mathcal{H}^d = \quad " \quad " \quad " \quad " \quad E_i$$

$$\mu'' := 1_{E \setminus E_i} \cdot \mathcal{H}^d = \quad " \quad " \quad " \quad " \quad E \setminus E_i = E \setminus S_i$$

We obseve that since S_i is a surface of class C^1 and dimension d

$$\lambda(B(x,r)) = \mathcal{H}^d(S \cap B(x,r)) \sim \omega_d r^d .$$

Note now that $\mu' := 1_{E_i} \cdot \mathcal{H}^d = 1_{E_i} \cdot \lambda$, and therefore the Lebesgue - Radon - Nikodym theorem yields that

$$\frac{\mu'(B(x,r))}{\lambda(B(x,r))} \xrightarrow[r\to 0]{} 1_{E_i}(x) \quad \text{for } \lambda\text{-a.e. } x ,$$

which implies

(2) $$\mu'(B(x,r)) \sim \lambda(B(x,r)) \sim \omega_d r^d \quad \text{for } \lambda\text{-a.e. } x \in E_i,$$
that is,
for $\mathcal{H}^d$-a.e. $x \in E_i$.

Consider now μ''. Since μ'' and λ are supported on the disjoint sets $E \setminus S_i$ and S_i they are mutually singular, and therefore the R.-N. density of μ'' w.r.t. λ is 0. Hence

$$\frac{\mu''(B(x,r))}{\lambda(B(x,r))} \xrightarrow[r\to 0]{} 0 \quad \text{for } \lambda\text{-a.e. } x$$

which implies

(3) $$\mu''(B(x,r)) \ll r^d \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in E_i .$$

We can now conclude the proof of (1).
Since S_i is a C^1 surface, $\forall \varepsilon > 0 \ \exists r_0 > 0$ s.t. for $r \leq r_0$ there holds $S \cap B(x,r) \subset \mathcal{C}(x, \mathrm{Tan}(S_i,x), \varepsilon)$ and therefore

(4) $$E_i \cap B(x,r) \subset \mathcal{C}(x, \mathrm{Tan}(S_i,x), \varepsilon)$$

Hence writing $E = E_i \cup (E\setminus E_i)$ we get (for $\mathcal{H}^d$-a.e. $x\in E_i$)

$$\mathcal{H}^d(E\cap B(x,r)\cap \mathcal{C}(x,\mathrm{Tau}(S_i,x),\varepsilon)) = \mathcal{H}^d(E_i\cap B(\dots)\cap \mathcal{C}(\dots)) + \mathcal{H}^d((E\setminus E_i)\cap B(\dots)\cap\mathcal{C}(\dots))$$

$$\mathcal{H}^d(E_i\cap B(\dots)\cap \mathcal{C}(\dots)) \overset{\text{by (4)}}{=} \mathcal{H}^d(E_i\cap B(\dots)) \overset{\text{by def. of }\mu'}{=} \mu'(B(\dots)) \overset{\text{by (2)}}{\geq} \omega_d r^d$$

$$\mathcal{H}^d((E\setminus E_i)\cap B(\dots)\cap\mathcal{C}(\dots)) \leq \mathcal{H}^d((E\setminus E_i)\cap B(\dots)) \overset{\text{by def. of }\mu''}{=} \mu''(B(\dots)) \overset{\text{by (3)}}{\ll} r^d$$

and we have proved the first line in (1).

Similarly we prove the second line in (1):

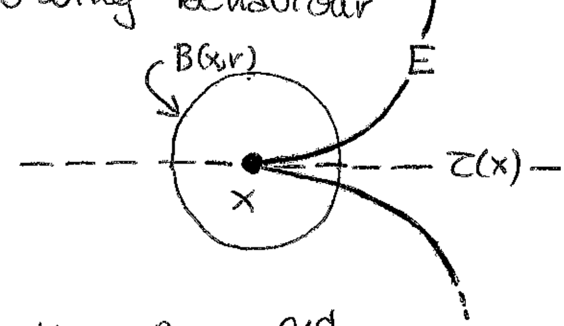
$$\mathcal{H}^d(E\cap(B(x,r)\setminus\mathcal{C}(x,\mathrm{Tau}(S_i,x),\varepsilon))) = \mathcal{H}^d(E_i\cap(B(\dots)\setminus\mathcal{C}(\dots))) + \mathcal{H}^d((E\setminus E_i)\cap(B(\dots)\setminus\mathcal{C}(\dots)))$$

because $E_i\cap(B(\dots)\setminus\mathcal{C}(\dots))$ is empty (by (4)): $\mathcal{H}^d(E_i\cap(B(\dots)\setminus\mathcal{C}(\dots))) = 0$

$$\mathcal{H}^d((E\setminus E_i)\cap(B(\dots)\setminus\mathcal{C}(\dots))) \leq \mathcal{H}^d((E\setminus E_i)\cap B(\dots)) = \mu''(B(\dots)) \ll r^d$$

$\square$

Theorem 4 is rather precise, but it can be improved. For instance, according to its statement the following behaviour



is admissible for $\mathcal{H}^d$-a.e. $x\in E$, while a close reading of the proof shows that it is not.

For every $x \in \mathbb{R}^n$ and $r > 0$ let $\psi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ be the homothety that takes the ball $B(x,r)$ onto the ball $B(0,1)$, that is

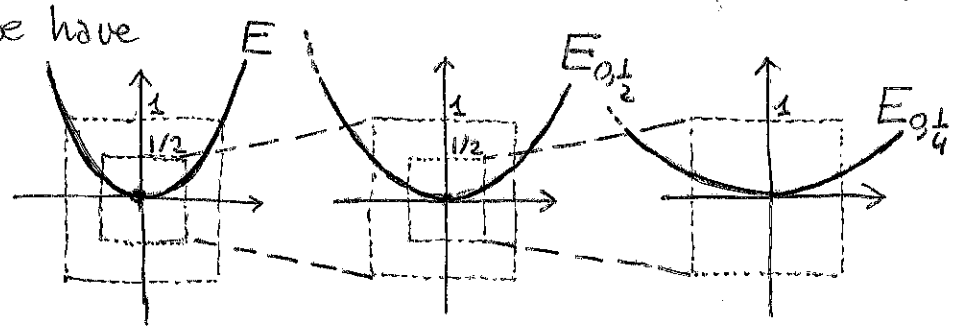
$$\psi_{x,r}(t) := \frac{1}{r}(t - x)$$

(or, if you prefer $\psi_{x,r} : x + rh \longrightarrow h$), and let

$$E_{x,r} := \psi_{x,r}(E)$$

be the blow-up of E around the point x by a scaling factor $\frac{1}{r}$.

For example, if $x = 0$ and E is given in the picture below we have



As the picture suggests, we expect that as $r \to 0$ $E_{x,r}$ "converge" in some sense to the tangent space of E at x.
If E is surface of class $\mathcal{C}^1$ this convergence occurs in the sense of the Hausdorff distance.

If E is a rectifiable set convergence cannot be established in terms of convergence of <u>sets</u> but only in terms of convergence of <u>measures</u>.
We have indeed the following:

Theorem 5

Let E be a d-rectifiable set in $\mathbb{R}^n$ with $\mathcal{H}^d(E)<+\infty$. For every $x\in\mathbb{R}^n$ and $r>0$ let $E_{r,x}$ be as above, and let $\mu_{x,r} := 1_{E_{x,r}}\cdot\mathcal{H}^d$ = restriction of $\mathcal{H}^d$ to $E_{x,r}$.

For every $x\in E$ let moreover $\mu_{\tau(x)} := 1_{\tau(x)}\cdot\mathcal{H}^d$ = restriction of $\mathcal{H}^d$ to $\tau(x)$, where τ is the tangent bundle of E defined above.

Then for $\mathcal{H}^d$-a.e. $x\in E$, as $r\to 0$ we have that

(*) $\mu_{x,r}$ converge to $\mu_{\tau(x)}$ in the sense of measures,

which means that for every $g:\mathbb{R}^n\to\mathbb{R}$ continuous and with compact support there holds

$$\begin{array}{ccc}
\displaystyle\int_{\mathbb{R}^n} g\, d\mu_{x,r} & \xrightarrow[r\to 0]{} & \displaystyle\int_{\mathbb{R}^n} g\, d\mu_{\tau(x)} \\
\| & & \| \\
\displaystyle\int_{h\in E_{x,r}} g(h)\, d\mathcal{H}^d(h) & & \displaystyle\int_{h\in\tau(x)} g(h)\, d\mathcal{H}^d(h) \\
\| & & \\
\displaystyle\frac{1}{r^d}\int_{t\in E} g\left(\frac{t-x}{r}\right) d\mathcal{H}^d(t) & &
\end{array}$$

Remarks • We called the above "convergence in the sense of measures", but actually it is NOT the usual sense, because the measures $\mu_{x,r}$ do not have uniformly bounded masses, and indeed the test functions g are required to be compactly supported, and not just to vanish at infinity. However the measures $\mu_{x,r}$ are locally uniformly bounded in mass, and therefore the appropriate space of test functions are the continuous ones with compact support.

- A consequence of this notion of convergence is that $\mu_{x,r}(F) \to \mu_{\tau(x)}(F)$ for every bounded Borel set F in $\mathbb{R}^n$ such that the topological boundary of F, ∂F, satisfies $\mu_{\tau(x)}(\partial F) = 0$. (Check that $\mu_{\tau(x)}(\partial F)=0$!)

By taking $F := B(0,1) \cap \mathcal{C}(0,\tau(x),\varepsilon)$ we obtain that

$$\begin{array}{ccc} \mu_{x,r}(F) & \longrightarrow & \mu_{\tau(x)}(F) \\ \| & & \| \\ \frac{1}{r^d}\mathcal{H}^d(E \cap B(x,r) \cap \mathcal{C}(x,\tau(x),\varepsilon)) & & \omega_d \end{array}$$

and therefore we recover the first line in (*) in Theorem 4.

By taking $F := B(0,1) \setminus \mathcal{C}(0,\tau(x),\varepsilon))$ we obtain instead

$$\begin{array}{ccc} \mu_{x,r}(F) & \longrightarrow & \mu_{\tau(x)}(F) \\ \| & & \| \\ \frac{1}{r^d}\mathcal{H}^d(E \cap (B(x,r) \setminus \mathcal{C}(x,\tau(x),\varepsilon))) & & 0 \end{array}$$

and we recover the second line in (*) in Theor. 4.

- Can we say something about the behaviour of $\mu_{x,r}$ as $r \to 0$ when x belongs to $\mathbb{R}^n \setminus E$?
- If the statement of Theorem 5 holds for a certain x, we say that $\tau(x)$ is the <u>APPROXIMATE</u> TANGENT PLANE to E at x. (According to some authors it suffices

that Theorem 4 holds).

Proof (of Theorem 5)

In essence, this proof is very similar to that of Theorem 4.

As in that proof we consider a countable family $\{S_i\}$ that covers $\mathcal{H}^d$-almost all of E, and we reduce to show that for every i and $\mathcal{H}^d$-a.e. $x \in S \cap E_i$ there holds

(1) $\mu_{x,r}$ converges to $\mu_{\mathrm{Tau}(S_i,x)}$ as $r \to 0$.

For every x and r we denote by

$$\lambda_{x,r} \; ; \; \lambda'_{x,r} \; ; \; \lambda''_{x,r}$$

the restriction of the measure $\mathcal{H}^d$ respectively to the sets

$$(S_i)_{x,r} \; ; \; (S_i \setminus E)_{x,r} \; ; \; (E \setminus S_i)_{x,r} .$$

Since $1_{E_{x,r}} = 1_{(S_i)_{x,r}} - 1_{(S_i \setminus E)_{x,r}} + 1_{(E \setminus S_i)_{x,r}}$ we have

(2) $$\mu_{x,r} = \lambda_{x,r} - \lambda'_{x,r} + \lambda''_{x,r} .$$

Now we study the convergence of these measures (as $r \to 0$) on the open ball $B := B(0,1)$

Claim 1: $\lambda_{x,r} \longrightarrow \mu_{\mathrm{Tau}(S_i,x)}$.

in the sense of measure on B

This claim can be easily verified by writing S_i as a graph (of a map from $\mathbb{R}^d$ to $\mathbb{R}^{n-d}$) in a neighbourhood of x, and using the area formula to represent the measure on S_i and on $(S_i)_{x,r}$. (Fill in the details!)

Claim 2 : $\lambda''_{x,r} \longrightarrow 0$.

We actually show that these measure converge to 0 in mass on B.

Indeed
$$\lambda''_{x,r}(B) = \mathcal{H}^d\left(((E\setminus S_i)_{x,r} \cap B(0,1)\right)$$
$$= \frac{1}{r^d}\mathcal{H}^d\left((E\setminus S_i) \cap B(x,r)\right) \xrightarrow[r\to 0]{} 0$$

The last convergence was already proved in the proof of Theorem 4 (formula (3)).

Claim 3 : $\lambda'_{x,r} \longrightarrow 0$.

Again, we show convergence to 0 of masses.

Indeed
$$\lambda_{x,r}(B) = \mathcal{H}^d\left((S_i\setminus E)_{x,r} \cap B(0,1)\right)$$
$$= \frac{1}{r^d}\mathcal{H}^d\left((S_i\setminus E) \cap B(x,r)\right)$$
$$= \Bigg[\underbrace{\frac{\mathcal{H}^d(S_i \cap B(x,r))}{r^d}}_{\downarrow\, r\to 0 \atop \omega_d} - \underbrace{\frac{\mathcal{H}^d((E\cap S_i)\cap B(x,r))}{r^d}}_{\downarrow\, r\to 0 \atop \omega_d}\Bigg] \xrightarrow[r\to 0]{} 0$$

And again, the last two limits were already given in the proof of Theorem 4 (formula (2) and above)

Putting together the decomposition (2) and Claims 1-3 we obtain that $\mu_{x,r} \to \mu_{\mathrm{Tan}(S_i,x)}$ on $B := B(0,1)$.

To conclude the proof it suffices to check that the same argument works for $B := B(0,R)$ for every $R>0$.

$\square$

We conclude this lecture with some "advanced" versions of the area formula.

Area formula (third version)

This is exactly the same as the second version, except that ϕ is a Lipschitz (and not C^1) map from $\mathbb{R}^d$ to $\mathbb{R}^n$. Then for every set $E \subset \mathbb{R}^d$ there holds

$$(*) \qquad \int_{\mathbb{R}^n} \#\left(\phi^{-1}(x) \cap E\right) \, d\mathcal{H}^d(x) = \int_E J\phi(t)\, dt \, .$$

As for the second version, this formula can be strengthened as follows: given a Borel function $h : \mathbb{R}^d \to [0,+\infty]$ then

$$(**) \qquad \int_{\mathbb{R}^n} \Big(\sum_{t \in \phi^{-1}(x) \cap E} h(t) \Big) \, d\mathcal{H}^d(x) = \int_E h(t)\, J\phi(t)\, dt \, .$$

Sketch of proof

The idea is to use the Lusin property of Lipschitz functions with C^1 functions.

Using this property we obtain a sequence of C^1 maps $\phi_i : \mathbb{R}^d \to \mathbb{R}^n$ and a sequence of (open) sets $A_i \subset \mathbb{R}^d$ such that $\mathcal{L}^d(A_i) \to 0$, ϕ is differentiable at every point of $\mathbb{R}^n \setminus A_i$, and $\phi_i = \phi$, $d\phi_i = d\phi$ at every such point.

Let now $E_i := E \setminus \big(A_i \cup (\bigcup_{j<i} E_j)\big)$; $E' := E \cap (\bigcap_i A_i)$.

Then E is the disjoint union of E' and all E_i.

Moreover, by applying (the second version of) the area formula to ϕ_i and E_i we get that (*) holds with E_i in place of E.

Concerning E', we have that (*) holds with E' instead of E, as well, and more precisely both sides of this identity are 0: the right-hand side because E' is $\mathscr{L}^d$-null, the left-hand side because $\phi(E')$ is $\mathcal{H}^d$-null (recall that ϕ is Lipschitz) and therefore $\#(\phi^{-1}(x)\cap E) = 0$ for $\mathcal{H}^d$-a.e. $x \in \mathbb{R}^n$.
By putting together the area formula for the sets E_i and E' we get it for the set E. $\square$

The area formula holds even if E is a d-rectifiable set in some $\mathbb{R}^m$ and $\phi : \mathbb{R}^m \to \mathbb{R}^n$ a Lipschitz map, but a precise statement requires some preliminary work.
Since E in general is $\mathscr{L}^m$-null, the map ϕ may be not differentiable at any point of E.
However it is tangentially differentiable, in the sense that for $\mathcal{H}^d$-a.e. $t \in E$ there exists a <u>linear map $d_\tau\phi(t) : \tau(t) \to \mathbb{R}^n$</u> such that the following first order Taylor expansion holds:

$$\phi(t+h) = \phi(t) + \langle d_\tau \phi(t) ; h \rangle + o(|h|)$$

for every $h \in \tau(x)$.

tangential differential of ϕ at t

In particular we can define the (tangential) Jacobian determinant $J_\tau\phi(t)$ as the determinant of the tangential differential $d_\tau\phi(t)$ as we did few lectures ago

<u>Remarks</u> ◦ The tangential differentiability of ϕ w.r.t. E is an (almost) immediate consequence of the tangential differentiability of ϕ w.r.t. d-dimensional surfaces of class $\mathcal{C}^1$ (use the fact that E can be covered by such surfaces....). Which in turn can be easily proved using the fact that surfaces can be parametrized by maps of class $\mathcal{C}^1$ defined on (open subsets of) $\mathbb{R}^d$.

◦ It is not difficult to show that the tangential differential $d_E f$ depends only on the restriction of ϕ to E (and not on the behaviour of f outside E); more precisely, given $\phi, \tilde{\phi} : \mathbb{R}^m \to \mathbb{R}^n$ lipschitz and such that $\phi = \tilde{\phi}$ ($\mathcal{H}^d$-a.e.) on E then for $\mathcal{H}^d$-a.e. $t \in E$ there holds $d_\tau \phi(t) = d_\tau \tilde{\phi}(t)$.
We can thus define the tangential differential even for lipschitz maps $\phi : E \to \mathbb{R}^n$.

◦ If E is a d-rectifiable in $\mathbb{R}^m$ and the map $\phi : E \to \mathbb{R}^n$ is lipschitz, then we already know that $\tilde{E} := \phi(E)$ is d-rectifiable, and that ϕ is tangentially differentiable at $\mathcal{H}^d$-a.e. $t \in E$. It remains to notice that the image of $d_\tau \phi(t)$ is contained in $\mathrm{Tan}(\tilde{E}, \phi(t))$ for $\mathcal{H}^d$-a.e. $t \in E$, that is, $d\phi(t) : \mathrm{Tan}(E,t) \to \mathrm{Tan}(\tilde{E}, \phi(t))$.
The proof of this fact, which is left as an exercise, can be easily obtained from the

definition of rectifiable sets and the following lemma: Let $S \subset \mathbb{R}^m$, $\tilde{S} \subset \mathbb{R}^n$ be d-dimensional surfaces of class $\mathcal{C}^1$, let $E \subset S$, and let $\phi : S \to \mathbb{R}^n$ be a map of class $\mathcal{C}^1$ s.t. $\phi(E) \subset \tilde{S}$.
Then $d_\tau\phi(x)$ maps $\mathrm{Tan}(S,x)$ into $\mathrm{Tan}(\tilde{S}, \phi(x))$ for $\mathcal{H}^d$-a.e. $x \in E$.

We can now state:

<u>Area formula (fourth version)</u>

Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map, and let E be a d-rectifiable set in $\mathbb{R}^m$.
Then

(*)
$$\int_{\mathbb{R}^n} \#(\phi^{-1}(x) \cap E) \, d\mathcal{H}^d(x) = \int_E J_\tau\phi(t) \, d\mathcal{H}^d(t)$$

(Try and fill in the details of the proof!)

<u>Remarks</u> • As pointed out in the remarks above, $d_\tau\phi(x)$ depends only on the restriction of ϕ to E, at least for $\mathcal{H}^d$-a.e. x. This means that we can assume that ϕ is defined only on E.

• As usual, we can strengthen (*) as follows: given $h : E \to [0,+\infty]$ Borel, then

$$\int_{\mathbb{R}^n} \Big(\sum_{t \in \phi^{-1}(x) \cap E} h(t) \Big) \, d\mathcal{H}^d(x) = \int_E h(t) \, J_\tau\phi(t) \, d\mathcal{H}^d(t).$$

Review of basic notions of multilinear algebra.

We begin with some very general definitions. The geometric meaning of these will be clarified later.

To begin with:

V is a (real) vector space.

V^* is the dual of V.

Definition

A k-linear alternating form on V (or a k-covector on V) is a function

$$\alpha : V^k \longrightarrow \mathbb{R}$$

such that:

(i) α is linear in each variable;

(ii) α is alternating, that is

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot \alpha(v_1, \dots, v_k)$$

for every $v_1, \dots, v_k \in V$ and every $\sigma \in S_k$ group of permutations of the indexes $\{1, \dots, k\}$.

OR EQUIVALENTLY:

if we swap any two variables in $\alpha(v_1, \dots, v_k)$ the value changes sign.

The vector space of k-covectors on V is denoted by

$$\Lambda^k(V)$$

Remarks

- We set $\Lambda^0(V) = \mathbb{R}$
- $\Lambda^1(V) = V^*$
- $\Lambda^k(V)$ has dimension 1 if $k = \dim V$.
  This is a reformulation of the well-known fact (from elementary linear algebra) that the determinant of $k \times k$ matrices is uniquely characterized as a k-linear functions on $\mathbb{R}^k$ (whose entries are the colums of the matrix) such that $\det I = 1$.
- If $\alpha \in \Lambda^k V$ and $v_1, \ldots, v_k$ are linearly dependent then $\alpha(v_1, \ldots, v_k) = 0$. Assume indeed that v_k can be written as linear combination of $v_1, \ldots, v_{k-1}$ and apply properties (i) and (ii) above...
- $\Lambda^k(V) = \{0\}$ if $k > \dim V$ (by the previous property).

Exterior Product

Given $\alpha \in \Lambda^h(V)$ and $\beta \in \Lambda^k(V)$ we define $\alpha \wedge \beta \in \Lambda^{h+k}(V)$ as follows

$$(\alpha \wedge \beta)(v_1, \ldots, v_{h+k})$$
$$:= \frac{1}{k!\,h!} \sum_{\sigma \in S_{h+k}} \operatorname{sgn}(\sigma)\ \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(h)}) \cdot \beta(v_{\sigma(h+1)}, \ldots, v_{\sigma(k+h)})$$

for every $v_1, \ldots, v_{h+k} \in V$.

Remarks

- Each addendum in this formula is clearly linear in each variable but not necessarily alternating; it is easy to see that the sum is alternating.
- The choice of the renormalizing factor $\frac{1}{h!k!}$ will play a role later.
- The exterior product is clearly linear in each factor. It is also associative (this requires a bit of proof). It is NOT COMMUTATIVE: Indeed

$$\beta \wedge \alpha = (-1)^{h \cdot k} \alpha \wedge \beta$$

In particular $\alpha \wedge \alpha = 0$ if h is odd (but not necessarily if h is even).

We now fix a basis $e_1, \dots, e_n$ on V.

Then $e_1^*, \dots, e_n^*$ is the corresponding basis of $V^* = \Lambda^1(V)$, that is,

$$e_i^*(e_j) = \delta_{ij} \qquad \forall\, i,j = 1, \dots, n.$$

Now, let $\underline{i} = (i_1, \dots, i_k)$ be a multiindex, and let $I_{n,k}$ be the set of all $\underline{i}$ such that

$$1 \le i_1 < i_2 < \dots < i_k \le n .$$

For every such $\underline{i}$ we set

$$e_{\underline{i}}^* := e_{i_1}^* \wedge \dots \wedge e_{i_k}^* .$$

<u>Theorem</u> The set

$$\{ e^*_{\underline{i}} \ : \ \underline{i} \in I_{n,k} \}$$

is a basis of $\Lambda^k(V)$, and more precisely for every $\alpha \in \Lambda^k(V)$ there holds

$$\alpha = \sum_{\underline{i} \in I_{n,k}} \alpha_{\underline{i}} \cdot e^*_{\underline{i}}$$

with $\alpha_{\underline{i}} := \alpha(e_{i_1}, \dots, e_{i_k}) \quad (=: \alpha(e_{\underline{i}}))$.

This theorem is the main result for this lecture. The proof requires some lemmas.

<u>Lemma 6</u> For every $\underline{i}$ and every $v_1, \dots, v_k \in V$ there holds

$$e^*_{\underline{i}}(v_1, \dots, v_k) = \det V$$

where V is the $k \times k$ matrix defined by

$$V_{je} := e^*_{i_j}(v_e)$$

that is, the matrix whose e-th column is given by the coordinates of the vectors v_e with respect to the elements $e_{i_1}, \dots, e_{i_k}$ of the basis of V (recall that $e^*_j(v)$ is the coordinate of v w.r.t. e_j).

<u>Remark</u> Here is were the choice of the renormalization factor $\frac{1}{\ell!\,k!}$ in the definition of Λ plays a role.

Proof By induction on k, if you really must.... $\square$

Lemma 2 If $\alpha \in \Lambda^k(V)$ and

$$(*) \qquad \alpha(e_{i_1}, \dots, e_{i_k}) = 0 \quad \forall\, \underline{i} \in I_{n,k}$$

then $\alpha \equiv 0$.

Proof From $(*)$ and from the fact tha α is alternating you get that

$$\alpha(e_{i_1}, \dots, e_{i_k}) = 0 \qquad \forall\, i_1, \dots, i_k \in \{1, \dots, n\}$$

Freeze all variable except the first one: by linearity you obtain that

$$\alpha(v_1, e_{i_2}, \dots, e_{i_k}) = 0 \quad \forall\, v_1 \in V, \; i_2, \dots, i_k \in \{1, \dots, n\}$$

Then by linearity in the second variable you get

$$\alpha(v_1, v_2, e_{i_3}, \dots, e_{i_k}) = 0 \quad \forall\, v_1, v_2 \in V, \; i_3, \dots, i_k \in \{1, \dots, n\}.$$

And so on.... $\square$

Lemma 3 For every $\underline{i}, \underline{j} \in I_{n,k}$ there holds

$$\underbrace{e^*_{\underline{i}}(e_{\underline{j}})}_{= \, e^*_{\underline{i}}(e_{j_1}, \dots, e_{j_k})} = \delta_{\underline{i}, \underline{j}}$$

Proof. Immediate from Lemma 1. $\square$

Proof of main theorem.

We first prove that

$$\alpha = \sum_{\underline{i}} \alpha_{\underline{i}} e^*_{\underline{i}} \quad \text{with} \quad \alpha_{\underline{i}} := \alpha(e_{\underline{i}}),$$

which shows that $\{e^*_{\underline{i}}\}$ spans $\Lambda^k(V)$.
Let indeed

$$\beta := \alpha - \sum_{\underline{i}} \alpha_{\underline{i}} e^*_{\underline{i}} .$$

Then, for every $\underline{j} \in I_{n,k}$,

$$\beta(e_{\underline{j}}) = \alpha(e_{\underline{j}}) - \sum_{\underline{i}} \alpha_{\underline{i}} \overbrace{e^*_{\underline{i}}(e_{\underline{j}})}^{\delta_{\underline{i}\underline{j}} \text{ by lemma 3}}$$

$$= \alpha(e_{\underline{j}}) - \alpha_{\underline{j}} \underset{\uparrow}{=} 0$$

and then $\beta \equiv 0$ by lemma 2. by def. of $\alpha_{\underline{j}}$

To prove that the elements of $\{e^*_{\underline{i}}\}$ are linearly independent it suffices to use lemma 3.

$\square$

A consequence of this theorem is that

$$\dim(\Lambda^k(V)) = \# I_{n,k} = \begin{cases} \binom{n}{k} & \text{if } k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Back to $V = \mathbb{R}^n$

In this case $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.

The corresponding dual basis is usally denoted by $dx_1, \dots, dx_n$.
This is in agreement with the notation for the differential of functions (the differential of the function $x \longmapsto x_i$ is indeed e_i^*).
Consequently, we write $dx_{\underline{i}}$ for $e_{\underline{i}}^*$, that is

$$dx_{\underline{i}} = dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

We can now prove the identity

$$\det(A^t A) = \sum_{\substack{M \ k\times k\text{-minor} \\ \text{of } A}} (\det M)^2$$

for every $A \in \mathbb{R}^{n \times k}$ that we used in a previous lecture. We actually prove more:

<u>Generalized</u> <u>Binet</u> <u>Identity</u>:
For every matrix $A \in \mathbb{R}^{n\times k}$ and every $\underline{i} \in I_{n,k}$ let $A^{\underline{i}}$ denote the $k\times k$ minor given by the <u>rows</u> of A with indexes $i_1, \dots, i_k$. Then

$$\det(B^t A) = \sum_{\underline{i} \in I_{n,k}} \det(B^{\underline{i}}) \cdot \det(A^{\underline{i}})$$

for every $A, B \in \mathbb{R}^{n \times k}$.

<u>Proof</u> Let $\alpha \in \Lambda^k(\mathbb{R}^n)$ be defined by

$$\alpha(v_1, \dots, v_k) := \det(B^t V)$$

where V is the $n \times k$-matrix with columns $v_1, \dots, v_k$.
(One has to verify that α belongs to $\Lambda^k(\mathbb{R}^n)$!).

Then $\alpha = \sum_{\underline{i}} \alpha_{\underline{i}} \, dx_{\underline{i}}$ with $\alpha_{\underline{i}} := \alpha(e_{\underline{i}})$.

And one checks that

- $\alpha_{\underline{i}} = \alpha(e_{\underline{i}}) = \det(\underbrace{B^t(e_{i_1}, \dots, e_{i_k})}_{= (B^{\underline{i}})^t}) = \det(B^{\underline{i}})$
- $dx_{\underline{i}}(v_1, \dots, v_k) = \det(V^{\underline{i}}) \longleftarrow$ this is Lemma 1 above.

Hence

$$\det(B^t V) = \alpha(v_1, \dots, v_k)$$
$$= \sum_{\underline{i}} \alpha_{\underline{i}} \, dx_{\underline{i}}(v_1, \dots, v_k) = \sum_{\underline{i}} \det(B^{\underline{i}}) \det(V^{\underline{i}}).$$

$\square$

Currents 13/14

Lecture 7

2/4/14

We proceed with the review of basic multilinear algebra. Today we define simple k-vectors in a way that makes the geometric meaning clear. Later we will identify simple k-vectors as elements of a more abstract linear space, that of k-vectors.

As in the previous lecture V is a vector space.

<u>Definition</u> On V^k we define the equivalence relation $\sim$ as follows:

$$(v_1, \ldots, v_k) \sim (v_1', \ldots, v_k')$$

iff

$$\alpha(v_1, \ldots, v_k) = \alpha(v_1', \ldots, v_k') \qquad \forall \alpha \in \Lambda^k(V).$$

We call the equivalence classes $[v_1, \ldots, v_k]$ of $V^k/\sim$ simple k-vectors on V. We write 0 for $[0, \ldots, 0]$.

<u>Proposition 1</u>

(i) $(v_1, \ldots, v_k) \sim (0, \ldots, 0)$ iff $v_1, \ldots, v_k$ are linearly dependent.

(ii) If $(v_1, \ldots, v_k) \sim (v_1', \ldots, v_k') \not\sim 0$ then

$$\underbrace{\mathrm{Span}(v_1, \ldots, v_k)}_{W} = \mathrm{Span}(v_1', \ldots, v_k').$$

Moreover the matrix of change of base M (from $v_1, \ldots, v_k$ to $v_1', \ldots, v_k'$, both bases of W)

has determinant 1. (Recall that M is defined by $v_i' = \sum_j M_{ij} v_j$).

Proof Step 1. If $v_1, \dots, v_k$ are linearly dependent then $\alpha(v_1, \dots, v_k) = 0$ $\forall \alpha$, as seen in the previous lecture, and therefore $(v_1, \dots, v_k) \sim (0, \dots, 0)$.

Step 2. Assume that $v_1, \dots, v_k$ are linearly independent (with span W) and choose $v_{k+1}, \dots, v_n$ to form a basis of V, with corresp. dual basis $v_1^*, \dots, v_n^*$. Let $\alpha := dv_1^* \wedge \dots \wedge dv_k^*$. Then $\alpha(v_1, \dots, v_k) = \det(I) = 1$ (see previous lecture) and this shows that $(v_1, \dots, v_k) \not\sim (0, \dots, 0)$. The proof of (i) is complete.

Step 3 Assume that $W := \text{span}(v_1, \dots, v_k) \neq W' := \text{span}(v_1', \dots, v_k')$. Then we can choose $v_{k+1}, \dots, v_n$ as before, adding the requirement that v_{k+1} is one of the v_i'. Then $\alpha(v_1, \dots, v_k) = 1$, as before, but $\alpha(v_1', \dots, v_k') = 0$ because the matrix of the coefficients of $v_1', \dots, v_k'$ w.r.t $v_1, \dots, v_k$ contains a 0 column (corresponding to v_i'). Hence $(v_1, \dots, v_k) \not\sim (v_1', \dots, v_k')$ and this proves the first part of (ii).

For the second part, note that since $W = W'$ then

$$\begin{aligned} \alpha(v_1', \dots, v_k') &= \det M \\ \parallel \\ \alpha(v_1, \dots, v_k) &= 1 \end{aligned}$$

and therefore $\det M = 1$.

Assume now that V is endowed with a scalar product.

For every $v_1, \dots, v_k$ let $R(v_1, \dots, v_k)$ be the rectangle spanned by $v_1, \dots, v_k$ that is

$$R(v_1, \dots, v_k) := \left\{ \sum_{i=1}^{k} t_i v_i \ : \ 0 \le t_i \le 1 \right\}$$

Now, if $(v_1, \dots, v_k) \sim (v_1', \dots, v_k') \neq (0, \dots, 0)$ then $v_1, \dots, v_k$ and $v_1', \dots, v_k'$ span the same subspace W and the linear map that takes each v_i in v_i' is associated with the change of variable matrix M.

Hence $\mathrm{vol}_k (R(v_1', \dots, v_k')) = \mathrm{vol}_k (R(v_1, \dots, v_k)) \cdot |\det M|$

$\mathcal{H}^k$

$$= \mathrm{vol}_k (R(v_1, \dots, v_k)).$$

Even if the two rectangles are different, they have the same k-dimensional volume (k-dim. Hausdorff measure).

We then call $\mathrm{vol}_k (R(v_1, \dots, v_k))$ the <u>norm</u> of the simple k-vector $[v_1, \dots, v_k]$ and denote it by $|[v_1, \dots, v_k]|$.

Why "norm"? simple k-vectors do not form a linear space! Well, this will become clear later.

Finally, recall that an orientation of a vector space W is an equivalence class of bases, where two bases are said to be equivalent if the corresponding change of basis matrix has positive determinant.

Thus if $(v_1,\dots,v_k)\sim(v_1',\dots,v_k')\neq 0$ then they induce of their span W the same orientation because the change of basis matrix has determinant one.

We have thus shown that the map defined for every $[v_1,\dots,v_k]\neq 0$ by

$$[v_1,\dots,v_k]\longmapsto\left(W,\ \text{orientation of } W,\ \|[v_1,\dots,v_k]\|\right)$$

is well-defined.

<u>Proposition 2</u> The map above is one-to-one.

<u>Proof</u> Injectivity. Assume that $(v_1,\dots,v_k)$ and $(v_1',\dots,v_k')$ span the same k-dimensional subspace W, induce the same orientation, and have the same norm. We claim that $(v_1,\dots,v_k)\sim(v_1',\dots,v_k')$. Take indeed $\alpha\in\Lambda^k(V)$.

Consider the restriction of α to W.

Since $\Lambda^k(W)$ has dimension 1 and is spanned by $dv_1^*\wedge\dots\wedge dv_k^*$, there exists c such that $\alpha=c\cdot dv_1^*\wedge\dots\wedge dv_k^*$. Hence

$$\alpha(v_1,\dots,v_k)=c\cdot\det I=c,$$

while

$$\alpha(v_1',\dots,v_k')=c\det M=c.$$

Hence $\alpha(v_1,\dots,v_k)=\alpha(v_1',\dots,v_k')$, and since this holds for every α we have that $(v_1,\dots,v_k)\sim(v_1',\dots,v_k')$.

Surjectivity is trivial. $\square$

We have thus proved that simple k-vectors are in one-to-one correspondance with oriented k-planes in V coupled with a multiplicity.

Or, if you prefer, unitary simple k-vectors are in one-to-one correspondance with oriented k-planes.

This is the key point of all this business involving k-covectors and (simple) k-vectors.

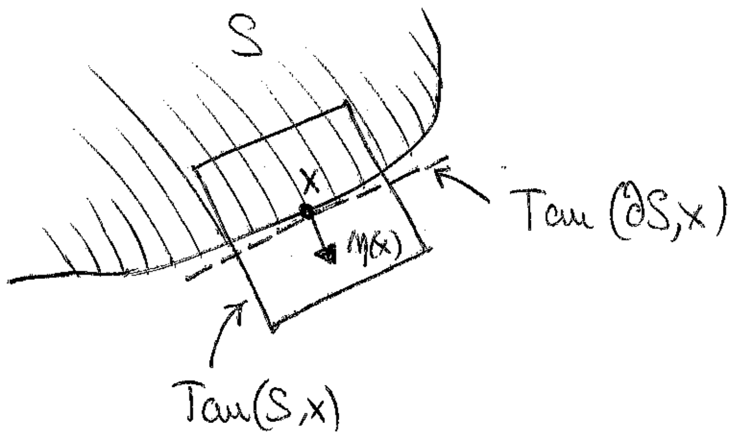
We conclude this lecture by recalling the basic notions that are needed to state Stokes theorem (we do not prove it).

<u>Orientation of a k-dimensional surface in $\mathbb{R}^n$.</u>
Let S be a k-dim. surface in $\mathbb{R}^n$ of class $\mathcal{C}^1$ (that is, a submanifold, possibly with boundary). An orientation of S is a map that to each $x \in S$ associate a <u>unit</u> simple k-vector $\tau(x) = [v_1(x), \dots, v_k(x)]$ which spans the tangent space $\mathrm{Tan}(S,x)$.
In this context it is assumed that τ is continuous. (Recall that the space of simple k-vectors, $(\mathbb{R}^n)^k/\sim$, being the quotient of a topological space is a topological space).

This definition of orientation differs from the usual one but is equivalent.

Orientation of the boundary

Let S be as above. Then for every $x \in \partial S$ we can define the exterior normal $\eta(x)$.



Then, if S is oriented by $\tau = [v_1, \ldots, v_k]$, we endow ∂S with the orientation $\tau' = [\,,\ldots, v'_{k-1}]$ defined so that

$$[v_1, \ldots, v_k] = [\eta, v'_1, \ldots, v'_{k-1}] \quad \forall x \in \partial S.$$

Differential forms

Let Ω be an open set in $\mathbb{R}^n$.
A differential k-form ω on Ω is a "map" that to each $x \in \Omega$ associate $\omega(x) \in \Lambda^k(\mathbb{R}^n)$.
We write ω in coordinates

$$\omega(x) = \sum_{\underline{i} \in I_{n,k}} \omega_{\underline{i}}(x)\, dx_{\underline{i}}$$

and we say that ω is of class C^k if all coefficients $\omega_{\underline{i}}$ are functions on Ω of class C^k.

Exterior derivative

If ω is a k-form of class $\mathcal{C}^1$ on Ω, we define the differential (or exterior derivative) of ω as the $(k+1)$-form

$$\omega(x) := \sum_{\underline{i} \in I_{n,k}} d\omega_{\underline{i}} \wedge dx_{\underline{i}}$$

where $df(x) := \sum \frac{\partial f}{\partial x_i}(x)\, dx_i$ for every function (or 0-form) f.

<u>Note that</u> one can define k-forms just on manifolds (or on surfaces) but that requires a different, and more intrinsic, definition of the exterior derivative.

Integration of forms

that is, a surface coupled with an orientation

If S is an <u>oriented surface</u> of dimension k in $\mathbb{R}^n$ and ω is a k-form defined on (an open set that contains) S we set

$$\int_S \omega := \int_S \langle \omega(x) ; \tau(x) \rangle \, d\mathcal{H}^k(x)$$

Recall the definition of simple k-vectors

where $\tau(x) = [v_1(x), \dots, v_k(x)]$ is the orientation of S, and $\langle \omega(x); \tau(x) \rangle$ stands for the k-covector $\omega(x)$ computed at $v_1(x), \dots, v_k(x)$ — we avoid to write $\omega(x)(v_1(x), \dots, v_k(x))$ — and is <u>well-defined</u>!

(Of course we must assume that the integral exists.)

Again, this definition is different from the usual one from geometry textbooks, but fits better our purposes (defining currents).

Stokes theorem

Let S be a compact oriented surface with dimension k in $\mathbb{R}^n$, and let ω be a $(k-1)$-form of class C^1 defined on (an open neighbourhood of) S. Then

$$\int_{\partial S} \omega = \int_S d\omega .$$

Lecture 8

3/4/14

We start this lecture by defining the linear space of k-vectors (on V) which includes all simple k-vectors defined in the previous lecture.

We construct k-vectors by analogy with the construction of k-covectors, exploiting the fact that V can be canonically identified with the dual of V.

More precisely we set

$$\Lambda_k(V) = \text{space of } k\text{-vectors on } V$$
$$:= \text{space of } k\text{-covectors on } V^*$$
$$= \Lambda^k(V^*).$$

note that the index is now a <u>sub</u>script

The duality between V and V^* extends to a duality between $\Lambda_k(V)$ and $\Lambda^k(V)$.
To define this duality we choose a basis $e_1, \dots, e_n$ of $V \simeq V^{**}$, let $e_1^*, \dots, e_1^*$ be the dual basis of V^*, and as we saw two lectures ago

$$\{e^*_{\underline{i}} : \underline{i} \in I_{n,k}\} \text{ is a basis of } \Lambda^k(V)$$

and also

$$\{e_{\underline{i}} : \underline{i} \in I_{n,k}\} \text{ is a basis of } \Lambda^k(V^*) = \Lambda_k(V).$$

We then define the duality pairing $\langle\ ;\ \rangle$ between $\Lambda^k(V)$ and $\Lambda_k(V)$ by setting

$$(*) \qquad \langle e^*_{\underline{i}}\,;\, e_{\underline{j}} \rangle := \delta_{\underline{i}\,\underline{j}} \qquad \forall\, \underline{j}, \underline{i} \in I_{n,k}$$

<u>Proposition 1</u> For every $\alpha \in \Lambda^k(V)$ and $v_1, \dots, v_k \in V$ there holds

$$(**) \qquad \langle \alpha\,;\, \underbrace{v_1 \wedge \dots \wedge v_k}_{\substack{\uparrow \\ \text{element of } \Lambda_k(V)}} \rangle = \alpha(v_1, \dots, v_k)$$

<u>Proof</u> From $(*)$ we obtain that $(**)$ holds when $\alpha \in \{e^*_{\underline{i}}\}$ and $v_i \in \{e_i\}$. Then using the linearity of both sides of $(**)$ w.r.t. α we obtain that $(**)$ holds when $\alpha \in \Lambda^k(V)$ and $v_i \in \{e_i\}$. Finally, using the linearity of both sides of $(**)$ w.r.t. each v_i we obtain $(**)$ in full generality. $\square$

<u>Remark</u> It follows from $(**)$ that the duality pairing defined above <u>does not</u> depend on the choice of the basis $(e_1, \dots, e_n)$.

Note now that

$$(v_1, \dots, v_k) \sim (v_1', \dots, v_k') \leftarrow \text{in the sense of previous lecture}$$
$$\Updownarrow$$
$$\alpha(v_1, \dots, v_k) = \alpha(v_1', \dots, v_k') \quad \forall \alpha \in \Lambda^k(V)$$
$$\langle \alpha\,;\, v_1 \wedge \dots \wedge v_k \rangle \qquad \langle \alpha\,;\, v_1' \wedge \dots \wedge v_k' \rangle$$
$$\Updownarrow$$
$$v_1 \wedge \dots \wedge v_k = v_1' \wedge \dots \wedge v_k'.$$

Therefore we can identify the simple vector $[v_1, \dots, v_k]$ with the product $v_1 \wedge \dots \wedge v_k$.

Question Are there non-simple k-vectors?

Answer: Yes! $e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda_2(\mathbb{R}^4)$ cannot be written as $v_1 \wedge v_2$ with $v_1, v_2 \in \mathbb{R}^4$ and therefore is not simple.

Note that all $(n-1)$-vector (and $(n-1)$-covectors) are simple. This fact is not immediate.

We assume now that V is equipped with a scalar product.

Next we choose $e_1, \dots, e_n$ orthonormal basis of V and endow $\Lambda^k(V)$ and $\Lambda_k(V)$ with the scalar products that make the bases $\{e^*_{\underline{i}}\}$ and $\{e_{\underline{i}}\}$ orthonormal.

Proposition 2 For every $v_1, \dots, v_k \in V$ we have

$$|v_1 \wedge \dots \wedge v_k| \longleftarrow \text{norm associated to the scalar}$$

$$\|$$

$$\sqrt{\sum_{\underline{i} \in I_{n,k}} (\det(V^{\underline{i}}))^2} \longleftarrow \text{where } V \text{ is the } n \times k \text{ matrix whose entries are the coordinates of } v_1, \dots, v_k \text{ (w.r.t. the basis } e_1, \dots, e_n)$$

$$\|$$

$$\mathrm{Vol}_k(R(v_1, \dots, v_k)) \longleftarrow k\text{-dim. volume } (\mathcal{H}^k) \text{ of the rectangle spanned by } v_1, \dots, v_k.$$

$$\overset{!!}{\cdot\cdot}$$

$$|[v_1, \dots, v_k]|$$

Proof For the first identity it suffices to notice that, setting $v := v_1 \wedge \dots \wedge v_k$, then the coordinates of v, $v_{\underline{i}}$, are given by

$$v_{\underline{i}} = \langle e^*_{\underline{i}} ; v \rangle \overset{\text{Proposition 1}}{=} e^*_{\underline{i}}(v_1, \dots, v_k) \overset{\text{two lectures ago}}{=} \det(V^{\underline{i}}).$$

The proof of the second identity has already been seen (essentially). Assume that $v_1, \dots, v_k$ are linearly independent and let W be their span, and $n_1, \dots, n_k$ an orthonormal basis of V; denote by N the matrix of coefficients of $n_1, \dots, n_k$ (w.r.t. $e_1, \dots, e_n$). Then

$$\mathrm{vol}_k(R(v_1, \dots, v_k)) = |\det(N^t V)|$$

we saw this ident. for the Jacobian $\longrightarrow$ $$= \left(\det(V^t V)\right)^{\frac{1}{2}}$$

by the generalized Binet formula $\longrightarrow$ $$= \left(\sum_{\underline{i}} (\det(V^{\underline{i}}))^2\right)^{\frac{1}{2}}. \qquad \square$$

Concluding remarks

- Even though we made $\Lambda_k(V)$ an Hilbert space, thus canonically isomorphic to its dual $\Lambda^k(V)$, we will never identify these two spaces.

- The choice of the Hilbert (or Euclidean) norm on $\Lambda_k(V)$ and $\Lambda^k(V)$ may sound natural, but for reasons that will be explained later, it is preferable to endow these spaces with different norms, defined as follows:

$$\|\cdot\| = \text{"mass norm" on } \Lambda_k(V)$$

$:=$ convex envelope of the restriction of the Euclidean norm to simple k-vectors

that is
$$\|v\| := \inf \sum_i t_i |v_i|$$
over all convex combin. $v = \sum t_i v_i$ with v_i <u>simple</u>.

Accordingly we define

$$\|\cdot\|_* = \text{"comass norm" of } \Lambda^k(V)$$
$$:= \text{dual norm of } \phi$$

that is

$$\|\alpha\|_* := \sup_{\|v\| \leq 1} \langle \alpha ; v \rangle$$
$$= \sup_{\substack{v \text{ simple} \\ |v| \leq 1}} \langle \alpha , v \rangle$$

Clearly $\|v\| = |v|$ if v is simple, but in general $\|v\| \geq |v|$. Conversely (or dually) $\|\alpha\|_* \leq |\alpha|$ and $\|\alpha\|_* = |\alpha|$ if α is simple (and only if!).

- In practice, the difference between the use of $\|\cdot\|$ and $|\cdot|$ in the following will not be seen except that in one theorem!

- By what we saw, the Jacobian determinant of a map $f : \underset{t}{\overset{\Psi}{A}} \subset \mathbb{R}^k \longrightarrow \underset{x}{\overset{\Psi}{\mathbb{R}^m}}$ can be written as

$$Jf(t) = \sqrt{\det(\nabla^t f \cdot \nabla f)}$$
$$= \left(\sum_{\underline{i} \in I_{n,k}} \left[\det((\nabla f)^{\underline{i}}) \right]^2 \right)^{\frac{1}{2}}$$
$$= \left| \frac{\partial f}{\partial t_1} \wedge \dots \wedge \frac{\partial f}{\partial t_k} \right| .$$

We begin now the theory of currents.

The definition of general currents follows strictly that of distributions: distributions acts on smooth test functions and similarly k-dimensional currents, which generalize the notion of k-dimensional oriented surface, act on smooth k-forms.

We define now currents on $\mathbb{R}^n$ (but the ambient space could be as well an opens set of $\mathbb{R}^n$ or a Riemannian manifold).

<u>Definition</u> (<u>of currents — de Rham</u>)

Let $\mathcal{D}^k(\mathbb{R}^n)$ denote the space of smooth k-forms on $\mathbb{R}^n$ with compact support.

Then the space of k-dimensional currents $\mathcal{D}_k(\mathbb{R}^n)$ is defined as the dual of $\mathcal{D}^k(\mathbb{R}^n)$.

Let be precise with <u>topologies</u> (just for one time!). By writing forms in coordinates we can identify $\mathcal{D}^k(\mathbb{R}^n)$ with $(\mathcal{D}(\mathbb{R}^n))^{I_{n,k}}$ where $\mathcal{D}(\mathbb{R}^n)$ is the space of smooth, compactly supported test functions o $\mathbb{R}^n$. The later is endowed with a topology of locally convex vector space, and so are $\mathcal{D}^k(\mathbb{R}^n)$ and its dual $\mathcal{D}_k(\mathbb{R}^n)$.

If you don't know the topology on $\mathcal{D}(\mathbb{R}^n)$, then for every K compact in $\mathbb{R}^n$ let $\mathcal{D}^k(K)$ be the space of smooth k-forms on $\mathbb{R}^n$ whose support is contained in K.

Then $\mathcal{D}^k(K)$ is a Frechet space with the seminorms given by the supremum norms of all partial derivative (of every order) of all coefficients of a given form.

Now each $\mathcal{D}^k(K)$ embeds in $\mathcal{D}^k(\mathbb{R}^n)$ and the topology on $\mathcal{D}^k(\mathbb{R}^n)$ is the smallest topology of locally convex vector space that makes this embedding continuous for all K.

(Thus $\mathcal{D}^k(\mathbb{R}^n)$ is the <u>direct limit</u> of the net $\{\mathcal{D}^k(K)\}$ in the cathegory of locally convex vector spaces and accordingly $\mathcal{D}_k(\mathbb{R}^n)$ is the <u>inverse limit</u> of the duals $(\mathcal{D}^k(K))^*$ in the same cathegory.)

! Of the topology of $\mathcal{D}_k(\mathbb{R}^n)$ we will only retain the notion of (sequential) convergence:

$$T_n \longrightarrow T \quad \text{in the sense of currents}$$

iff

$$\langle T_n ; \omega \rangle \longrightarrow \langle T, \omega \rangle \quad \forall\, \omega \in \mathcal{D}^k(\mathbb{R}^n).$$

<u>Oriented</u> <u>k-dimensional</u> <u>surfaces</u> <u>as</u> <u>currents</u>

Given S <u>closed</u> k-dimensional surface n $\mathbb{R}^n$ of class $\mathcal{C}^1$ and <u>oriented</u>, we define the associated current as

$$T_S : \underset{\mathcal{D}^k(\mathbb{R}^n)}{\underset{\cap}{\omega}} \longmapsto \int_S \omega$$

Note that the integral is well defined because ω has compact support. Moreover the map $S \mapsto T_S$ is <u>injective</u> (the current T_S determines S)!

Boundary of a current

Given $T \in \mathcal{D}_k(\mathbb{R}^n)$ we define its boundary $\partial T \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ by

$$\langle \partial T ; \omega \rangle := \langle T ; d\omega \rangle \quad \forall \omega \in \mathcal{D}^{k-1}(\mathbb{R}^n)$$

Then Stokes theorem shows that this notion of boundary is compatible with the usual one from geometry, that is,

(Check!)

$$\partial(T_S) = T_{\partial S} .$$

Note that ∂ is the adjoint of d, and $\partial^2 = 0$ because $d^2 = 0$.

Mass of a current (extends the notion of area of a surface)

The mass of a current $T \in \mathcal{D}_k(\mathbb{R}^n)$ is defined as

$$\mathbb{M}(T) := \sup_{\substack{\omega \in \mathcal{D}^k(\mathbb{R}^n) \\ \|\omega(x)\| \leq 1 \ \forall x}} \langle T, \omega \rangle$$

Here we use the comass norm. Some prefer to use the Euclidean norm $|\omega(x)|$ (Simon, Krantz & Parks)

If S is a surface then $\mathbb{M}(T_S) = \mathcal{H}^k(S)$.

Indeed for every ω with $\|\omega(x)\| \leq 1 \ \forall x$ there holds $\langle \omega(x) ; \tau(x) \rangle \leq 1 \ \forall x \in S$, where $\tau(x)$ is the orientation of S at x, and then

$$\langle T_S, \omega \rangle = \int_{x \in S} \langle \omega(x) ; \tau(x) \rangle \, d\mathcal{H}^k(x) \leq \int_{x \in S} 1 \cdot d\mathcal{H}^k(x) = \mathcal{H}^k(S),$$

which gives

$$\mathbb{M}(T_S) \leq \mathcal{H}^k(S)$$

To prove the opposite inequality one would like to choose ω so that $\langle \omega(x) ; \tau(x) \rangle = |\omega(x)| = 1$ for every $x \in S$. Such $\omega(x)$ exists for all $x \in S$, but the map $\underset{\in S}{x} \longmapsto \omega(x)$ is continuous (recall that S is of class $\mathcal{C}^1$).

This is, however, not a problem: extend ω to a continuous k-form on $\mathbb{R}^n$ such that $\|\omega(x)\| \leq 1$ everywhere, then approximate ω by smooth k-forms with compact support using regularization by convolution and multiplication by suitable cutoff functions.

<u>Other examples of currents with finite mass</u>

Let μ be any finite measure of $\mathbb{R}^n$, and let τ be a map in $L^1(\mu)$ with values in $\Lambda_k(\mathbb{R}^n)$. Define the current $T = \tau\mu$ as follows

$$\langle T ; \omega \rangle = \int_{\mathbb{R}^n} \langle \omega(x) ; \tau(x) \rangle \, d\mu(x) .$$

Then one easily check that for every ω s.t. $\|\omega(x)\| \leq 1$ everywhere there holds $\langle \omega(x) ; \tau(x) \rangle \leq \|\tau(x)\|$ and then

$$\langle T, \omega \rangle \leq \int \|\tau(x)\| \, d\mu(x) = \|\tau\|_{L^1(\mu)} ;$$

thus $\mathbb{M}(T) = \mathbb{M}(\tau\mu) \leq \|\tau\|_{L^1(\mu)}$, and one can prove as above that equality holds, that is,

$$\mathbb{M}(T) = \mathbb{M}(\tau\mu) = \|\tau\|_{L^1(\mu)} .$$

Currents with finite mass

There are no other currents with finite mass beyond those in the previous paragraph.

Indeed if $\mathbb{M}(T) < +\infty$ then T is a linear functional on $\mathcal{D}^k(\mathbb{R}^n)$ which is <u>bounded</u> with respect to the supremum norm on forms.
Hence T can be extended by density to a linear functional on the closure (completion) of $\mathcal{D}^k(\mathbb{R}^n)$ w.r.t. the supremum norm, which is the space of continuous k-form that vanish at infinity $\mathcal{C}_0(\mathbb{R}^n; \Lambda^k(\mathbb{R}^n))$. Hence T is represented by a vector measure with values in the dual of $\Lambda^k(\mathbb{R}^n)$, which is $\Lambda_k(\mathbb{R}^n)$.
And all such measures can be written as $\tau\mu$ as in the previous paragraph.

Finally, note that the mass agrees with the norm of such measures (also called mass).

Currents 13/14 | Lecture 9 30/4/14

Today I will define the main classes of currents that we will consider in this course and <u>state</u> the main compactness results.

Currents with finite mass

As pointed out in the previous lecture, d-currents T on $\mathbb{R}^n$ such that $\mathbb{M}(T) < +\infty$ can be viewed as (finite) measures with values in d-vectors, and can be represented as $T = \tau\mu$ with μ (finite) positive measure on $\mathbb{R}^n$ and $\tau \in L^1(\mu, \Lambda_d(\mathbb{R}^n))$, that is

$$\langle T ; \omega \rangle = \int_{x \in \mathbb{R}^n} \langle \omega(x) ; \tau(x) \rangle \, d\mu(x) \quad \forall \omega \in \ldots$$

(If needed, we can also assume $\|\tau\| = 1$ a.e., under this assumption μ and τ are "essentially" unique).

The following proposition is a direct consequence of the standard compactness theorem for measures:

<u>Proposition 1</u> Let (T_n) be a sequence of currents with finite mass such that

$$\mathbb{M}(T_n) \leq C < +\infty.$$

convergence in the sense of currents

Then, up to subsequence, $T_n \to T$ and $\mathbb{M}(T) \leq \liminf_{n \to +\infty} \mathbb{M}(T_n)$.

Remark Since τ and μ can be arbitrarily chosen, there is in general no connection between the tangent d-vectorfield τ and the geometric structure (if any) of the "support" of the measure μ. In particular μ could be a Dirac mass, which is supported on a point (a zero-dimensional set), and still $T = \tau\mu$ is a d-current.

This suggest that currents of finite mass do not have any geometric structure (in general).

The next class of currents, even though still quite general, is (geometrically) more interesting.

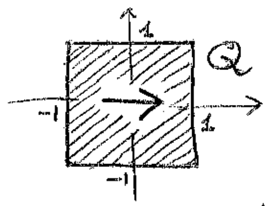
Normal currents

We say that a d-current T on $\mathbb{R}^n$ is normal if both T and ∂T have finite mass, that is,

$$\mathbb{M}(T)\ ;\ \mathbb{M}(\partial T) < +\infty .$$

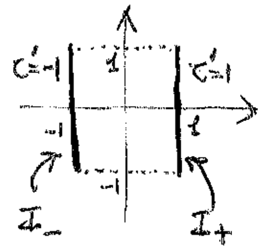
Thus T and ∂T can be represented as $T = \tau\mu$; $\partial T = \tau'\mu'$.

Example Let T be the 1-current on $\mathbb{R}^2$ given by $T = \tau\mu$ where μ is the Lebesgue measure on the square $Q := [-1,1]^2$ and $\tau := e_1 = (1,0)$, that is,

" $T =$  " (in quotes, because it does not really make sense...)

Then $\partial T = \tau'\mu'$ where μ' is the length measure ($\mathcal{H}^1$) restricted to the union of the segments $I_\pm = \{\pm 1\} \times [-1,1]$

and $\tau' = +1$ on I_+, $\tau' = -1$ on I_-, that is,

" $\partial T =$  ",

Thus T is normal.

Let us prove this claim rigorously: given any test 0-form (that is, function!) ϕ we have

$$\langle \partial T ; \phi \rangle = \langle T ; d\phi \rangle = \int \langle d\phi(x) ; \overset{e_1}{\overset{\parallel}{\tau(x)}} \rangle \, d\overset{\mathcal{L}^2 \text{ on } Q}{\overset{\parallel}{\mu}}(x)$$

$$= \int_Q \frac{\partial \phi(x)}{\partial x_1} \, dx \leftarrow \text{Lebesgue measure}$$

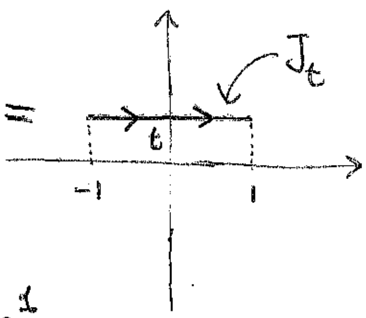
$$= \int_{-1}^{1} \left(\int_{-1}^{1} \frac{\partial \phi(x)}{\partial x_1} \, dx_1 \right) dx_2$$

$$= \int_{-1}^{1} \phi(1, x_2) - \phi(-1, x_2) \, dx_2$$

$$= \int \phi \cdot \tau' \, d\mu' .$$

<u>Remark</u> The formula for ∂T could be obtained also in a different way.

For every $t \in [-1,1]$, let T_t be the 1-current associated with the horizontal segment $J_t := [-1,1] \times \{t\}$ (intended as a 1-dimensional (smooth) submanifold oriented by e_1) that is

" $T_t :=$  "

The $T = \int_{-1}^{1} T_t \, dt$, where this means that

$$\langle T ; \omega \rangle = \int_{-1}^{1} \langle T_t ; \omega \rangle \, dt \qquad \forall \omega \in \ldots .$$

In other words, we can view T as a "superposition" of the currents T_t, and accordingly we expect ∂T to be a superposition of ∂T_t, that is, $\partial T = \int \partial T_t \, dt$ (provided that the integral makes sense — we do not discuss the details here).

Indeed, since $\partial T_t = \delta_{x_t^+} - \delta_{x_t^-}$ where $x_t^\pm := (\pm 1, t)$, we get

$$\partial T = \int_{-1}^{1} \delta_{x_t^+} \, dt - \int_{-1}^{1} \delta_{x_t^-} \, dt$$

and more precisely

$$\langle \partial T ; \phi \rangle = \int_{-1}^{1} \phi(+1, t) \, dt - \int_{-1}^{1} \phi(-1,t) \, dt \quad \forall \phi \in \dots$$

as before ....

<u>Example</u> Here is an example of 1-current T which has finite mass but it is not normal, that is, $\mathbb{M}(\partial T) = +\infty$. In $\mathbb{R}^2$, let $T := e_1 \delta_0$ where $e_1 := (1,0)$.

For every function (0-form) ϕ we have

$$\langle \partial T ; \phi \rangle = \langle T, d\phi \rangle$$
$$= \int \langle d\phi ; e_1 \rangle \, d\delta_0 = \frac{\partial \phi}{\partial x_1}(0)$$

Now, $\phi \mapsto \frac{\partial \phi}{\partial x_1}(0)$ is the distributional derivative of the Dirac mass δ_0 in the direction e_1 (up to a change of sign), which is known to be a distribution that cannot be represented by a measure; thus ∂T is not a measure.

Alternatively we can prove directly that

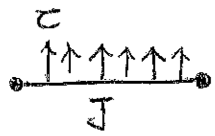
$$\mathbb{M}(\partial T) = \sup_{\substack{|\phi|\leq 1 \\ \text{everywhere}}} \langle \partial T ; \phi \rangle = \sup_{\substack{|\phi|\leq 1 \\ \text{everywhere}}} \frac{\partial \phi}{\partial x_1}(0) \; .$$

(Take ϕ of the form $\phi(x) = m x_1 \cdot \sigma(x)$ where σ is a suitable cut-off function equal to 1 in a neighbourhood of 0, null outside $B(0, 1/m)$....)

<u>Remark</u> The first example <u>shows</u> that a 1-dimensional normal current (in $\mathbb{R}^2$) may be supported on a 2-dimensional set, while the second example <u>suggests</u> that it cannot be supported on a 0-dimensional set.

We will see that indeed a d-dimensional normal current (in $\mathbb{R}^n$) cannot be supported on a set with Hausdorff dimension strictly less than d (while of course it is easy to construct examples which are supported on sets with dimension larger than d).

<u>Example</u> Let T be the 1-current in $\mathbb{R}^2$ given by $T = \tau\mu$ where μ is the length measure ($\mathcal{H}^1$) on the <u>horizontal segment</u> $J := [-1,1]$ while τ agrees with the <u>vertical vector</u> $e_2 := (0,1)$:

" $T =$  "

Then $\langle \partial T ; \phi \rangle = \langle T ; d\phi \rangle = \int_{-1}^{1} \frac{\partial \phi}{\partial x_2}(t, 0)\, dt$

and again one can show that $\mathbb{M}(\partial T) = +\infty$, and T is not normal.

<u>Remark</u> One can generalize the previous example and show the following: let $T := \tau\mu$ where μ is as above and $\tau : \underset{\substack{\shortparallel \\ J}}{[-1,1]} \longrightarrow \mathbb{R}^2$ is a smooth vectorfield,

and assume that τ is NOT everywhere tangent to J, that is τ_2 is not identically zero; then $\mathbb{M}(\partial T) = +\infty$, and T is not normal.

This suggests that for a <u>normal</u> <u>current</u> $T = \tau\mu$ the "orientation" τ must be tangent (in some sense) to the support of the measure μ. We will indeed prove some statement in this direction later.

(that is, $\tau = m \cdot e_1$)

On the other hand, if τ is tangent to J, then $\mathbb{M}(\partial T) < +\infty$ and T is normal.

Indeed

$$\langle \partial T; \phi \rangle = \langle T, d\phi \rangle = \int \langle d\phi; \overset{m\cdot e_1}{\overset{\shortparallel}{\tau}} \rangle \, d\mu$$

$$= \int_{-1}^{1} \frac{\partial \phi}{\partial x_1}(t,0) \; m(t)\, dt$$

$$= \phi(1,0) - \phi(-1,0) - \int_{-1}^{1} m'(t)\, \phi(t,0)\, dt,$$

which means that

$$\partial T = \delta_{(1,0)} - \delta_{(-1,0)} - m'\mu$$

and in particular

$$\mathbb{M}(\partial T) = 2 + \int_{-1}^{1} |m'(t)|\, dt = 2 + \|m'\|_1 .$$

<u>Proposition 2</u> (Compactness for normal currents).

Let (T_k) be a sequence of normal d-currents in $\mathbb{R}^n$ such that

$$\mathbb{M}(T_k);\ \mathbb{M}(\partial T_k) \leq C < +\infty .$$

Then T_k converges (up to subsequence) to a normal current T.

Moreover $\partial T_k \to \partial T$ and $\liminf_{k\to+\infty} \mathbb{M}(T_k) \geq \mathbb{M}(T)$; $\liminf_{k\to+\infty} \mathbb{M}(\partial T_k) \geq \mathbb{M}(\partial T)$.

<u>Proof</u>

[applied to the sequences (T_k) and (∂T_k) !]

From Proposition 1 we get that, up to subseq., $T_k \to T$, $\partial T_k \to U$, $\liminf_{k\to+\infty} \mathbb{M}(T_k) \geq \mathbb{M}(T)$ and $\liminf_{k\to+\infty} \mathbb{M}(\partial T_k) \geq \mathbb{M}(U)$.

It remains to show that $U = \partial T$.

And indeed for every (admissible) $(d-1)$-form ω we have

$$\begin{aligned}\langle \partial T ; \omega \rangle &= \langle T ; d\omega \rangle \\ &= \lim_{k\to+\infty} \langle T_k ; d\omega \rangle \\ &= \lim_{k\to+\infty} \langle \partial T_k ; \omega \rangle = \langle U ; \omega \rangle\end{aligned}$$

and then $\partial T = U$ (actually this proof amounts to say that the boundary operator ∂ is continuous). $\square$

Remark. An immediate corollary of Proposition 2 is the existence of a solution of the Plateau problem for normal currents: given a normal current T_0, among all normal current T s.t. $\partial T = \partial T_0$ there exists one that minimizes the mass.

This solution, however, is not satisfactory, because the class of normal currents is, in some sense, too large. We will show indeed that there is a closed subclass of normal currents which contains all smooth surfaces and all polyhedral chains, namely the class of integral currents. That one will provide the "right" solution of the Plateau problem.

Rectifiable currents

Let E be a d-rectifiable set in $\mathbb{R}^n$, τ an orientation of E, that is, a (Borel) map that to ($\mathcal{H}^k$-almost every) $x \in E$ associates a unit simple d-vector $\tau(x)$ that spans the approximate tangent space $\mathrm{Tan}(E,x)$ (that is $\tau(x) = \tau_1(x) \wedge \dots \wedge \tau_d(x)$ and $\mathrm{Tan}(E,x) = \mathrm{span}\{\tau_1(x), \dots, \tau_d(x)\}$).

Let m be a multiplicity on E, that is, a real-valued function in $L^1(1_E \mathcal{H}^d)$

restriction of $\mathcal{H}^d$ to E.

Then we define the d-current $[E,\tau,m]$ by

$$\langle [E,\tau,m]; \omega\rangle := \int_{x\in E} \langle \omega(x); \tau(x)\rangle \, m(x) \, d\mathcal{H}^k(x)$$

for every $\omega \in \ldots$

Every current T that can be written as $T=[E,\tau,m]$ with E,τ,m as above is called <u>rectifiable</u>. If in addition m takes values in $\mathbb{Z}$, T is called <u>rectifiable with integral multiplicity</u>.

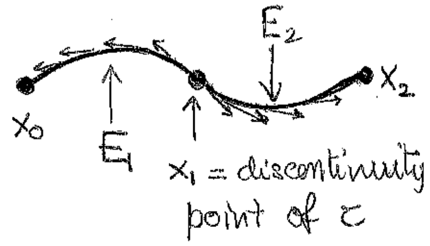
<u>Remarks</u> • If $T=[E,\tau,m]$ then it is easy to prove that

$$M(T) = \int_E |m| \, d\mathcal{H}^d .$$

- What really matters in the definition of $[E,\tau,m]$ is the product $m\tau$ rather than m and τ separately. In particular, by changing the sign of m and τ on the same set we get the same current, and therefore we can also assume that $m>0$ $\mathcal{H}^k$-a.e. on E.
  Under this additional assumption the set E, the orientation τ and the multiplicity m are "essentially" determined by the current.
  (that is, up to $\mathcal{H}^d$-null subsets of E.)
- Note that the dimension of the supporting set E agrees with the "algebraic" dimension of the current. And no regularity is assumed on τ.

- If S is a $\mathcal{C}^1$ surface of dim. d oriented by τ (and $\mathcal{H}^d(S)<+\infty$) then the rectifiable current $[S,\tau,1]$ is the canonical current associated to S...

- A slightly less obvious example of 1-rectifiable current is the following: let E be an arc and let τ be a discontinuous orientation of E,



$E = E_1 \cup E_2$.

and let $T := [E,\tau,1]$.

Note that $\partial T = \delta_{x_2} + \delta_{x_0} - 2\delta_{x_1}$.

(This can be proved by using that $T = T_1 + T_2$ where $T_i := [E_i,\tau,1]$ and E_1, E_2 are the subarcs of E given above; since we already know that $\partial T_1 = \delta_{x_0} - \delta_{x_1}$ and $\partial T_2 = \delta_{x_2} - \delta_{x_1}$, we obtain $\partial T = \partial T_1 + \partial T_2 = \delta_{x_2} + \delta_{x_0} - 2\delta_{x_1}$.)

Thus the discontinuities of τ affect the boundary of T...

Integral Currents

A d-current T is called <u>integral</u> if both T and ∂T can be represented as rectifiable currents with integral multiplicity. That is, there exist $\underbrace{E,\tau,m}_{d\text{-rectif.}}$ and $\underbrace{E',\tau',m'}_{(d-1)\text{-rectif.}}$ such that

$T = [E,\tau,m]$ and $\partial T = [E',\tau',m']$.

Remark

One would expect that the supporting sets E and E' (for T and ∂T, respectively) should be geometrically related. But the relation is not quite clear....

We can now state one of the main result of this course:

Theorem (Competness of integral currents, Federer & Fleming)
Let (T_n) be a sequence of integral d-currents such that

$$\mathbb{M}(T_n) \ ; \ \mathbb{M}(\partial T_n) \leq C < +\infty .$$

Then, up to subsequence, T_n converge to an integral current T.
(Moreover $\partial T_n \to \partial T$, $\liminf_{n\to+\infty} \mathbb{M}(T_n) \geq \mathbb{M}(T)$, etc. etc.)

The proof of this resul will take a large part of this course.

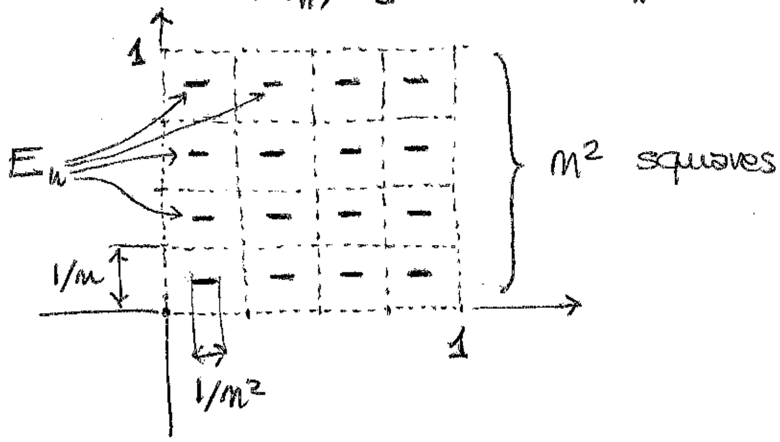
Remarks
- Using this theorem we obtain (immediately) the solution of Plateau problem for integral currents: let T_0 be an integral d-current in $\mathbb{R}^n$, then, among all integral currents T such that $\partial T = \partial T_0$ there exists one that minimizes the mass.
- Using the compactness theorem for normal currents (Proposition 2 above) we immediately obtain that there exists a converging subseq. of (T_n). What this theorem adds is that

the limit current T is integral.
Thus this result is essentially a closure theorem, and it is often referred to as such.

We discuss now few examples aimed at showing the necessity of the assumptions in the theorem of Federer and Fleming.

Example

Consider the following sequence of integral 1-currents in $\mathbb{R}^2$: $T_n = [E_n, \tau_n, 1]$ where E_n is



(that is, E_n is the union of n^2 horizontal segments with length $1/n^2$) and $\tau_n := e_1 = (1,0)$.

(*) [Then T_n converges to the normal current $T = \tau\mu$ where μ is the Lebesgue measure on $Q := [0,1]^2$ and $\tau := e_1$.]

Note that T is not rectifiable.

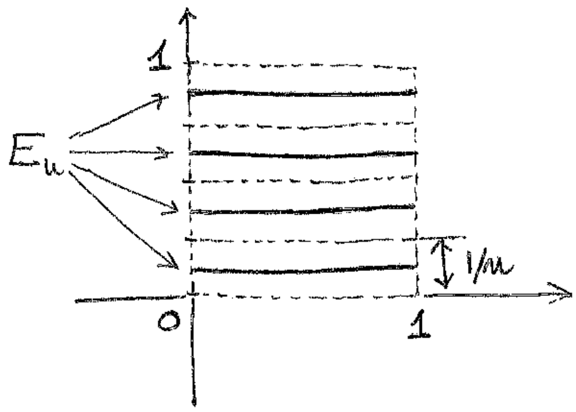
Indeed F&F theorem does not apply in this case because $\mathbb{M}(T_n) = 1$ but $\mathbb{M}(\partial T_n) = 2n^2 \to +\infty$.

To prove the claim (*) we use that the positive measures $\mu_n := 1_{E_n} \cdot \mathcal{H}^1$ converge (in the sense of measures) to μ (check the details!)

and therefore the vector measures $e_1 \cdot \mu_n$ converge to $e_1\mu$, which means exactly that T_n converge to T.

Example

Consider the following sequence of rectifiable 1-currents on $\mathbb{R}^2$: $T_n := [E_n, \tau_n, m_n]$, where



(that is, E_n is the union of n horizontal segments); $\tau_n = e_1 = (1,0)$; $m_n = \frac{1}{n}$.

Thus each T_n (and ∂T_n) is rectifiable but not rectifiable with integral multiplicity.

Then $T_n \to T$ where T is as in the previous example.

Note that $\mathbb{M}(T_n) = 1$ and $\mathbb{M}(\partial T_n) = 2$ for every n, but F & F theorem does not apply because the currents T_n are not integral.

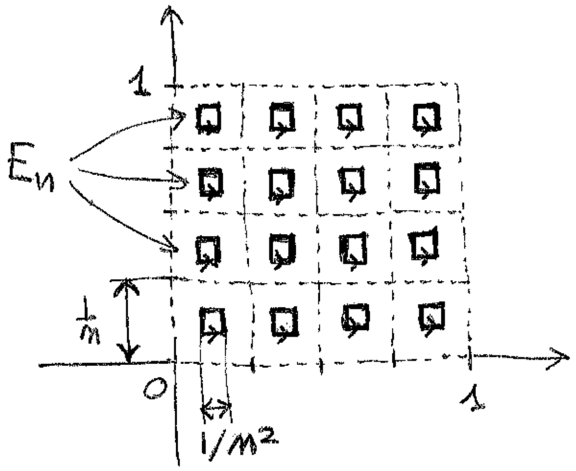
This example shows that there is no variant of F & F theorem for rectifiable currents with real multiplicities, even assuming the boundaries are rectifiable currents....

(This is not completely correct: as we will see later there exists some more general compactness result, where the key point is that the multiplicities

m_n of the currents T_n stay bounded away from zero.)

<u>Example</u> / <u>Question</u>

Let T_n be the integral 1-current in $\mathbb{R}^2$ given by $T_n = [E_n, \tau_n, 1]$ where E_n is



that is, E_n is the union of the boundaries of n^2 squares with side length $\frac{1}{n^2}$, and τ_n is chosen so that each of these boundaries is oriented counter-clockwise.

Then T_n is integral and $\mathbb{M}(T_n) = 4$, while $\partial T_n = 0$, and in particular $\mathbb{M}(\partial T_n) = 0$.

Thus F & F theorem applies.

On the other hand the measures $\mu_n := 1_{E_n} \cdot \mathcal{H}^1$ converge to the Lebesgue measure on $Q := [0,1]^2$.

What is the limit of T_n?

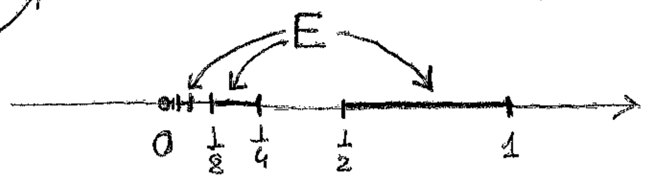
Lecture 10
6/5/14

Today I will complete the list of the main results in the theory of currents that we will prove in the rest of the course (besides Federer and Fleming compactness result, which is presumably <u>the</u> main result).

<u>Theorem</u> (boundary rectifiability)
Let T be a rectifiable d-current on $\mathbb{R}^n$ with integral multiplicity. If $\mathbb{M}(\partial T) < +\infty$ then ∂T is a rectifiable current with integral multiplicity, and in particular T is an integral current.

<u>Remarks</u>
- The proof of this result, as that of F & F theorem, is highly non trivial.
- Note that not all rectifiable currents have a boundary with finite mass. An example is the following 1-current in $\mathbb{R}$:
  $T := [E, e, 1]$ where $E := \bigcup_{n=0}^{\infty} [2^{-2n-1}, 2^{-2n}]$

canonical orientation of $\mathbb{R}$ ($e := 1$)



One can show that ∂T is the 0-current (distribution) given by $\partial T = \sum_{n=0}^{\infty} \delta_{2^{-2n}} - \delta_{2^{-2n-1}}$, which has infinite mass. (To check this claim requires some care.)

- It is somehow essential that T has integer multiplicity. Let indeed T be the 1-current in $\mathbb{R}$ given by $T = [\mathbb{R}, \underset{\uparrow}{e}, m]$ where $m \in \mathcal{E}^1_c(\mathbb{R})$.

  canonical orientation of $\mathbb{R}$

  Then $\partial T = -m' \cdot \mathcal{L}^1$ which is not a rectifiable 0-current (but has finite mass).

  (Verification: $\langle \partial T ; \phi \rangle = \langle T ; d\phi \rangle = \int_{\mathbb{R}} \langle d\phi ; e \rangle m \, dx =$
  $= \int_{\mathbb{R}} \phi' \cdot m \, dx = - \int_{\mathbb{R}} \phi \cdot m' \, dx$.)

- Using the boundary rectifiability theorem we can restate F & F theorem as follows:

  [Let (T_n) be a sequence of rectifiable currents with integral multiplicity such that
  $$\mathbb{M}(T_n) ; \mathbb{M}(\partial T_n) \leq C < +\infty .$$
  Then T_n converge (up to subsequence) to a current T which is rectifiable with integral multiplicity.]

  (This statement may look weaker than F & F but it is actually equivalent!)

  This statement admits an interesting generalization (due to Ambrosio & Kirchheim):

  [Let $(T_n = [E_n, \tau_n, m_n])$ be a sequence of rectifiable currents with real-valued multiplicities m_n such that
  $$\mathbb{M}(T_n) ; \mathbb{M}(\partial T_n) \leq C < +\infty ; \ |m_n| \geq \delta > 0 .$$
  Then T_n converge to a rectifiable current $T = [E, \tau, m]$ with $|m| \geq \delta$.]

We conclude with two staments that show that currents can be approximated by "regular" currents. Contrary to expectations, however, "regular" currents are not those associated to smooth surfaces but polyhedral currents, which we define next.

Note that the situation here is different from what happens with distributions or Sobolev functions, where approximation by smooth functions is quickly obtained using regularization by convolution. The fact is that the convolution of currents by a smooth kernel is well-defined but produces d-currents which are "diffuse" (and more precisely of the form $\tau \cdot \mathcal{L}^n$ where $\mathcal{L}^n$ is Lebesgue measure and $\tau : \mathbb{R}^n \to \Lambda_d(\mathbb{R}^n)$ is smooth) thus nowhere close to any notion of d-dimensional object.

<u>Polyhedral currents</u>

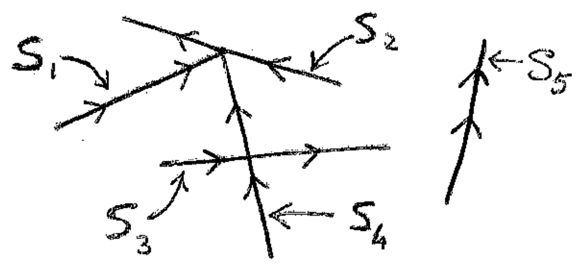
A d-current T in $\mathbb{R}^n$ is called <u>polyhedral</u> if it can be written in the form

$$T = \sum_i [S_i, \tau_i, m_i]$$

where:

- the sum is finite;
- S_i is a d-dimensional simplex (the convex envelope of $(d+1)$ points which are affinely independent);
- τ_i is a <u>constant</u> orientation of S_i;
- m_i is a <u>constant</u> multiplicity.

Example of a polyhedral 1-current:



If the multiplicities m_i are integers we say that T is an <u>integral polyhedral current</u>.

<u>Remarks</u> ∘ Polyhedral currents agree with the polyhedral chains with real coefficients in algebraic topology; integral polyhedral currents are polyhedral chains with integral coefficients.

∘ If needed one can "rewrite" a polyhedral current in order to "improve" the collection of the simplexes $\{S_i\}$, e.g. by requiring that the intersection of any two simplexes is a k-dimensional face of both, with $k<d$....

<u>Approximation of normal currents</u>

Let T be a normal current in $\mathbb{R}^n$.
Then there exists a sequence of <u>polyhedral currents</u> T_n (with <u>real</u> multiplicities) such that

$$T_n \to T; \quad \partial T_n \to \partial T$$

and

$$\mathbb{M}(T_n) \to \mathbb{M}(T); \quad \mathbb{M}(\partial T_n) \to \mathbb{M}(\partial T).$$

<u>Approximation of integral currents</u>

Let T be an integral current in $\mathbb{R}^n$.
Then there exists a sequence of <u>integral polyhedral</u> currents T_n that approximate T as in the previous statement.

Remarks
- One can improve these approximation statements in many ways: for instance, if $\partial T = 0$ we may require that $\partial T_n = 0$ for every n. In case the ambient space is a manifold (whatever the definition of polyhedral current is in this context) and $\partial T = 0$ we may require that T and T_n are cobordant, that is $T - T_n = \partial U_n \dots$ Moreover we can "improve" the convergence of T_n to $T \dots$
All this is possible because the tool behind these approximation results, the polyhedral deformation theorem, is highly flexible.
- The problem of approximating a d-current T with (the currents associated to) d-dimensional smooth surfaces is much more complicated and largely unexplored; by the previous results the point is the approximation of a polyhedral current by smooth surfaces, and under certain assumptions it is known that such an approximation is not always possible.

Question / exercise
In $\mathbb{R}$ consider the 1-current $T := [\mathbb{R}, \underset{\substack{\uparrow \\ \text{canonical orient. of } \mathbb{R}}}{e}, m]$ where $m \in C^1_c$. We have seen that T is normal and $\partial T = m' \cdot \mathscr{L}^1$.
What the polyhedral approximation of T should look like?

I conclude this lecture with an outline of the topics treated in the next lectures...

- Constancy lemma (and its variants);
- elementary constructions with currents: product and push-forward according to a map;
- cone construction and homotopy formula;
- flat distance;
- polyhedral deformation theorem (and applications);
- slicing of currents;
- characterization of rectifiable currents by slicing (after B. White, R. Jerrard, L. Ambrosio and B. Kirchheim);
- proofs of the boundary rectifiability theorem and of F&F compactness theorem.

Here the core of the theory of currents will have been completed.

At least the first five points I will try to explain in all details, while I might have to be more sketchy for the last three....

Currents 13/14 — Lecture 11 — 7/5/14

Constancy lemma, first version

Let T be a d-current in $\mathbb{R}^d$ with $\partial T = 0$.
Then $T = [\mathbb{R}^d, e, m]$ where m is a constant.

$e = e_1 \wedge \dots \wedge e_d$ canonical orientation of $\mathbb{R}^d$

Proof

We associate to T a distribution $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ by setting

$$\langle \Lambda; \phi \rangle := \langle T; \phi \, dx \rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

where $dx = dx_1 \wedge \dots \wedge dx_d$

(More abstractly, we write $T = e\Lambda$).

We claim that the distributional derivative $D\Lambda$ vanishes.

Fix indeed $i = 1, \dots, d$ and let

$$\omega := \phi \, \widehat{dx_i} \quad \longleftarrow (d-1)\text{-form}$$

where $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $\widehat{dx_i} := \bigwedge_{j \neq i} dx_j$. Then

$$0 \underset{\text{because } \partial T = 0}{=} \langle \partial T; \omega \rangle = \langle T; d\phi \rangle$$

$$= \left\langle T; \sum_{j=1}^{d} \frac{\partial \phi}{\partial x_j} dx_j \wedge \widehat{dx_i} \right\rangle$$

because $dx_j \wedge \widehat{dx_i} = 0$ for $j \neq i$ while $dx_i \wedge \widehat{dx_i} = (-1)^{i-1} dx$

$$= \left\langle T; (-1)^{i-1} \frac{\partial \phi}{\partial x_i} dx \right\rangle$$

by the defin. of Λ

$$= \left\langle \Lambda; (-1)^{i-1} \frac{\partial \phi}{\partial x_i} \right\rangle = \left\langle (-1)^i \frac{\partial \Lambda}{\partial x_i}; \phi \right\rangle.$$

To conclude the proof we recall the known fact that a distribution Λ on $\mathbb{R}^d$ with $D\Lambda=0$ is (represented by) a constant function m.

One way to prove this fact (if you really need to) is to consider the regularizations $\Lambda_\varepsilon := \Lambda * \rho_\varepsilon$ (with a kernel $\rho \in \mathcal{D}(\mathbb{R}^d)$) and observe that the Λ_ε are smooth functions with gradient

$$\nabla \Lambda_\varepsilon = \nabla(\Lambda * \rho_\varepsilon) = (D\Lambda) * \rho_\varepsilon = 0$$

and therefore are constant. Hence Λ is constant, too, being the limit of constant functions. $\square$

<u>Remark</u> we will actually apply the constancy lemma to currents defined on an open subset Ω of $\mathbb{R}^d$, which so far we carefully avoided. The statement runs as follows:

[Let T be a d-current on Ω, open subset of $\mathbb{R}^d$, such that $\partial T = 0$. Then $T = [\Omega, e, m]$ where $m : \Omega \to \mathbb{R}$ is constant on every connected component of Ω.]

The proof is essentially the same as before, and relies on the following known fact: if Λ is a distribution on Ω with $D\Lambda = 0$, then Λ is (represented by) a locally constant function.

As a by-product of the proof above we obtain the following useful fact:

Proposition 1

Let T be a d-current on $\mathbb{R}^d$ such that $\mathbb{M}(\partial T) < +\infty$. Then $T = [\mathbb{R}^d, e, m]$ where $m : \mathbb{R}^d \to \mathbb{R}$ is a function in $L^1_{loc}(\mathbb{R}^d)$ whose distributional derivative Dm is a (finite) measure (with values in $\mathbb{R}^d$). In particular $m \in BV_{loc}(\mathbb{R}^d)$.

Remark This result shows that every d-current in $\mathbb{R}^d$ such that the boundary has finite mass, and in particular every normal d-current, has to be rectifiable, that is, "absolutely continuous w.r.t. Lebesgue measure" and in particular it cannot be supported on a set with dimension strictly less than d.
We will later extend this last statement to normal d-currents in $\mathbb{R}^n$...

Proof

Define Λ as in the proof of the constancy lemma.
Since ∂T has finite mass, it can be written as $\partial T = \tau\mu$ with μ positive measure etc...
Then, given $i = 1, \dots, d$, and $\phi \in \mathcal{D}(\mathbb{R}^d)$ we have (from the proof of the constancy lemma)

$$\langle \partial T ; \phi \widehat{dx_i} \rangle = \langle (-1)^i \frac{\partial \Lambda}{\partial x_i} ; \phi \rangle$$

and then, writing $\tau = \sum_i \tau_i \hat{e}_i$ where $\hat{e}_i := \bigwedge_{j \neq i} e_j$,

$$\begin{aligned} \langle \frac{\partial \Lambda}{\partial x_i} ; \phi \rangle \quad & \langle \partial T ; (-1)^i \phi \widehat{dx_i} \rangle \\ &= \int \langle \phi \widehat{dx_i} ; \tau \rangle (-1)^i \, d\mu \\ &= \int \phi \, (-1)^i \tau_i \, d\mu \end{aligned}$$

Hence $\frac{\partial \Lambda}{\partial x_i}$ is (represented by) the measure $(-1)^i \tau_i \mu$.
Thus $D\Lambda$ is a measure, and we conclude the proof using the following lemma. $\square$

Lemma

Let Λ be a distribution of $\mathbb{R}^d$ such that $D\Lambda$ is a measure. Then Λ is (represented by) a function in $L^p_{loc}(\mathbb{R}^d)$ where $p := \frac{d}{d-1}$.

Proof

We start from a (more or less) known fact.
Let Ω be a bounded regular open set in $\mathbb{R}^d$, T a bounded linear functional on $L^1(\Omega)$ which does not vanishes on constants.
Then $\|\nabla u\|_1 + |Tu|$ is a norm on $W^{1,1}(\Omega)$ which is equivalent to the usual norm.
In particular $\|u\|_p \leq c\,(\|\nabla u\|_1 + |Tu|)$.

constant depending on Ω

Sobolev embedding exponent $\frac{d}{d-1}$

Now we apply the last estimate with

- Ω = ball in $\mathbb{R}^d$;
- $Tu := \int u\phi\, dx$ with $\phi \in \mathcal{C}^\infty_c(\Omega)$ chosen so that the average of ϕ does not vanishes;
- $u := \Lambda_\varepsilon := \Lambda * \rho_\varepsilon$, regularization of Λ;

and we obtain

$$\left(\int_\Omega |\Lambda_\varepsilon|^p dx\right)^{1/p} \leq c\left[\int_\Omega |\nabla\Lambda_\varepsilon|\, dx + \left|\int_\Omega \Lambda_\varepsilon \phi\, dx\right|\right]$$
$$\leq c\left[\int_{\mathbb{R}^d} |\nabla\Lambda_\varepsilon|\, dx + \left|\int_{\mathbb{R}^d} \Lambda_\varepsilon \phi\, dx\right|\right]$$
$$= c\left[\|\nabla\Lambda_\varepsilon\|_1 + |\langle\Lambda_\varepsilon; \phi\rangle|\right]$$

Now as $\varepsilon \to 0$ we have that $\|\nabla \Lambda_\varepsilon\|_1 \to \|D\Lambda\|$ and $\langle \Lambda_\varepsilon, \phi\rangle \to \langle \Lambda, \phi\rangle$ and therefore the functions Λ_ε are uniformly bounded in $L^p(\Omega)$.
Hence the restriction of Λ to Ω belongs to $L^p(\Omega)$, and since Ω is arbitrary, Λ belongs to $L^p_{loc}(\mathbb{R}^d)$. $\square$

One can extend Proposition 1 as follows (we omit the proof):

Proposition 2

Let S be d-dimensional surface of class C^1 in $\mathbb{R}^n$ oriented by τ_S, and let T be a d-current with $\mathbb{M}(\partial T) < +\infty$ which is supported on S (that is, $\langle T, \omega\rangle = 0$ for every ω s.t. $\mathrm{spt}(\omega) \cap S = \emptyset$).
Then $T = [S, \tau_S, m]$ where $m : S \to \mathbb{R}$ is a function in $BV_{loc}(S)$.

An immediate corollary is the following.

Constancy lemma, second version

Let S be as in Proposition 2 and connected, and let T be a d-current supported on S with $\partial T = 0$.
Then $T = [S, \tau_S, m]$ with m a constant.

Product of currents

We define now an operation between currents that in case of smooth surfaces corresponds to the usual Cartesian product (in the sense of product of sets).

However, since currents are not sets, the abstract definition of product of currents requires some care....

Let T be an h-current in $\mathbb{R}^m$ and U a k-current in $\mathbb{R}^n$. Then there exists a unique $(h+k)$-current in $\mathbb{R}^{m+n} \simeq \mathbb{R}^m \times \mathbb{R}^n$, denoted by $T \times U$, which satisfies the following identities:

(i) $\langle T \times U \, ; \, \phi(x) \, \psi(y) \, dx_{\underline{i}} \wedge dy_{\underline{j}} \rangle =$

variable in $\mathbb{R}^m$

variable in $\mathbb{R}^n$

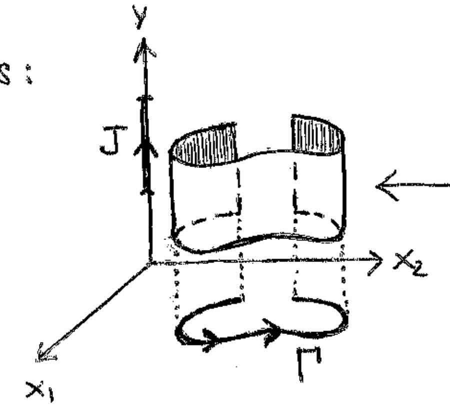
$$= \langle T ; \phi(x) \, dx_{\underline{i}} \rangle \cdot \langle U ; \psi(y) \, dy_{\underline{j}} \rangle$$

for every h-index $\underline{i}$ (that is, every $\underline{i} \in I_{m,h}$), every k-index $\underline{j}$ (that is, every $\underline{j} \in I_{n,k}$), every $\phi \in \mathcal{D}(\mathbb{R}^m)$ and every $\psi \in \mathcal{D}(\mathbb{R}^n)$.

(ii) $\langle T \times U \, ; \, \rho(x,y) \, dx_{\underline{i}} \wedge dy_{\underline{j}} \rangle = 0$

whenever $\underline{i}$ is <u>not</u> an h-index (and accordingly $\underline{j}$ is not a k-index).

To understand conditions (i) and (ii) consider for simplicity the following case: T is the 1-current associated to a curve Γ in $\mathbb{R}^2$ while U is the 1-current associated to a segment J in $\mathbb{R}$. It is easy to see that in this case both (i) and (ii) holds:



$T \times U$ is the current associated to the surface $\Gamma \times J$, oriented by the product of the orientations of Γ and J, namely $\tau_\Gamma \wedge \tau_J$

In particular (ii) states that $\int_{\Gamma \times J} \omega = 0$ for every form $\omega = \rho \, dx_1 \wedge dx_2$, which is obvious because the 2-vector $\tau_\Gamma \wedge \tau_J$ has zero component with respect to $e_{x_1} \wedge e_{x_2}$ and therefore $\langle \omega ; \tau_\Gamma \times \tau_J \rangle \equiv 0$.

To make sure that the current $T \times U$ is well-defined we should check that a current satisfying (i) and (ii) exists and is unique.

Let us begin with uniqueness.

Property (i) determines the action of $T \times U$ on all forms $\phi(x) \cdot \psi(y) \, dx_{\underline{i}} \wedge dy_{\underline{j}}$ with ϕ, ψ smooth, $\underline{i}$ an h-index and $\underline{j}$ a k-index.

By linearity we pass to all forms $\rho(x,y) \, dx_{\underline{i}} \wedge dy_{\underline{j}}$ with $\rho(x,y) = \sum_e \phi_e(x) \, \psi_e(y)$ and $\underline{i}, \underline{j}$ as before.

By density we pass to all $\rho(x,y) \, dx_{\underline{i}} \wedge dy_{\underline{j}}$ with $\underline{i}$ and $\underline{j}$ as above. (Here we use that finite sums $\sum_e \phi_e(x) \psi_e(y)$ with $\phi_e \in \mathcal{D}(\mathbb{R}^m)$ and $\psi_e \in \mathcal{D}(\mathbb{R}^n)$ are dense in $\mathcal{D}(\mathbb{R}^m \times \mathbb{R}^n)$, in the appropriate topology. And that $T \times U$ is continuous...)

Thus (i) and (ii) determine the action of $T \times U$ on all forms $\rho(x,y) \, dx_{\underline{i}} \wedge dy_{\underline{j}}$ with $\rho \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^m)$ and $\underline{i}, \underline{j}$ multi de es of order h' and k' so that $h' + k' = h + k$. And by linearity on all forms in $\mathcal{D}^{k+h}(\mathbb{R}^m \times \mathbb{R}^n)$.

For existence, one should show that there exists a linear functional on $\mathcal{D}^{h+k}(\mathbb{R}^m \times \mathbb{R}^n)$ which satisfies (i) and (ii) and is <u>continuous</u>.
Continuity may be tricky because we didn't really studied the topology on $\mathcal{D}^{h+k}(\mathbb{R}^m \times \mathbb{R}^n)$....
So I will skip this part.

<u>Boundary of the product</u>

Given T and U as above, we have that

$$(*) \qquad \partial(T\times U) = \partial T \times U + (-1)^{\underset{\substack{\uparrow \\ \text{dimension of } T}}{h}} T \times \partial U .$$

<u>Proof</u>

To prove this formula we must start from a formula for the differential of the product of two forms.
Given ω h-form on $\mathbb{R}^N$ and σ k-form on $\mathbb{R}^N$, both of class C^1, we have

$$(1) \qquad d(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^h \omega\wedge d\sigma .$$

The proof of this formula is easily obtained by writing ω and σ in coordinates and then applying the definition of differential.

We now prove that $(*)$ holds for certain classes of forms, and then show that from these classes we obtain $(*)$ in full generality....

<u>Step 1.</u> Let $\omega = \phi(x)\,\psi(y)\, dx_{\underline{i}} \wedge dy_{\underline{j}}$ with $\underline{i}$ an $(h-1)$-index and j a k-index.
Then
$$\langle \partial(T\times U) ; \omega\rangle = \langle T\times U ; d\omega\rangle$$
$$= \langle T\times U ; d\big(\phi(x) dx_{\underline{i}} \wedge \psi(y) dy_{\underline{j}}\big)\rangle$$

and using (1)

$$\langle \partial(T\times U);\omega\rangle = \langle T\times U;\ \overbrace{d(\phi(x)\,dx_{\underline{i}})}^{h\text{-form}} \wedge \overbrace{\psi(y)\,dy_{\underline{j}}}^{k\text{-form}}\rangle$$

$$+(-1)^{h-1}\underbrace{\langle T\times U;\ \underbrace{\phi(x)\,dx_{\underline{i}}}_{(h-1)\text{-form}} \wedge \underbrace{d(\psi(y)dy_{\underline{j}})}_{(k+1)\text{-form}}\rangle}_{\substack{\text{by (ii) in the definition of}\\ \text{product } T\times U,\ \text{this term is } 0}}$$

$$= \langle T\times U;\ d(\phi(x)\,dx_{\underline{i}}) \wedge \psi(y)\,dy_{\underline{j}}\rangle$$

by (i) in the definition of $T\times U$ $\longmapsto$ $= \langle T;\ d(\phi(x)dx_{\underline{i}})\rangle\cdot\langle U;\ \psi(y)\,dy_{\underline{j}}\rangle$

$$= \langle \partial T;\ \phi(x)\,dx_{\underline{i}}\rangle\cdot\langle U;\ \psi(y)\,dy_{\underline{j}}\rangle$$

by (i) in the definition of $\partial T\times U$ $\longmapsto$ $= \langle \partial T\times U;\ \omega\rangle$

Notice moreover that property (ii) applied to the definition of $T\times\partial U$ implies that $\langle T\times\partial U;\omega\rangle = 0$ and therefore we have

(2) $$\langle \partial(T\times U);\omega\rangle = \langle \partial T\times U + (-1)^h\, T\times\partial U;\ \omega\rangle$$

whenever $\omega = \phi(x)\,\psi(y)\;dx_{\underline{i}}\wedge dy_{\underline{j}}$ with $\phi,\psi\in\mathcal{D}$ and $\underline{i}$ an $(h-1)$-index, $\underline{j}$ a k-index.

By linearity and continuity we can then extend (2) to all $\omega = \rho(x,y)\;dx_{\underline{i}}\wedge dy_{\underline{j}}$ with $\rho\in\mathcal{D}$ and $\underline{i},\underline{j}$ as before.

Step 2. Let $\omega = \phi(x)\,\psi(y)\;dx_{\underline{i}}\wedge dy_{\underline{j}}$ with $\underline{i}$ an h-index and $\underline{j}$ a $(k-1)$-index. Proceeding as in Step 1 we get

$$\langle \partial(T\times U);\omega\rangle = (-1)^h\,\langle T\times\partial U;\omega\rangle$$

and from this we obtain that (2) holds for ω of this type, and then also for every ω of the form $\omega = \rho(x,y)\;dx_{\underline{i}}\wedge dy_{\underline{j}}$ with $\rho\in\mathcal{D}$, $\underline{i}$ an h-index and $\underline{j}$ a $(k-1)$-index.

Step 3. Consider now $\omega = \rho(x,y)\, dx_{\underline{i}} \wedge dy_{\underline{j}}$ where $\underline{i}$ is neither an $(h-1)$-index nor an h-index. Then

$$d\omega = d\rho \wedge dx_{\underline{i}} \wedge dy_{\underline{j}}$$

$$= \sum_{i'=1}^{m} \frac{\partial \rho}{\partial i'} \underbrace{(dx_{i'} \wedge dx_{\underline{i}})}_{\text{form of order} \neq h \text{ because } \underline{i} \text{ is not an } (h-1)\text{-index}} \wedge dy_{\underline{j}}$$

$$+ \sum_{j'=1}^{n} \frac{\partial \rho}{\partial y_j} \underbrace{dx_{\underline{i}}}_{\text{form of order} \neq h} \wedge (\pm dy_{j'} \wedge dy_{\underline{j}})$$

and by property (ii) in the definition of $T \times U$ we get that $T\times U$ applied to any of the addenda in the last term gives 0, and then

$$\langle \partial(T\times U) ; \omega \rangle = \langle T\times U ; d\omega \rangle = 0 .$$

On the other hand, property (ii) in the definition of $\partial T \times U$ and $T \times \partial U$ implies that

$$\langle \partial T \times U ; \omega \rangle = \langle T \times \partial U ; \omega \rangle = 0 .$$

Putting together the last two formulas we get that (2) holds also for this type of form ω.

Putting together Steps 1–3 we obtain that formula (2) holds (by linearity) for all forms ω, and therefore we have proved $(*)$.

$\square$

Product of currents with finite mass

If $T := \tau\mu$ and $U := \sigma\lambda$ are currents with finite mass, then one readily checks that $T \times U$ is the current with finite mass given by

$$T \times U = (\tau \wedge \sigma)(\mu \times \lambda)$$

where $\mu \times \lambda$ is the product measure in $\mathbb{R}^m \times \mathbb{R}^n$.
(Note that the product $\tau \wedge \sigma$ makes sense provided that we identify the h-vector (field) τ on $\mathbb{R}^m$ with an h-vector (field) in $\mathbb{R}^m \times \mathbb{R}^n$, and the same for the k-vector (field) σ .....)
In particular $\mathbb{M}(T \times U) = \mathbb{M}(T) \cdot \mathbb{M}(U)$.

Product of rectifiable currents

If $T := [E, \tau, m]$ and $U := [F, \sigma, n]$ are rectifiable currents, then $T \times U$ is the rectifiable current given by

$$T \times U = [E \times F ; \tau \wedge \sigma ; m \cdot n]$$

$(h+k)$-rectifiable set in $\mathbb{R}^m \times \mathbb{R}^n$ (check!)
orientation of $E \times F$ (chek!)

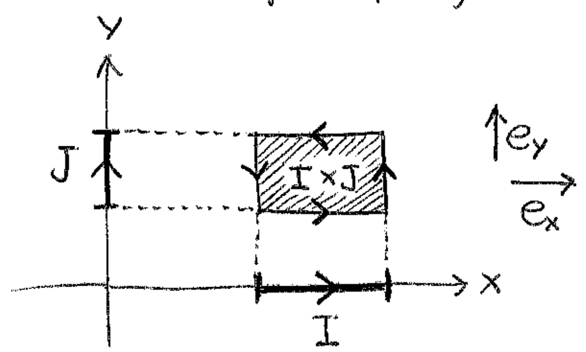
This formula is an immediate consequence of the formula for the product of currents with finite mass, and relies on the the fact that

$$1_{E\times F} \cdot \mathcal{H}^{h+k} = (1_E \cdot \mathcal{H}^h) \times (1_F \cdot \mathcal{H}^k)$$

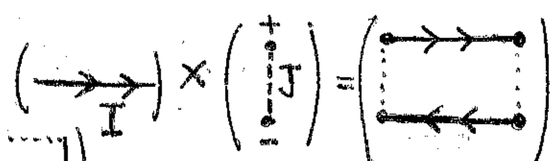
(restriction of $\mathcal{H}^{h+k}$ to $E \times F$)
Note that the last identity holds because E, F are <u>rectifiable</u>!!

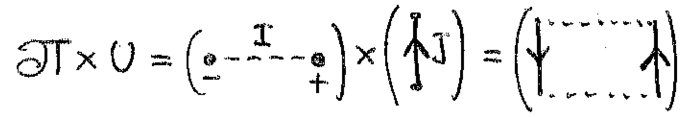
<u>Remark</u> Using the results in the previous paragraph we can easily prove the following: if T and U are normal then $T\times U$ is normal; if they are rectifiable with integral multiplicity so is $T\times U$; and finally, if the are integral currents so is $T\times U$.

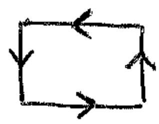
<u>Example/exercise</u> Let us check the meaning of the formula for the boundary of $T\times U$ in a simple case: let $T := [I, e_x, 1]$ be the 1-current in $\mathbb{R}_x$ associated to an interval I and the standard orientation e_x or $\mathbb{R}_x$, and let $U := [J, e_y, 1]$ be a similarly defined 1-current in $\mathbb{R}_y$ (we write $\mathbb{R}_x$ and $\mathbb{R}_y$ instead of $\mathbb{R}$ to distinguish these two copies of $\mathbb{R}$).



Then $T\times U = [I\times J, e_x\wedge e_y, 1]$ and in particular the boundary of $T\times U$, according to the classical formula for surfaces, should be oriented as in the picture.

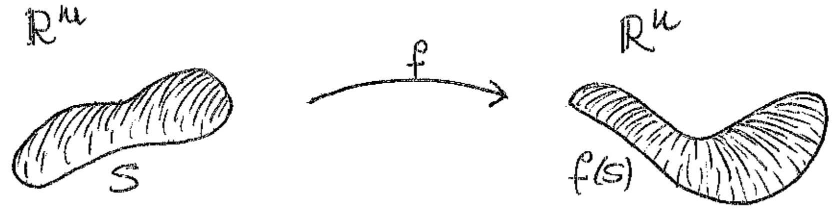
On the other hand $T\times\partial U =$ 

and $\partial T\times U =$ ,

and therefore $\partial T\times U - T\times\partial U =$  as in the figure above.

Currents 13/14 — Lecture 12 8/5/14

In this lecture we discuss the notion of push-forward of a current according to a map.
The elementary geometric meaning to keep in mind is the following: given a surface S in $\mathbb{R}^m$ of dim. d and a map $f : \mathbb{R}^m \to \mathbb{R}^n$, if f satisfies certain conditions $f(S)$ will also be a surface, and if T is the current associated to S, its push-forward according to f is the current associated to $f(S)$.
This definition, however, can be hardly extended to a general current T (or to a less regular map f) and therefore the definition of push-forward has to be completely different.



Note that given a d-form on $\mathbb{R}^n$ (the codomain of f) then

$$\int_{f(S)} \omega = \int_S f^\# \omega$$

where $f^\# \omega$ is the pull-back of ω according to f, and is a d-form on $\mathbb{R}^m$ (the domain of f).
(We define $f_\# \omega$ presently.) This identity can be extended to currents and is used to define the push-forward

Pull-back of k-covector

Let $T : V \longrightarrow W$ be a linear map between linear spaces, and let $\alpha \in \Lambda^k(W)$. We define the pull-back of α according to T as the k-covector $T^\# \alpha \in \Lambda^k(V)$ given by

$$T^\# \alpha(v_1, \dots, v_k) := \alpha(Tv_1, \dots, Tv_k) \qquad \forall\, v_1, \dots, v_k \in V.$$

Assume now that V and W are endowed with a scalar product, and let $|T|$ be the operator norm of T (that is, its Lipschitz constant). Then

$$\| T\alpha \| \leq |T|^k \, \|\alpha\|$$

where $\|\cdot\|$ is the comass norm, that is

$$\|\alpha\| := \sup_{|v_1 \wedge \dots \wedge v_k| \leq 1} \underset{\alpha(v_1, \dots, v_k)}{\underset{\|}{\langle \alpha ; v_1 \wedge \dots \wedge v_k \rangle}}$$

Note that for every $v_1, \dots, v_k$ s.t. $|v_1 \wedge \dots \wedge v_k| \leq 1$ there holds $|Tv_1 \wedge \dots \wedge Tv_k| \leq |T|^k$. Indeed the mass (or Euclidean) norm of a simple k-vector $v_1 \wedge \dots \wedge v_k$ is the volume ($\mathcal{H}^k$) of the rectangle $R(v_1, \dots, v_k)$ spanned by $v_1, \dots, v_k$ and then

$$\begin{aligned} |Tv_1 \wedge \dots \wedge Tv_k| &= \mathcal{H}^k(R(Tv_1, \dots, Tv_k)) \\ &= \mathcal{H}^k(T(R(v_1, \dots, v_k))) \\ &\leq |T|^k \, \mathcal{H}^k(R(v_1, \dots, v_k)) \\ &= |T|^k \, |v_1 \wedge \dots \wedge v_k| \leq |T|^k . \end{aligned}$$

Hence

$$\begin{aligned} \|T^\# \alpha\| &= \sup_{|v_1 \wedge \dots \wedge v_k| \leq 1} T^\#_\alpha(v_1, \dots, v_k) \\ &= \sup_{|v_1 \wedge \dots \wedge v_k| \leq 1} \alpha(Tv_1, \dots, Tv_k) \end{aligned}$$

$$\leq \sup_{|w_1\wedge\cdots\wedge w_k|\leq |T|^k} \alpha(w_1,\dots,w_k) = |T|^k \,\|\alpha\|.$$

Push-forward of a k-vector

Let $T : V \to W$ as before, and let $v \in \Lambda_k(V)$. Then we define the push-forward of v according to T as the k-vector $T_\# v \in \Lambda_k(W)$ defined by the duality relation

$$\langle T_\# v; \alpha\rangle := \langle v; T^\# \alpha\rangle \qquad \forall \alpha \in \Lambda^k(W).$$

If V and W are endowed with scalar products, then this formula and the inequality $\| T^\#\alpha\| \leq |T|^k \|\alpha\|$ yield

$$\| T_\# v\| \leq |T|^k \,\|v\|$$

where now $\|\cdot\|$ stands for the mass norm (whose dual norm is the comass norm).

Remarks
- It follows immediately from the definition of $T_\#$ and $T^\#$ that

$$T_\#(v_1\wedge\cdots\wedge v_k) = Tv_1\wedge\cdots\wedge Tv_k$$

for every simple vector $v_1\wedge\dots\wedge v_k$.
More generally, if $v \in \Lambda_k(V)$ and $\tilde{v}\in\Lambda_h(V)$ then

$$T_\#(v\wedge\tilde{v}) = T_\# v \wedge T_\#\tilde{v}$$

and if $\alpha\in\Lambda^k(W)$ and $\tilde{\alpha}\in\Lambda^h(W)$ then

$$T^\#(\alpha\wedge\tilde{\alpha}) = T^\#\alpha\wedge T^\#\tilde{\alpha}.$$

- If $k=1$ then $T_\# = T$ and $T^\#$ is the adjoint of T.

More generally $T_\# : \Lambda_k(V) \to \Lambda_k(W)$ is the adjoint of $T^\# : \Lambda^k(W) \to \Lambda^k(V)$ (just by the definition of $T_\#$).

Pull-back of a form

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a map of class C^1 at least and let ω be a k-form on $\mathbb{R}^n$. The pull-back of ω according to f is the k-form $f^\#\omega$ on $\mathbb{R}^m$ defined by

$$\left(f^\#\omega\right)(x) := \left(df(x)\right)^\#\left(\omega(\underbrace{y}_{=f(x)})\right) \quad \forall x \in \mathbb{R}^m.$$

Thus $$\underbrace{\|(f^\#\omega)(x)\|}_{\substack{\text{comass}\\ \text{norm of } (f^\#\omega)(x)}} \le \underbrace{|df(x)|}_{\substack{\text{operator}\\ \text{norm of } df(x)}}{}^{k} \; \underbrace{\|\omega(y)\|}_{\cdots}$$

k is not enough!

Moreover, if ω is of class C^k and f is of class C^{k+1} then $f^\#\omega$ is of class C^k.
In particular, if ω and f are of class C^∞, so is $f^\#\omega$.

Push-forward of a current

Given $f : \mathbb{R}^m \to \mathbb{R}^n$ and T k-current on $\mathbb{R}^m$, the push-forward of T according to f is the k-current $f_\# T$ on $\mathbb{R}^n$ defined by

$$\langle f_\# T ; \omega \rangle := \langle T ; f^\#\omega \rangle \qquad \forall \omega \in \mathscr{D}^d$$

Thus $f_\#$ is, by definition, the adjoint of $f^\#$.

push-forward operator on k-currents — pull-back operator on k-forms

Note that the definition of $f_\# T$ is well-posed only if f satisfies certain regularity assumptions, depending on the "regularity" of T.
We discuss now some relevant cases.

1) If T is a general current then f must be smooth ($\mathcal{E}^\infty$) and proper (that is, $f^{-1}(K)$ is compact for every K compact, or, equivalently, $|f(x)| \longrightarrow +\infty$ as $|x| \longrightarrow +\infty$).
Under these assumptions $f^\# \omega$ belongs to $\mathcal{D}^k$, that is, $f^\# \omega$ is smooth (for this we need f smooth) and compactly supported (for this we need that f is proper). And it is clear that such assumptions cannot be really weakened....

2) If T has compact support then f must be smooth (but we no longer need that f is proper). The key point is the following: if T has compact support we can define $\langle T; \omega \rangle$ for every k-form ω of class $\mathcal{E}^\infty$ (e.g., use that $\langle T; \omega \rangle := \langle T; \rho_0 \omega \rangle$ for a given $\rho_0 \in \mathcal{D}$ such that $\rho_0 = 1$ on $\mathrm{spt}(T)$, and then notice that the right-hand side make sense for all ω of class $\mathcal{E}^\infty$...).
But then $\langle T; f^\# \omega \rangle$ is well-defined if $f^\# \omega$ is smooth, and for this it suffices that f is smooth.

3) If T has compact support and finite mass then it suffices that f is of class $\mathscr{C}^1$.
Since $T = \tau\mu$ where μ is a measure with finite mass and compact support, then
$\langle T; \omega \rangle = \int \langle \omega; \tau \rangle \, d\mu$ is well-defined
(and continuous) on the space of continuous forms ω; so $\langle T; f^\# \omega \rangle$ is defined if $f^\# \omega$ is continuous, and for this it suffices that $f \in \mathscr{C}^1$.

<u>Mass of push-forward</u>

Moreover in the last case we have that

$$(*) \qquad \mathbb{M}(f_\# T) \leq \int |df|^k \, \|\tau\| \, d\mu \leq \Big(\sup_{x \in \operatorname{spt}(\mu)} |df(x)| \Big)^k \cdot \mathbb{M}(T)$$

<u>Proof</u> Note that

$$\langle f_\# T; \omega \rangle = \langle T; f^\# \omega \rangle = \int \langle f^\# \omega; \tau \rangle \, d\mu$$

and then

$$|\langle f_\# T; \omega \rangle| \leq \int \underbrace{\| f^\# \omega(x) \|}_{\text{comass norm}} \, \underbrace{\| \tau(x) \|}_{\text{mass norm}} \, d\mu(x)$$

$$\leq \int |df(x)|^k \, \| \omega(f(x)) \| \, \| \tau(x) \| \, d\mu(x),$$

and if $\|\omega\|_\infty \leq 1$

$$\leq \int |df(x)|^k \, \| \tau(x) \| \, d\mu(x),$$

and taking the supremum over all ω s.t. $\|\omega\|_\infty \leq 1$

$$\mathbb{M}(f_\# T) \leq \int |df(x)|^k \, \| \tau(x) \| \, d\mu(x).$$

which is the first inequality in $(*)$ (the second follows). $\square$

Boundary of the push-forward

Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a map such that both $f_\# T$ and $f_\#(\partial T)$ are well-defined (according to the previous discussion). Then

$$\partial(f_\# T) = f_\#(\partial T) .$$

This identity is the "dual version" of a similar identity for the pull-back of forms (which is taken for granted) namely that

$$d(f^\# \omega) = f^\#(d\omega) .$$

Indeed $\langle \partial(f_\# T) ; \omega \rangle = \langle f_\# T ; d\omega \rangle = \langle T ; f^\#(d\omega) \rangle =$
$= \langle T ; d(f^\# \omega) \rangle = \langle \partial T ; f^\# \omega \rangle = \langle f_\#(\partial T) ; \omega \rangle \dots$

Push-forward of a rectifiable current

Let $T = [E, \tau, m]$ be a rectifiable k-current with compact support (that is, E is bounded) in $\mathbb{R}^m$, and let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a map of class C^1. Then $f_\# T$, which is well-defined by the discussion above, is rectifiable, and more precisely

$$f_\# T = [\tilde{E}, \tilde{\tau}, \tilde{m}]$$

where
- $\tilde{E} := f(E)$ (which is k-rectifiable!)
- $\tilde{\tau}$ is any (fixed) orientation of $\tilde{E}$
- $\tilde{m}$ is given by the following formula (for $\mathcal{H}^k$-a.e. $y \in \tilde{E}$):

$$(*) \qquad \tilde{m}(y) = \sum_{x \in f^{-1}(y) \cap E} \pm\, m(x)$$

where $\pm$ means $+$ if $df(x)$, viewed as a map from $\mathrm{Tan}(E,x)$ to $\mathrm{Tan}(\hat{E},y)$, preserves the orientation, and $\pm$ means $-$ otherwise.

Proof

We have two issues here: first showing that the definition of $\tilde{m}(y)$ makes sense for (at least) $\mathcal{H}^k$-a.e. $y \in \tilde{E}$; and secondly proving that $f_\# T = [\tilde{E}, \tilde{\tau}, \tilde{m}]$.

To this end we recall that given f as above, then $df(x)$ maps $\mathrm{Tan}(E,x)$ into $\mathrm{Tan}(\tilde{E},y)$ for $\mathcal{H}^k$-a.e. $x \in E$, and therefore $df(x)$ defines a linear map from $\mathrm{Tan}(E,x)$ to $\mathrm{Tan}(\hat{E},y)$, the tangential differential $d_\tau f(x)$.
(This ~~is~~ explained at the end of the notes for lecture 5, and it actually applies even to f Lipschitz).

We recall moreover the area formula (third version):

$$(1) \qquad \int_{\tilde{E}} \#(f^{-1}(y) \cap E)\, d\mathcal{H}^k(y) = \int_E \underbrace{J_\tau f(x)}\, d\mathcal{H}^k(x)$$

tangential
Jacobian of f

Now we assume for simplicity that $\mathcal{H}^k(E) < +\infty$.
Since $J_\tau f$ is bounded on E (which is bounded) the right-hand side of last identity is finite.
It follows that

$$(2) \qquad \int_{\tilde{E}} \#(f^{-1}(y) \cap E) \, d\mathcal{H}^k(y) < +\infty$$

and in particular $\#(f^{-1}(y) \cap E) < +\infty$ for $\mathcal{H}^k$-a.e. $y \in \tilde{E}$, i.e.,

(3) $\quad f^{-1}(y) \cap E$ is a finite set for $\mathcal{H}^k$-a.e. $y \in \tilde{E}$.

Finally we consider the set S of all $x \in E$ s.t. $df(x)$ does not maps $\mathrm{Tan}(E,x)$ into $\mathrm{Tan}(\tilde{E}, f(x))$, or it does but $\mathrm{rank}(d_\tau f(x)) < k$.
We claim that

$$(4) \qquad \mathcal{H}^k(f(S)) = 0 .$$

Apply indeed formula (1) with S in place of E:

$$\mathcal{H}^k(f(S)) \le \int_{f(S)} \#(f^{-1}(y) \cap S) \, d\mathcal{H}^k(y)$$

$$= \int_S J_\tau f(x) \, d\mathcal{H}^k(x)$$

but $J_\tau f(x) = 0$ for $\mathcal{H}^k$-a.e. $x \in S$ because $\mathrm{rank}(d_\tau f(x)) < k$.
Hence we get (4).

Putting together what we have seen so far, and in particular (3) and (4), we obtain that for $\mathcal{H}^k$-a.e. $y \in \tilde{E}$ there holds:

- $f^{-1}(y) \cap E$ is a finite set
- $d_\tau f(x)$ is a well-defined linear map from $\mathrm{Tan}(E,x)$ to $\mathrm{Tan}(\tilde{E},y)$ with rank $= k$, which means that it is a linear isomorphism (and therefore it either preserves or reverses the orientation).

We have thus proved that $\tilde{m}(y)$ is well-defined for $\mathcal{H}^k$-a.e. $y \in \tilde{E}$.

We prove now that $\int_{y\in\tilde{E}} |\tilde{m}(y)| \, d\mathcal{H}^k(y)$ is finite.

We start from the obvious inequality

$$|\tilde{m}(y)| \leq \sum_{x\in f^{-1}(y)\cap E} |m(x)|$$

and the following variant of the area formula (1):

(5)
$$\int_{\tilde{E}} \Big(\sum_{x\in f^{-1}(y)\cap E} h(x) \Big) \, d\mathcal{H}^k(y) = \int_E h(x) \, J_\tau f(x) \, d\mathcal{H}^k(x).$$

where $h: E \to [0,+\infty]$ is Borel.

By applying (5) with $|m|$ in place of h we get

$$\int_{\tilde{E}} |\tilde{m}(y)| \, d\mathcal{H}^k(y) \leq \int_{\tilde{E}} \sum_{x\in f^{-1}(y)\cap E} |m(x)| \, d\mathcal{H}^k(y)$$
$$= \int_E |m(x)| \, J_d f(x) \, d\mathcal{H}^k(x) < +\infty$$

(the last inequality holds because $m \in L^1(1_E \mathcal{H}^k)$ by assumption, while $J_d f$ is bounded on E).

We can finally prove that $f_\# T = [\tilde{E}, \tilde{\tau}, \tilde{m}]$:
Given $\omega \in \ldots$ we have

$$\langle f_\# T ; \omega \rangle = \langle T ; f^\# \omega \rangle$$

(6) $$= \int_{x\in E} \langle f^\# \omega(x) ; \underbrace{\tau(x)}_{\tau_1(x)\wedge\ldots\wedge\tau_k(x)} \rangle \, m(x) \, d\mathcal{H}^d(x)$$

$$= \int_{x \in E} \langle \omega(f(x)) ; (df(x))_\# \tau(x) \rangle \, m(x) \, d\mathcal{H}^k(x)$$

Now, $(df(x))_\# \tau(x)$ is a k-vector in $\mathrm{Tan}(\tilde{E}, \overset{y}{f(x)})$ (at least for a.e. x) and therefore is a multiple of $\tilde{\tau}(y)$, that is

$$(df(x))_\# \tau(x) = a \cdot \tilde{\tau}(y).$$

Let us compute a. Since $\|\tilde{\tau}(y)\| = 1$ by definition,

$$|a| = \|(df(x))_\# \tau(x)\|$$

Write $\tau = \tau_1 \wedge \ldots \wedge \tau_k$ $\longrightarrow$ $$= \|(df(x)\,\tau_1(x)) \wedge \ldots \wedge (df(x)\,\tau_k(x))\|$$

$$= \mathcal{H}^k \big(\underbrace{df(x)\,(R(v_1, \ldots, v_k))}_{\text{image according to } df(x) \text{ of the rectangle spanned by } v_1,\ldots,v_k} \big)$$

this is the <u>defining</u> property of the determinant of $df(x)$ $\longrightarrow$ $$= J_\tau f(x) \,.\, \mathcal{H}^k(R(v_1, \ldots, v_k))$$

$$= J_\tau f(x) \, \|v_1 \wedge \ldots \wedge v_k\| = J_\tau f(x).$$

Thus $|a| = J_\tau f(x)$, and therefore

$$a = \pm J_\tau f(x)$$

where $\pm$ means $+$ if $d_\tau f(x)$ is orientation preserving,

and $\pm$ means $-$ otherwise.
Therefore, going back to (6) we get

$$\langle f_\# T ; \omega \rangle = \int_{x\in E} \langle \omega(f(x)) ; (df(x))_\# \tau(x) \rangle \, m(x) \, d\mathcal{H}^k(x)$$

$$= \int_{x\in E} \langle \omega(f(x)) ; \pm J_\tau f(x) \cdot \tilde\tau(f(x)) \rangle \, m(x) \, d\mathcal{H}^k(x)$$

$$= \int_{x\in E} \underbrace{\langle \omega(f(x)) ; \tilde\tau(f(x)) \rangle \, (\pm m(x))}_{h(x)} \, J_\tau f(x) \, d\mathcal{H}^k(x)$$

we apply now (5) with h as above

$$= \int_{y\in\tilde E} \Big(\sum_{x\in f^{-1}(E)\cap y} h(x) \Big) \, d\mathcal{H}^k(y)$$

$$= \int_{y\in\tilde E} \langle \omega(y) ; \tilde\tau(y) \rangle \underbrace{\Big(\sum_{x\in f^{-1}(y)\cap E} \pm m(x) \Big)}_{\tilde m(y)} \, d\mathcal{H}^k(y)$$

$$= \langle [\tilde E, \tilde\tau, \tilde m] ; \omega \rangle$$

and since ω is arbitrary the proof is complete. $\square$

<u>Final Remarks</u> ◦ Looking at the formula for $f_\# T$ when $T = [E, \tau, m]$, we notice that it makes sense even if f is a Lipschitz map from E to $\tilde E := f(E)$, and indeed it is used to define $f_\# T$ in this case.
This definition makes sense because of the following stability property: given a sequence of maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$ of class C^1 such that

$f_n \to f$ uniformly on E, and $\mathrm{lip}(f_n) \le C < +\infty$, then $(f_n)_\# T \to f_\# T$.

- The estimate for $\mathbb{M}(f_\# T)$ can be slightly improved when $T = [E, \tau, m]$ is rectifiable:
$$\mathbb{M}(f_\# T) \le \int_E |d_\tau f(x)|^k \, |m(x)| \, d\mathcal{H}^k(x)$$
(the original estimate had $df(x)$ in place of $d_\tau f(x)$).

- Note that if T is rectifiable with integral multiplicity then so is $f_\# T$.
Indeed it is obvious that $\widehat{m}(y) \in \mathbb{Z}$ if $m(x) \in \mathbb{Z}$ ....

- Using the fact that the push-forward operator $f_\#$ takes currents with finite mass into currents with finite mass, and commutes with the boundary operator, we immediately obtain that $f_\#$ takes normal currents into normal currents.

- In the same spirit, it is easy to check that $f_\#$ takes integral currents into integral currents.

- We conclude with an example showing that for currents T with non-compact support there are problems to define $f_\# T$ if f

is not proper.

Let indeed T be the 0-current on $\mathbb{R}$ given by

$$T := \sum_{n=0}^{\infty} \delta_n$$

and let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := e^{-x}$.

Note that T is a well-defined current: indeed $\langle T; \phi \rangle = \sum_{n=0}^{\infty} \phi(n)$ is a finite sum if ϕ has compact support (that is $\phi(n)=0$ except for finitely many n).

Moreover f is smooth, but not proper.

Now, $f_{\#}T$ should be $\sum_{n=0}^{\infty} \delta_{e^{-n}}$, but this infinite sum does not define a 0-current (that is, it does not define a distribution on $\mathbb{R}$).

To be precise, one can prove that the distributions $U_N := \sum_{n=0}^{N} \delta_{e^{-n}}$ do not converge to any distribution on $\mathbb{R}$ as $N \to +\infty$ (just consider $\langle U_N, \phi \rangle$ where ϕ is a test function with $\phi(0) \neq 0$ ....)

Currents 13/14

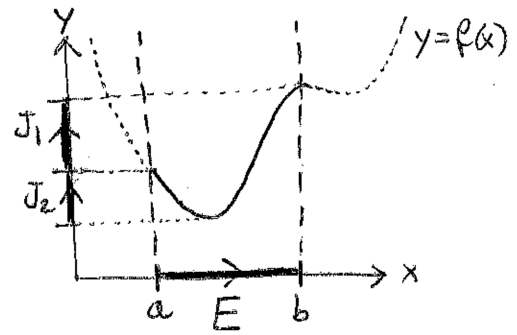
Lecture 13
13/5/14

This lecture and the next one are devoted to minor details

Examples of push-forward

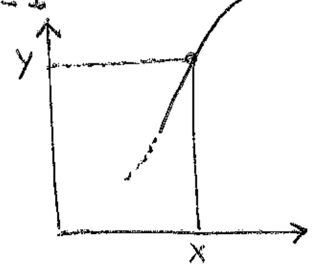
1| Let $f : \mathbb{R} \to \mathbb{R}$ and $T := [E, e, 1]$ the 1-current in $\mathbb{R}$ as in the figure

standard orientation of $\mathbb{R}$



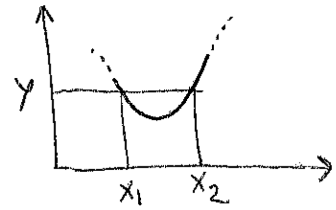
Then $f_\# T = [\tilde{E}, \tilde{\varepsilon}, \tilde{m}]$ where

- $\tilde{E} = f(E) = J_1 \cup J_2$;
- $\tilde{\varepsilon}$ we choose to be e, the standard orientation of $\mathbb{R}$;
- $\tilde{m}$ has to be computed according to the formula seen in the last lecture.
  For $y \in J_1$; $f^{-1}(y) \cap E$ is only one point x, and since $f'(x) > 0$, $df(x)$ preserves the orientation (as a linear map from $\mathbb{R}$ to $\mathbb{R}$) thus $\tilde{m}(y) = 1$



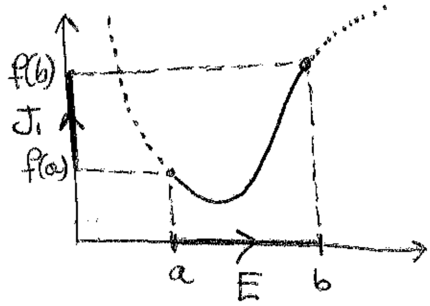
On the other hand, for $y \in J_2$, $f^{-1}(y) \cap E$ consists of two points, $x_1 < x_2$, and since $f'(x_1) < 0 < f'(x_2)$, we have that $df(x_1)$ reverses the orientation, while

$df(x_2)$ preserves it. So $\tilde{m}(y)=0$.



Therefore $f_\# T = [\tilde{E}, e, \tilde{m}] = [J_1, e, 1]$.

Let us now check the boundary. We have seen that $\partial T = \delta_b - \delta_a$, then $f_\#(\partial T) = \delta_{f(b)} - \delta_{f(a)}$ (note that ∂T is a 0-current, and there is no orientation involved, in particular the computation of the multiplicity of $f_\#(\partial T)$ is trivial....). As expected, $f_\#(\partial T) = \partial(f_\# T)$

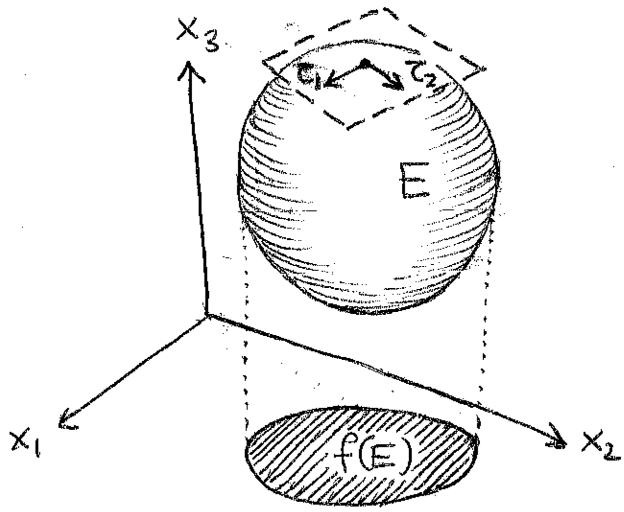


<u>Minor issues to settle</u>

A 0-current T with finite mass in $\mathbb{R}^n$ is a measure with values in $\Lambda_0(\mathbb{R}^n) = \mathbb{R}$, that is, a real-valued measure. Show that $f_\# T$ is just the push-forward of T as a measure.

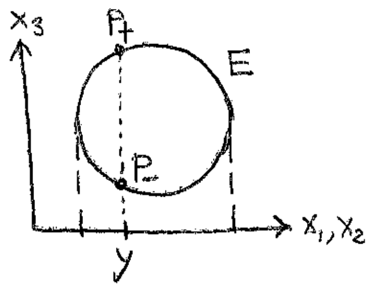
Note that a 0-dimensional vector space (that is $\{0\}$) does not admit any orientation. What then should be the definition of rectifiable 0-current?

2] Let T be the 2-current in $\mathbb{R}^3$ given by $T := [E, \tau, 1]$ where E is a sphere, endowed with the continuous orientation τ as in the figure, and let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $f(x_1, x_2, x_3) := (x_1, x_2)$.



Thus $f_\# T = [\tilde{E}, \tilde{\tau}, \tilde{m}]$ where

- $\tilde{E} = f(E)$ is the disk in the figure;
- $\tilde{\tau}$ we choose to be the standard orientation of $\mathbb{R}^2$, $e_1 \wedge e_2$;
- $\tilde{m}$ has to be computed according to the usual formula: note that every point y in (the interior of) the disc is the projection of two different points on the sphere, p_+ and p_-; and that $df(p_+)$ preserves the orientation while $df(p_-)$ reverses it.



Thus $\tilde{m}(y) = 0$ and $f_\# T = 0$.

3| Take T as in the previous example, and let now f be any map from $\mathbb{R}^3$ to $\mathbb{R}^2$ (of class C^1).
I claim that $f_\# T = 0$ also in this case.
There are two ways to prove this claim.

First proof (Using degree theory).
We know that $f_\# T = [\tilde{E}, \tilde{\tau}, \tilde{m}]$ where $\tilde{E} := f(E)$, $\tilde{\tau} = e$ (the standard orientation of $\mathbb{R}^2$) and $\tilde{m}$ is given by the formula

$$\tilde{m}(y) = \sum_{x \in f^{-1}(y) \cap E} \pm 1 \longleftarrow \left| \begin{array}{l} + \text{ if } df(x) : \mathrm{Tan}(E,x) \to \mathbb{R}^2 \\ \text{is orientation preserving,} \\ - \text{ otherwise.} \\ \text{Recall that } m = 1 \ldots \end{array} \right.$$

but this is exactly the definition of Brower degree of f at the point y (considering f as a map from the sphere E to $\mathbb{R}^2$), and we know from the theory that this degree is constant in y, and must be 0 because the map f from E to $\mathbb{R}^2$ is not surjective ($f(E)$ is compact). Thus $\tilde{m}(y) = 0$. $\square$

Second proof (Using the constancy lemma).
We know that $f_\# T$ is a normal 2-current in $\mathbb{R}^2$ without boundary (because $\partial T = 0$) and therefore the constancy lemma yields $f_\# T = [\mathbb{R}^2, e, \tilde{m}]$ with $\tilde{m}$ constant. But then the only possibility is $\tilde{m} = 0$, because $f_\# T$ has finite mass. $\square$

The last example suggests that there should be a connection between degree theory and the theory of currents...

Links to degree theory

Let $M, \tilde{M}$ be d-dimensional oriented manifolds of class $\mathscr{C}^1$ and let $f : M \to \tilde{M}$ be a map of class $\mathscr{C}^1$.

Assume moreover that M is compact and $\partial M = \emptyset$.

Let $T := [M, \tau_M, 1]$ and consider $f_\# T$.

orientation of M

Then $f_\# T$ is a d-current in $\tilde{M}$ with $\partial(f_\# T) = f_\#(\partial T) = 0$.

Now, the constancy lemma states that $f_\# T = [\tilde{M}, \tau_{\tilde{M}}, m]$ with m a constant.

On the other hand the formula for the push-forward of rectifiable currents yields that $f_\# T = [\tilde{M}, \tau_{\tilde{M}}, \tilde{m}]$ where $\tilde{m}(y)$ is the degree of f at y for $\mathscr{H}^d$-a.e. $y \in \tilde{M}$.

Putting together these results we obtain that $\tilde{m}(y)$ agrees (a.e.) with a constant, that is, the degree does not depend on the point. (This is one of the basic results of degree theory; another fundamental result states that if f is homotopic to another map $g : M \to \tilde{M}$ then f and g have the same degree; this result can also be obtained via the theory of currents, but we will need the homothopy formula.)

1-current associated to a Lipschitz path

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a Lipschitz map (path).

Then we associate to γ the 1-current T_γ in $\mathbb{R}^n$ given by

standard orient. of $\mathbb{R}$.

$$T_\gamma := \gamma_\# ([\![a,b]\!], e, 1])$$

Being the push-forward of an integral current, T_γ is integral, too.

In this case the formula that defines the push-forward reduces to the classical formula that defines the integration of a 1-form on a Lipschitz path:

$$\langle T_\gamma ; \omega \rangle = \int_a^b \langle \omega(\gamma(t)) ; \dot\gamma(t) \rangle \, dt ,$$

which in turn yields

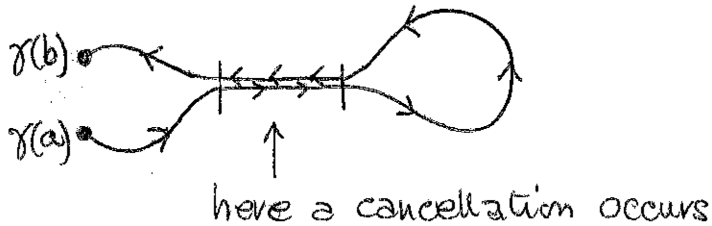
$$\mathbb{M}(T_\gamma) \leq \int_a^b |\dot\gamma(t)| \, dt = \text{Length of } \gamma .$$

Moreover $\partial T_\gamma = \delta_{\gamma(b)} - \delta_{\gamma(a)}$.

Recall that given any increasing bijection $\sigma : [\tilde a, \tilde b] \to [a,b]$, the reparametrized path $\tilde\gamma : [\tilde a, \tilde b] \to \mathbb{R}^n$ given by $\tilde\gamma := \gamma \circ \sigma$ defines the same current, that is, $T_\gamma = T_{\tilde\gamma}$.

In particular we can always choose σ so that $[\tilde a, \tilde b] = [0,1]$ and $|\dot{\tilde\gamma}| =$ constant a.e. (or, alternatively, $|\dot{\tilde\gamma}| = 1$ a.e.).

Note that the inequality in the estimate of $\mathbb{M}(T_\gamma)$ above may be strict: this happens for example when γ "goes through" the same arc in $\mathbb{R}^n$ more than once, but not with the same orientation



Consider now a sequence of paths $\gamma_i : I_i \to \mathbb{R}^n$ such that:

(i) $\sum_i \text{length}(\gamma_i) < +\infty$

(ii) γ_i is closed for every i.

Then one can prove (it's an exercise) that

$$T := \sum_{i=1}^{\infty} T_{\gamma_i}$$

is a well-defined integral current without boundary.

Now we have the following compactness result

<u>Proposition 1</u>: Let $T_n = \sum_i T_{\gamma_{n,i}}$ be a sequence of integral currents as above, and assume that

$$(*) \qquad \sum_i \mathrm{length}(\gamma_{n,i}) \leq L < +\infty \quad \forall n.$$

Then T_n converge, up to subsequence, to a current T_∞ of the same type (that is $T_\infty = \sum_i T_{\gamma_{\infty,i}}$).

This statement is related to the compactness result by Federer and Fleming (according to which T_n must converge to an integral current) and the proof reveals some of the intricacies of the proof of F&F theorem.

<u>Sketch of proof</u>

Let $\ell_{n,i} := \mathrm{length}(\gamma_{n,i})$. Since $\sum_i \ell_{n,i} < +\infty$, we can rearrange w.r.t. the i index so that $\ell_{n,i}$ is decreasing in i.

<u>Step 1</u> For every i, $T_{\gamma_{n,i}}$ converge up to subseq. to $T_{\gamma_{\infty,i}}$ for a suitable $\gamma_{\infty,i}$. Indeed we can assume that all $\gamma_{n,i}$ are defined on $[0,1]$ and have constant speed $|\dot\gamma_{n,i}|$. Then the bound $(*)$ yields $L \geq \ell_{n,i} = \|\dot\gamma_{n,i}\|_\infty = \mathrm{Lip}(\gamma_{n,i})$, and then $\gamma_{n,i}$ converge up to subsequence to some Lipschitz path $\gamma_{\infty,i} : [0,1] \to \mathbb{R}^n$. It is then easy to

check that $T_{\gamma_{n,i}} \to T_{\gamma_{\infty,i}}$.

Step 2 We extract a subseq. so that for every i there holds $T_{\gamma_{n,i}} \to T_{\gamma_{\infty,i}}$. It remains to show that T_n converge to

$$T_\infty := \sum_{i=1}^{\infty} T_{\gamma_{\infty,i}} .$$

We first prove this convergence result under the additional assumption that

(1) for every $\varepsilon>0$ there exists i_ε such that $\sum_{i\geq i_\varepsilon} e_{n,i} \leq \varepsilon$ for every n.

Step 3 If (1) does not hold then we can find a sequence $i_n \to +\infty$ such that the truncated sequences

$$\tilde{e}_{n,i} := \begin{cases} e_{n,i} & \text{if } i\leq i_n \\ 0 & \text{if } i>i_n \end{cases}$$

satisfy (1). Then by Step 2 the currents

$$\tilde{T}_n := \sum_{i\leq i_n} T_{\gamma_{n,i}}$$

converge to T_∞.

Step 4 It remains to show that the currents

$$\hat{T}_n := \sum_{i>i_n} T_{\gamma_{n,i}}$$

converge to 0. We recall that all $\gamma_{n,i}$ are closed and that since $i_n \to \infty$ then

$$\Big(\sup_{i>i_n} e_{n,i}\Big) \xrightarrow[n\to+\infty]{} 0 .$$

Note that if either of these properties were not satisfied, the claim would not hold! $\square$

<u>Remarks</u> • The assumption that the currents T_n in Proposition 1 can be represented as $T_n = \sum_i T_{\gamma_{n,i}}$ with $\gamma_{n,i}$ <u>closed</u> is essential.
Construct an example of sequence T_n of this form in $\mathbb{R}^2$, that satisfy the bound (*) in Proposition 1 but not the assumption that all $\gamma_{n,i}$ are closed, and such that T_n converge to a current with finite mass $T_\infty = \tau\mu$ where μ is the Lebesgue measure on a square.

• The only point in the proof of Proposition 1 where we use the assumption that all $\gamma_{n,i}$ are closed is Step 4.

• Proposition 1 has an interesting consequence: let X be the class of all integral 1-current T in $\mathbb{R}^n$ such that $\partial T = 0$, and let Y be the subclass of all T that can be represented as $T = \sum_i T_{\gamma_i}$ as above.
Now, we will prove that every T in X is the limit of a sequence of polyhedral integral 1-currents T_n such that $\partial T_n = 0$ and $\mathbb{M}(T_n) \to \mathbb{M}(T)$ (this is a variant of an approximation result for integral currents stated earlier in this course and yet to be proved).
Notice now that each polyhedral current T_n belongs to Y, and more precisely in can be represented as

$$T_n = \sum_i T_{\gamma_{n,i}} \quad \text{in such a way that}$$

finite sum!

$$\mathbb{M}(T_n) = \sum_i \mathbb{M}(T_{\gamma_{n,i}}) = \sum_i \text{length}(\gamma_{n,i})$$

(roughly speaking, we choose the representation so that there are no cancellations....)

But now Proposition 1 implies that the limit of such T_n must belong to Y. We have thus obtained that $Y = X$, that is, every integral 1-current T with $\partial T = 0$ can be represented as $T = \sum_{i=1}^{\infty} T_{\gamma_i}$. Additionally we also obtain that such representation satisfies

$$\mathbb{M}(T) = \sum_i \mathbb{M}(T_{\gamma_i}) = \sum_i \text{length}(\gamma_i).$$

- One can easily generalize Proposition 1 by replacing the assumption that <u>all</u> $\gamma_{n,i}$ are closed with: all $\gamma_{n,i}$ are closed for (at most) N indexes i, where N does not depend on n.

Starting from this result one obtains that every integral 1-current T on $\mathbb{R}^n$ can be represented as $T = \sum_{i=1}^{\infty} T_{\gamma_i}$ where all γ_i are closed except N, and

$$\mathbb{M}(T) = \sum_i \mathbb{M}(T_{\gamma_i}) = \sum_i \text{length}(\gamma_i);$$

$$\mathbb{M}(\partial T) = 2N.$$

Note that this representation result has no counterpart for integral k-current with $k > 1$.

Currents 13/14 — Lecture 14 14/5/14

We begin this lecture with the discussion of some minor points (as in the previous lecture).
The last result in this lecture, however, is relevant for the rest of the course.

About the support of a current

I recall that the support of a distribution Λ on $\mathbb{R}^n$, $\mathrm{supp}(\Lambda)$, is by definition the complement of the largest open set A such that the restriction of Λ to A is null, that is, $\langle \Lambda; \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $A \supset \mathrm{supp}(\varphi) := \overline{\{x : \varphi(x) \neq 0\}}$. Thus $\mathrm{supp}(\Lambda)$ is closed.
Similarly, the support $\mathrm{supp}(T)$ of a k-current $T \in \mathcal{D}_k(\mathbb{R}^n)$ is the complement of the largest open set A such that the restriction of T to A is null, that is, $\langle T; \omega \rangle$ for all $\omega \in \mathcal{D}^k(\mathbb{R}^n)$ with $A \supset \mathrm{supp}(\omega)$.

Consider now the measure (or 0-current) μ in $\mathbb{R}$ given by

$$\mu := \sum_{n=0}^{\infty} 2^{-n} \delta_{q_n}$$

where $\{q_n\}$ is the set of all rational numbers.
According to the definition above $\mathrm{supp}(\mu) = \mathbb{R}$.
However we convey more information by saying that "μ is supported on the set of rational numbers".

More generally, we say that a measure μ on $\mathbb{R}^n$ is supported on the Borel set E if $\mu(\mathbb{R}^n \setminus E) = 0$
The example above show that E does not necessarily contain the support of μ.

Similarly, given a current with finite mass $T = \tau\mu$, we say that T is supported on the Borel set E if μ does (here we might want to assume that $\tau \neq 0$ μ-a.e.).

Support of the pushforward of a current

It is immediate to prove that given a map $f : \mathbb{R}^m \to \mathbb{R}^n$ and a k-form ω on $\mathbb{R}^n$, then

$$\mathrm{supp}(f^{\#}\omega) \subset f^{-1}(\mathrm{supp}(\omega))$$

Accordingly, we obtain that given a k-current T on $\mathbb{R}^m$

$$\mathrm{supp}(f_{\#}T) \subset f(\mathrm{supp}(T))$$

(here we assume either that f is proper or that T has compact support; in former case $f(\mathrm{supp}(T))$ is closed, in the latter it is compact).
(Check the details ...)

One can actually prove more: if $T = \tau\mu$ is a current with finite mass supported on the Borel set E, then $f_{\#}T$ is supported on $f(E)$.
One way of proving this result is to show that $f_{\#}T$, viewed as a measure, is absolutely continuous w.r.t. the push-forward of μ, that is, $f_{\#}T$ can be written

as $f_\# T = \tilde{\tau}\tilde{\mu}$ where $\tilde{\mu} = f_\#\mu$, that is, $\tilde{\mu}(E) = \mu(f^{-1}(E))$ for every Borel set E.
Note that there is a measurability issue around here, since $f(E)$ is not necessarily Borel...

Multiplication of a current by a function

Let T be a k-current in $\mathbb{R}^n$ and $\rho : \mathbb{R}^n \to \mathbb{R}$ a smooth function. Then we denote by ρT the current defined by

$$\langle \rho T ; \omega \rangle := \langle T ; \rho\omega \rangle \quad \forall \omega \in \mathcal{D}^k(\mathbb{R}^n).$$

If T is a current with finite mass, that is, $T = \tau\mu$, and ρ is bounded on the support of T, then ρT is the current with finite mass given by

$$\rho T = \rho \tau \mu$$

(and this identity justifies the notation ρT....)

Note moreover that the last formula makes sense even if ρ is continuous (and actually even Borel) and

$$\mathbb{M}(\rho T) = \int |\rho| \|\tau\| \, d\mu \leq \sup_{x \in \operatorname{supp}(T)} |\rho(x)| \cdot \mathbb{M}(T).$$

Proposition 2

If T is a normal k-current and ρ is a C^1 function such that ρ and $d\rho$ are bounded on $\operatorname{supp}(T)$ then ρT is a normal current.

Moreover

$$\mathbb{M}(\partial(\rho T)) \le \sup_{x\in \mathrm{supp}(T)} |\rho(x)| \cdot \mathbb{M}(\partial T) \qquad (*)$$
$$\qquad + \sup_{x\in \mathrm{supp}(T)} |d\rho(x)| \cdot \mathbb{M}(T).$$

<u>Proof</u> We already know that ρT has finite mass, and then it remains to show that also $\partial(\rho T)$ has finite mass and that $(*)$ holds.

Consider then $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^n)$:

$$\begin{aligned}\langle \partial(\rho T) ; \omega\rangle &= \langle \rho T ; d\omega\rangle \\ &= \langle T ; \rho\, d\omega\rangle \\ &= \langle T ; d(\rho\omega) - d\rho\wedge\omega\rangle \\ &= \langle \partial T ; \rho\omega\rangle - \langle T ; d\rho\wedge\omega\rangle .\end{aligned}$$

Here we use that $d(\rho\omega) = d\rho\wedge\omega + \rho d\omega$

Hence

$$\langle \partial(\rho T) ; \omega\rangle \le \sup_{\mathrm{supp}(T)} |\rho| \cdot \sup \|\omega\| \cdot \mathbb{M}(\partial T)$$
$$\qquad + \sup_{\mathrm{supp}(T)} |d\rho| \cdot \sup \|\omega\| \cdot \mathbb{M}(T)$$

and taking the supremum over all ω s.t. $\|\omega\| \le 1$ everywhere we get $(*)$, which implies in particular that $\partial(\rho T)$ has finite mass. $\square$

<u>Remark</u> The proof above suggests that there should be an explicit formula for $\partial(\rho T)$. More precisely, if $T = \tau\mu$ and $\partial T = \tau'\mu'$, then we expect that

$$\partial(\rho T) = \rho\tau'\mu' + (d\rho \circ \tau)\mu$$

except that it is not clear what the product $\circ$ should be. We leave it as an exercise to find the correct definition of $\alpha \circ v \in \Lambda_{k-1}(\mathbb{R}^n)$ for every $\alpha \in \Lambda^1(\mathbb{R}^n)$ and $v \in \Lambda_k(\mathbb{R}^n)$ and prove the formula above.

We can now state and prove a result the we already mentioned before.

Theorem 3

Let $T \neq 0$ be a normal k-current on $\mathbb{R}^n$. Then $\dim_{\mathcal{H}}(\mathrm{supp}(T)) \geq k$. In fact, we can prove more: if $T = \tau\mu$ with $\tau \neq 0$ μ-a.e. and E is a Borel set such that $\mu(E) > 0$, then at least one of the projections of E on the coordinate k-planes has positive ($\mathcal{H}^k$) measure. In particular $\mathcal{H}^k(E) > 0$, and therefore μ is absolutely continuous w.r.t. $\mathcal{H}^k$. (Note that the second part of this statement implies the first.)

Proof

We first explain the idea by proving the first part of the statement.

We choose a cut-off function ρ with compact support so that $\rho T \neq 0$. Let now $f : \mathbb{R}^n \to \mathbb{R}^k$ be a map of class C^1; since ρT has compact support $f_\#(\rho T)$ is well-defined, and since ρT is normal, so is $f_\#(\rho T)$. Now a lemma proved in the previous lectures yields

$$f_\#(\rho T) = [\mathbb{R}^k, \underset{\text{standar orientation in } \mathbb{R}^k}{e}, m]$$

where m is a suitable BV function. Now, if $f_\#(\rho T) \neq 0$ then

$$k = \dim_H(\mathrm{supp}(f_\#(\rho T)))$$

because $f_\#(\rho T)$ is absolutely continuous w.r.t. $\mathcal{L}^k$

$$= \dim_H(f(\mathrm{supp}(T)))$$

because $f(\mathrm{supp}(T)) \supset f(\mathrm{supp}(\rho T)) \supset \mathrm{supp}(f_\#(\rho T))$

$$\leq \dim_H(\mathrm{supp}(T))$$

because lipschitz maps (such as f) do not increase Hausdorff dimension

It remains to show that we can choose ρ and f so that $f_\#(\rho T) \neq 0$. Note that we cannot take $\rho := 1$ even if T has compact support, because f may not exists (for instance if T is a compact surface without boundary ....)

The idea is to take ρ so that $\tau(x)$ is "close" to some given $\tau_0 \in \Lambda^k(\mathbb{R}^n)$ for "most" $x \in \mathrm{supp}(\rho)$, then choose $\underline{i} \in I_{n,k}$ so that $(\tau_0)_{\underline{i}} \neq 0$, and take as f the projection of $\mathbb{R}^n$ on the $\underline{i}$-th coordinate plane, that is, $f(x) = (x_{i_1}, \dots, x_{i_k})$.

We work out the detaies of this last step in the proof of the second part of the statement (which is completely independent from the proof above).

Given E such that $\mu(E) > 0$, we choose $x_0 \in E$ such that

E has density 1 at x_0 and τ is approximately continuous at x_0. Since $\tau(x_0)\neq 0$, there exists $\underline{i}\in I_{n,k}$ such that the coordinate $\tau_{\underline{i}}(x_0)$ is $\neq 0$.

Then for every $r>0$ we can choose $\rho:\mathbb{R}^n\to[0,1]$ smooth with $\mathrm{supp}(\rho)\subset\overline{B(x_0,r)}$ so that, as $r\to 0$,

(1) $$\int_{\mathbb{R}^n}\tau_{\underline{i}}\,\rho\,d\mu \sim \int_{B(x_0,r)}\tau_{\underline{i}}\,d\mu \sim \tau_{\underline{i}}(x_0)\,\mu(B(x_0,r))$$

this follows from the fact that $\tau_{\underline{i}}$ is approx. continuous at x_0.

As before, let $f(x):=(x_{i_1},\dots,x_{i_k})$.

We prove that $\mathscr{L}^k(f(E))>0$.

Assume by contradiction that $\mathscr{L}^k(f(E))=0$.

Then, recalling that $f_\#(\rho T)=[\mathbb{R}^k,e,m]$ is a.c. w.r.t $\mathscr{L}^k$, we have that $f_\#(\rho T)=f_\#(\rho T)\cdot(1-1_{f(E)})=f_\#((1-1_{\tilde{E}})\rho T)$ where $\tilde{E}:=f^{-1}(f(E))$. Hence

$$\langle f_\#(\rho T);\underset{dy_1\wedge\dots\wedge dy_k}{\underset{\parallel}{dy}}\rangle = \langle (1-1_{\tilde{E}})\rho T;dx_{\underline{i}}\rangle$$

pull-back of dy according to f

$$= \int (1-1_{\tilde{E}})\,\rho\,\tau_{\underline{i}}\,d\mu$$

and then

$$|\langle f_\#(\rho T);dy\rangle| \le \int |1-1_{\tilde{E}}|\,\rho\,|\tau_{\underline{i}}|\,d\mu$$

(2) $$\le \int_{B(x_0,r)}|1-1_E|\,d\mu \ll \mu(B(x_0,r))$$

because E has density 1 at x_0

On the other hand

(3) $$\langle f_\#(\rho T);dy\rangle = \langle \rho T;dx_{\underline{i}}\rangle = \int \tau_{\underline{i}}\,\rho\,d\mu \sim \tau_{\underline{i}}(x_0)\,\mu(B(x_0,r)).$$

by (1)

But (2) and (3) are in contradiction. $\square$

If a current T on $\mathbb{R}^n$ is a boundary ($T=\partial U$) then $\partial T=0$ (because $\partial T=\partial^2 U$). We conclude this lecture by proving that the converse holds as well.

Theorem 4 (Cone construction)

Let T be a K-current in $\mathbb{R}^n$, $0<K<n$, with compact support and such that $\partial T = 0$.

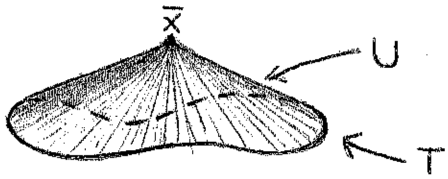
Then there exists a $(K+1)$-current U in $\mathbb{R}^n$ such that $T = \partial U$.

If in addition T has finite mass, then we can find U so that

$$(*) \qquad \mathbb{M}(U) \leq 2^{K+1} \operatorname{diam}(\operatorname{supp}(T)) \cdot \mathbb{M}(T) \quad .$$

Proof

The proof is based on the so-called "cone construction".
The geometric idea, in case T is a closed curve in $\mathbb{R}^3$, is to take as U a cone as in the figure:



What we have to do now is to turn this purely geometric construction about sets into a construction about currents

To do this we notice that the cone in the figure is the image of the cylinder $[0,1] \times T$ in $\mathbb{R} \times \mathbb{R}^3$ according to the map $f : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $f(t,x) = tx + (1-t)\bar{x}$, where $\bar{x}$ is the vertex of the cone.

But we know what the cylinder is in the context of currents (the product of T and a segment) and we know what the image of a current according to a map is (the push-forward).

More precisely, let I be the 1-current in $\mathbb{R}$ given by $I := [[0,1], e, 1]$, let $f : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be given by $f(t,x) := tx + (1-t)\bar{x}$, and let

$$U := f_\#(I\times T) .$$

First notice that $I\times T$ has compact support and f is smooth, and therefore U is well-defined.
Moreover

$$\begin{aligned}
\partial U &= \partial(f_\#(I\times T))\\
&= f_\#(\partial(I\times T))\\
&= f_\#(\partial I\times T - \overset{\text{by assumption}}{\overset{0}{\overset{\shortparallel}{\cancel{I\times\partial T}}}})\\
&= f_\#((\delta_1-\delta_0)\times T)\\
&= f_\#(\delta_1\times T) - f_\#(\delta_0\times T)
\end{aligned}$$

Now we claim that $f_\#(\delta_1\times T) = T$ because $f(1,x) = x$ and that $f_\#(\delta_0\times T) = 0$ because $f(0,x) = 0$, and this would conclude the proof of $\partial U = T$.

This claim is correct, but not as immediate as it may look at first glance.

Let us prove that $f_\#(\delta_1\times T) = T$. Given a k-form ω on $\mathbb{R}^n$, we write

$$f^\#\omega = \sum_{\underline{i}\in I_{n,k}} (f^\#\omega)_{\underline{i}}\, dx_{\underline{i}} + \overset{\substack{\text{sum of the components of } f^\#\omega \\ \text{that contains } dt \\ \downarrow}}{\tilde{\omega}}$$

$$e_i^x = (0, e_i) \in \mathbb{R}\times\mathbb{R}^n \quad \forall n$$

Then

$$(f^\# \omega)_{\underline{i}}(t,x) = \langle (f^\#\omega)(t,x)\,;\, e_{i_1}^x \wedge \dots \wedge e_{i_k}^x \rangle$$

$$= \langle \omega(f(t,x))\,;\, df(t,x)e_{i_1}^x \wedge \dots \wedge df(t,x)e_{i_k}^x \rangle$$

Now, the fact that $f(1,x) = x$ for every x implies that $df(1,x)\cdot e_i^x = e_i$ for every $i = 1,\dots,n$, and then

$$(f^\#\omega)_{\underline{i}}(1,x) = \langle \omega(x)\,;\, e_{i_1}\wedge\dots\wedge e_{i_k}\rangle = \omega_{\underline{i}}(x)$$

Thus

$$(f^\#\omega)(1,x) = \sum_{\underline{i}} \omega_{\underline{i}}(x)\, dx_{\underline{i}} + \tilde{\omega}(1,x)$$

Now, going back to the definition of product of currents we see that the action of $\delta_1 \times T$ on $\tilde{\omega}$ is zero, because $\tilde{\omega}$ is a sum of components that "contain dt". Hence

$$\langle f_\#(\delta_1\times T)\,;\,\omega\rangle = \langle \delta_1\times T\,;\, f^\#\omega\rangle$$

$$= \langle \delta_1 \times T\,;\, \sum_{\underline{i}} \omega_{\underline{i}}\, dx_{\underline{i}}\rangle$$

$$= \langle T\,;\, \sum_{\underline{i}} \omega_{\underline{i}}\, dx_{\underline{i}}\rangle = \langle T\,;\,\omega\rangle$$

and then $f_\#(\delta_1\times T) = T$.

In the same way one shows that $(f^\#\omega)_{\underline{i}}(0,x) = 0$ which implies that $\langle f_\#(\delta_0\times T)\,;\,\omega\rangle = 0$, that is $f_\#(\delta_0\times T) = 0$.

We now pass to the proof of estimate (*)

To this end, however we define U in a slightly different way.

More precisely we take $d>0$ (to be chosen later) and set
$$U = f_\#(I \times T)$$
where $I := [[0,d],e,1]$ and $f(t,x) := \frac{t}{d}x + \left(1-\frac{t}{d}\right)\bar{x}$.

The fact that $\partial U = T$ can be proved as above.

Moreover
$$M(U) \le \Big[\sup_{\substack{t\in[0,d]\\ x\in \mathrm{supp}(T)}} |df(t,x)|\Big]^{k+1} \cdot \underset{\substack{\uparrow \\ \text{mass of } I}}{d} \cdot M(T)$$

Now $df(t,x) = \frac{x-x_0}{d}\,dt + \frac{t}{d}\,dx$ and then
$$|df(t,x)| \le \frac{|x-x_0|}{d}|dt| + \frac{|t|}{d}|dx| \le \frac{|x-x_0|}{d} + 1 .$$

If we then take $x_0 \in \mathrm{supp}(T)$ and $d := \mathrm{diam}(\mathrm{supp}(T))$ we obtain that
$$\sup_{\dots} |df(t,x)| \le \frac{1}{d} \underbrace{\sup_{x\in \mathrm{supp}(T)} |x-x_0|}_{\substack{\wedge\!\backslash \\ \mathrm{diam}(\mathrm{supp}(T))}} + 1 \le 2$$
and the estimate above yields $(*)$.

$\square$

<u>Remarks on the cone construction</u>

- Note that is T is rectifiable then $I\times T$ is rectifiable and so is $U := f_\#(I\times T)$.

  If T is rectifiable with integral multiplicity then $I\times T$ is rectifiable with integral multiplicity and so is U. Thus U is also integral (recall that $\partial U = T$).

- The assumption $k < n$ is irrelevant, because the only n-current in $\mathbb{R}^n$ with no boundary and compact support is the trivial one (this follows from the constancy lemma).
- The assumption $k > 0$ is needed. Indeed the 0-current $T = \delta_0$ (the Dirac mass at 0) has compact support and no boundary (as all 0-current) but cannot be obtained as a boundary of any 1-current U with compact support (try and prove rigorously this claim). The proof above breaks down because $f_\#(\delta_0 \times T)$ is not 0 but the Dirac mass at $\bar{x}$.

  However, one can show (using the same proof as above with few modifications) that the theorem holds if T is a 0-current that satisfies the additional assumption $\langle T ; 1 \rangle = 0$

  (Recall that T is a distribution with compact support and therefore $\langle T ; \phi \rangle$ is extended to all smooth $\phi : \mathbb{R}^n \to \mathbb{R}$, including the constant function 1.)

Currents 13/14 — Lecture 15 — 15/5/14

In this lecture we define and describe the properties of a rather useful tool, the flat norm.
(Note that "flat" has no geometric connotation, but refers to musical notation ♭.)

Flat norm

For every k-current T in $\mathbb{R}^n$ we set

$$\mathbb{F}(T) := \inf\{\mathbb{M}(R)+\mathbb{M}(S) : T = R+\partial S\}$$

$\mathbb{F}$ is obviously a seminorm (at least if we allow norms and seminorm to take the value $+\infty$) but it is also a norm, that is, $\mathbb{F}(T)=0 \Rightarrow T=0$.

Proof

We begin with a simple computation: let $T = R+\partial S$ and let $\omega \in \mathcal{D}^k(\mathbb{R}^n)$. Then

$$\langle T;\omega\rangle = \langle R;\omega\rangle + \langle \partial S;\omega\rangle$$

$$= \langle R;\omega\rangle + \langle S; d\omega\rangle$$

and then

$$|\langle T;\omega\rangle| \leq \mathbb{M}(R)\cdot\|\omega\|_\infty + \mathbb{M}(S)\cdot\|d\omega\|_\infty$$

$$\leq (\mathbb{M}(R)+\mathbb{M}(S))(\|\omega\|_\infty + \|d\omega\|_\infty)$$

and taking the infimum over all R, S st. $T = R+\partial S$

$$|\langle T;\omega\rangle| \leq \mathbb{F}(T)\,(\|\omega\|_\infty + \|d\omega\|_\infty).$$

Therefore, if $\mathbb{F}(T)=0$ then $\langle T;\omega\rangle=0\ \forall\omega$, which means $T=0$ $\square$

We prove now some other properties of the flat norm.

- $\mathbb{F}(T)\leq \mathbb{M}(T)$ (it suffices to take $R:=T$, $S:=0$).
- If $\mathbb{F}(T_n-T)\to 0$ then $\langle T_n;\omega\rangle\to\langle T;\omega\rangle$ for every $\omega\in\mathcal{D}^k(\mathbb{R}^n)$. In other words convergence in the flat norm implies convergence in the sense of currents.
  This claim follows from the estimate (proved above) $$|\langle T_n-T;\omega\rangle|\leq \mathbb{F}(T_n-T)\,(\|\omega\|_\infty+\|d\omega\|_\infty).$$

Since the convergence in the sense of currents is the weakest of all possible convergences, this result is not surprising. What is perhaps surprising is that, under certain assumptions, the flat norm metrizes the convergence in the sense of currents

<u>Proposition 1</u>

Let (T_n) be a sequence of currents such that for every n

$$\mathrm{supp}(T_n)\subset K \text{ bounded};\quad \mathbb{M}(T_n),\ \mathbb{M}(\partial T_n)\leq C<+\infty.$$

Then T_n converge to some T in the sense of currents if and only if T_n converge to T in the flat norm.

The proof of this result requires some advanced tools that we do not have yet, and has to be postponed. Anyhow, this result, even if very interesting, will not be used in the following.

We present now some examples.

<u>1</u>| There exists T such that $\mathbb{F}(T)=+\infty$.
In $\mathbb{R}$, a 0-current (distribution) T satisfies $\mathbb{F}(T)<+\infty$ only if $T=R+\partial S$ with $\mathbb{M}(R), \mathbb{M}(S)<+\infty$, that is, $T=\mu+D\lambda$ where μ and λ are (finite) real valued measures, and D is the distributional derivative. This means that T is a distribution of order 1 (for those who know what this means...).
Therefore any distribution T of higher order, such as $T:\phi\longrightarrow\phi''(0)$ (that is $T=D^2\delta_0$) must have $\mathbb{M}(T)=+\infty$.

Dirac masses at $x, \bar{x}\in\mathbb{R}^n$

<u>2</u>| If $T=\delta_x-\delta_{\bar{x}}$ then $\mathbb{F}(\delta_x-\delta_{\bar{x}})\leq\min\{2, |x-\bar{x}|\}$.
We get the bound $\mathbb{F}(\dots)\leq|x-\bar{x}|$ by taking $\delta_x-\delta_{\bar{x}}=R+\partial S$ with $R:=0$ and $S:=[I,\tau,1]$ where I is the segment joining x and $\bar{x}$, oriented from x to $\bar{x}$.
On the other hand $\mathbb{F}(\delta_x-\delta_{\bar{x}})\leq\mathbb{M}(\delta_x-\delta_{\bar{x}})=2$.

Thus the flat norm has a more geometric meaning than the mass.

<u>3</u>| There exists T such that $\mathbb{M}(T)=+\infty$ but $\mathbb{F}(T)<+\infty$.

Let indeed T be the 0-current in $\mathbb{R}$ given by

$$T := \sum_{n=0}^{\infty} \left(\delta_{2^{-2n}} - \delta_{2^{-2n-1}} \right)$$

It is indeed easy to show that $T = \partial S$ with

$$S := \sum_{n=0}^{\infty} [[2^{-2n-1}, 2^{-2n}], e, 1]$$

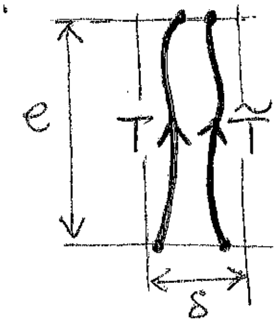
e ← standard orient. of $\mathbb{R}$

and $\mathbb{M}(S) = \sum_{n=0}^{\infty} 2^{-2n-1} = \frac{2}{3}$ and then $\mathbb{F}(T) \leq \frac{2}{3}$.

(The same proof shows that T is actually well-defined, which is not immediate.)

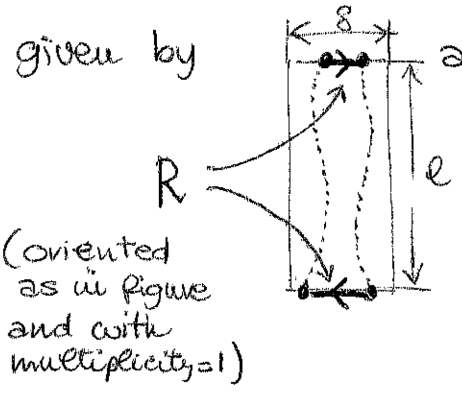
On the other hand one can prove that $\mathbb{M}(T) = +\infty$.

4] Consider the 1-currents T and $\tilde{T}$ associated to the oriented curves in the figure (and multiplicities equal to 1):

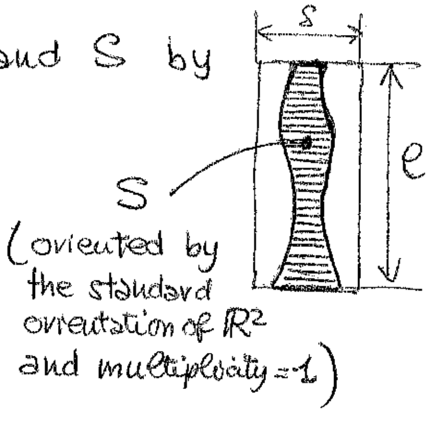


Then $\mathbb{F}(\tilde{T} - T) \leq (e+2)\delta$.

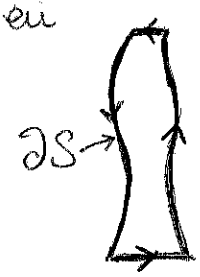
Indeed we write $\tilde{T} - T = R + \partial S$ where R is given by



and S by

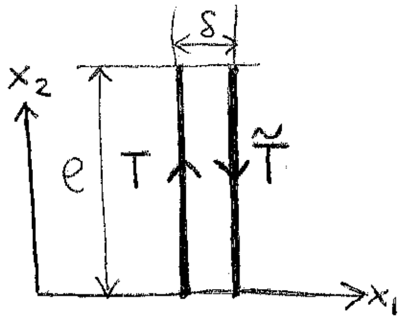


and then



Hence $\mathbb{F}(\tilde{T}-T) \leq \mathbb{M}(R)+\mathbb{M}(S) \leq 2\delta+\ell\delta=(2+\ell)\delta$.

5] Note however that if T and $\tilde{T}$ are given as follows



Then $\mathbb{F}(\tilde{T}-T)$ does not become small, no matter how small is δ.

So the flat distance is not just measure of the distance between the supports of T and $\tilde{T}$, but also the orientations matter.

6] In one of the previous lectures the following problem arose: Let (T_n) be a sequence of integral 1-currents in $\mathbb{R}^m$ of the form

$$T_n := \sum_{i=1}^{\infty} T_{\gamma_{n,i}}$$

where $\gamma_{n,i}$ are <u>closed</u> lipschitz paths satisfying the following conditions:

(i) $\sum_i \text{length}(\gamma_{n,i}) \leq C < +\infty \quad \forall n$;

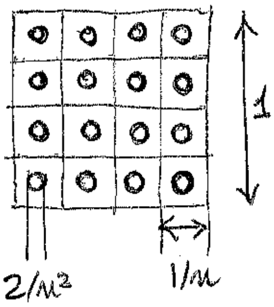
(ii) $\underbrace{\sup_i \text{length}(\gamma_{n,i})}_{= \ell_n} \longrightarrow 0$ as $n \to +\infty$.

Then prove that $T_n \to 0$.

Now from (i) we get that each T_n is well-defined and

$$\mathbb{M}(T_n) \leq \sum_i \mathbb{M}(T_{\gamma_{n,i}}) \leq \sum_i \text{length}(\gamma_{n,i}) \leq C$$

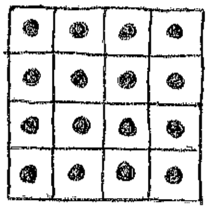
But this is clearly not enough to infer that $T_n \to 0$. A simple example is given in the figure

$T_n =$ 

that is $\{\gamma_{n,i}\}_i$ consists of n^2 circles of radius $1/n^2$ "uniformly distributed" in the square $Q = [0,1]^2 \subset \mathbb{R}^2$, all oriented counterclockwise, say.

Clearly $\quad \mathbb{M}(T_n) = n^2 \cdot \underbrace{2\pi \frac{1}{n^2}}_{\text{length of each circle}} = 2\pi .$

One can easily prove that $\mathbb{F}(T_n) \to 0$ (and then $T_n \to 0$) using the fact that $T_n = \partial S_n$ where S_n is the rectifiable two current $S_n = [A_n, e, 1]$ and
A_n is $\qquad$ (e: standard orient. in $\mathbb{R}^2$)



that is, the union of the discs enclosed by the previous circles. Then

$$\mathbb{F}(T_n) \le \mathbb{M}(S_n) = \mathcal{L}^2(A_n) = n^2 \, \pi \left(\frac{1}{n^2}\right)^2 = \frac{\pi}{n^2} \to 0 .$$

We can extend this proof to the general case: since each $\gamma_{n,i}$ is closed, $\partial T_{\gamma_{n,i}} = 0$, and therefore using the cone construction from previous lecture we can find a 2-current $S_{n,i}$ such that $\partial S_{n,i} = T_{\gamma_{n,i}}$ and

$$\begin{aligned} \mathbb{M}(S_{n,i}) &\le 4 \cdot \underbrace{\operatorname{diam}(\operatorname{supp}(T_{\gamma_{n,i}}))} \cdot \underbrace{\mathbb{M}(T_{\gamma_{n,i}})} \\ &\le 4 \cdot \quad \operatorname{length}(\gamma_{n,i}) \cdot \operatorname{length}(\gamma_{n,i}) \\ &\le 4 \cdot \varepsilon_n \cdot \operatorname{length}(\gamma_{n,i}) . \end{aligned}$$

Hence $T_n = \partial S_n$ where

$$S_n := \sum_i S_{n,i}$$

(if the sum is infinite this requires a check!) and then

$$\begin{aligned} \mathbb{F}(T_n) \le \mathbb{M}(S_n) &\le \sum_i \mathbb{M}(S_{n,i}) \\ &\le 4 \varepsilon_n \sum_i \operatorname{length}(\gamma_{n,i}) \\ &\le 4 C \varepsilon_n \longrightarrow 0 \end{aligned}$$

Summarizing, what you should keep in mind (because it will be often used) is that convergence in flat norm implies convergence in the sense of currents and the flat norm $\mathbb{F}(T)$ is often estimated from above quite precisely by exhibiting a suitable decomposition $T = R + \partial S$.

The rest of this lecture is devoted to minor points concerning the flat norm.

Variants of the flat norm

1| When dealing with integral currents (or rectifiable currents with integral multiplicity, it makes sense to consider the following variant of the flat distance (that is, the distance induced by the flat norm)

$$d_f(T_1, T_2) := \inf \{ \mathbb{M}(R) + \mathbb{M}(S) \}$$

where the infimum is taken over all decompositions $T_1 - T_2 = R + \partial S$ with R, S rectifiable currents with integral multiplicity.

Remarks • Note that d_f does not correspond to a norm, because of the constraint that R and S have integral multiplicity, which is not linear.

• Clearly $d_f(T_1, T_2) \le \mathbb{F}(T_1 - T_2)$; it is not known if d_f is equivalent to the usual flat distance, that is, if it exists some δ (depending on the dimensions n and k) such that $d_f(T_1, T_2) \geq \delta\, \mathbb{F}(T_1 - T_2)$.

• Note that in many textbooks (Krantz & Parks, Simon) only this flat distance is defined (with a different notation).

2| When dealing with currents T that are boundaries (which in $\mathbb{R}^n$ is essentially equivalent to say that $\partial T = 0$), it makes sense to define the following variant of the flat norm:

$$\widetilde{\mathbb{F}}(T) := \inf\{\mathbb{M}(S) : T = \partial S\}.$$

Remarks
- Within the class of currents that are boundaries, the norms $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ do not agree, but are equivalent. This is a consequence of a result known as "isoperimetric inequality" which we will prove later (not to be confused with the usual isoperimetric inequality).
- Of course one can combine variants 1 and 2 above, to fit the class of boundaries of integral currents...

Lower bounds for the flat norm

At the beginning of this lecture we proved that if $T = R + \partial S$ then

$$\langle T; \omega\rangle \leq \mathbb{M}(R)\cdot\|\omega\|_\infty + \mathbb{M}(S)\cdot\|d\omega\|_\infty ;$$

hence

$$\sup_{\substack{\|\omega\|_\infty \leq 1 \\ \|d\omega\|_\infty \leq 1}} \langle T; \omega\rangle \leq \mathbb{M}(R) + \mathbb{M}(S)$$

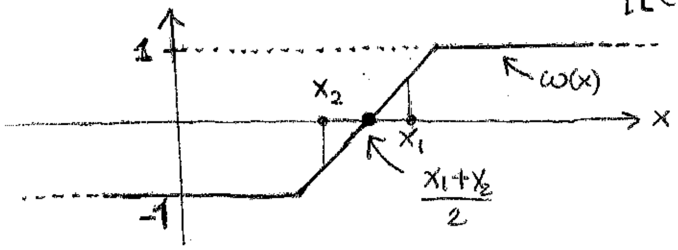
and taking the infimum over all decompositions $T = R + \partial S$

$$(*) \qquad \sup_{\substack{\|\omega\|_\infty \leq 1 \\ \|d\omega\|_\infty \leq 1}} \langle T; \omega\rangle \leq \mathbb{F}(T)$$

Similarly one can prove that if T is a boundary then

$$(**) \qquad \sup_{\|d\omega\|_\infty \leq 1} \langle T; \omega\rangle \leq \widetilde{\mathbb{F}}(T).$$

<u>Remarks</u> ∘ Inequality (*) can be used to prove lower bounds on the flat norm $\mathbb{F}$ (and similarly (**) for $\tilde{\mathbb{F}}$). For example, we proved that given two points $x_1, x_2 \in \mathbb{R}^n$ then $\mathbb{F}(\delta_{x_1} - \delta_{x_2}) \leq \min\{2; |x_1 - x_2|\}$, and one can show that the equality holds by considering the 0-form (function) $\omega(x) := \left[\left[\left(x - \frac{x_1+x_2}{2}\right) \cdot \underset{\frac{x_1-x_2}{|x_1-x_2|}}{\underset{\parallel}{e}}\right] \wedge 1\right] \vee (-1)$



Similarly one can use the form $\omega(x) := dx_2$ to prove that $\mathbb{F}(T - \tilde{T}) = 2\ell$ when T and $\tilde{T}$ are given in Example 5 at page 5 of these notes

∘ Concerning the examples we just discussed, one might argue that the forms ω that we used are not in the right space. Let be a bit more precise: if T has compact support then (*) holds by taking all forms ω of class $\mathcal{E}^\infty$ such that $\|\omega(x)\| \leq 1$ and $\|d\omega(x)\| \leq 1$ for every $x \in \operatorname{supp}(T)$ (and not all $x \in \mathbb{R}^n$), and if in addition T has finite mass we can take ω of class $\mathcal{C}^1$. The use of a form ω that is Lipschitz (as above in this page) can be easily justified by regularization....

<u>Proposition 2</u> (Characterization of the flat norm(s))

Both inequalities (*) and (**) are actually equalities.

Proof of (**) (the proof of (*) is similar, just a bit more complicated, and we omit it).

We want to prove that $L \geq \widetilde{\mathbb{F}}(T)$ where

$$L := \sup_{\|d\omega\|_\infty \leq 1} \langle T ; \omega \rangle$$

(we already know that $L \leq \widetilde{\mathbb{F}}(T)$).

Of course we can assume that L is finite.

Now, let X be the subspace of $\mathcal{C}_0(\mathbb{R}^n ; \Lambda^{k+1}(\mathbb{R}^n))$ of all α of the form

that is, the space of $(k+1)$-forms of class $\mathcal{C}_0$, endowed with the supremum norm

$$\alpha = d\omega$$

with $\omega \in \mathcal{D}^k(\mathbb{R}^n)$.

On X we define the linear functional S by

$$(1) \qquad \langle S ; \alpha \rangle := \langle T ; \omega \rangle \text{ with } \omega \text{ s.t. } \alpha = d\omega .$$

Note that S is well-defined: given ω and ω' s.t. $\alpha = d\omega = d\omega'$, then we must show that $\langle T ; \omega \rangle = \langle T ; \omega' \rangle$ and indeed, taking U s.t. $\partial U = T$ (recall that T is a boundary),

$$\begin{aligned}
\langle T ; \omega \rangle - \langle T ; \omega' \rangle &= \langle T ; \omega - \omega' \rangle \\
&= \langle \partial U ; \omega - \omega' \rangle \\
&= \langle U ; d\omega - d\omega' \rangle = 0
\end{aligned}$$

Notice moreover that the norm of S is

$$(2) \qquad \|S\| := \sup_{\substack{\alpha \in X \\ \|\alpha\|_\infty \leq 1}} \langle S ; \alpha \rangle = \sup_{\substack{\omega \in \mathcal{D}^k \\ \|d\omega\|_\infty \leq 1}} \langle T ; \omega \rangle := L .$$

Now we use Hahn-Banach theorem to extend S to a linear functional on the entire space $\mathcal{E}_0(\mathbb{R}^n; \Lambda^{k+1}(\mathbb{R}^n))$ keeping the norm equal to L.

But then S is (now!) a $(k+1)$-current, and (1) means that $\partial S = T$, while (2) means that $\mathbb{M}(S) = L$. Hence $\widetilde{\mathbb{F}}(T) \leq \mathbb{M}(S) = L$. $\square$

<u>Remark</u>

Let μ^+, μ^- be probability measures on $\mathbb{R}^n$. Then the 0-current $\mu^+ - \mu^-$ is a boundary, and therefore we can compute

$$\widetilde{\mathbb{F}}(\mu^+ - \mu^-) = \sup_{\|d\omega\|_\infty \leq 1} \langle \mu^+ - \mu^- ; \omega \rangle$$

$$= \sup_{\substack{\omega \text{ is} \\ 1\text{-Lipschitz}}} \left\{ \int_{\mathbb{R}^n} \omega \, d\mu^+ - \int_{\mathbb{R}^n} \omega \, d\mu^- \right\}$$

but the last expression is known as Wasserstein distance (with exponent 1) between μ^+ and μ^-.

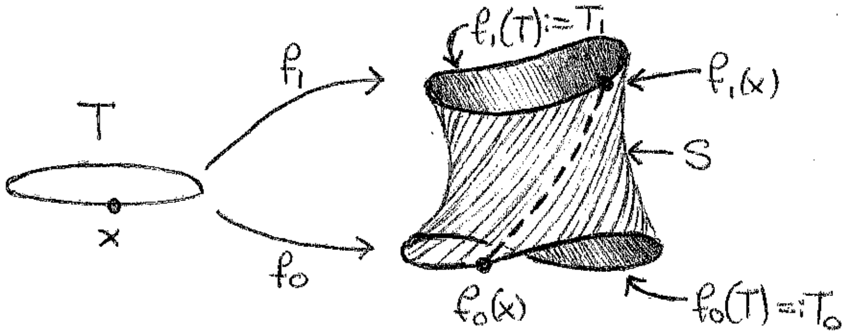
Let T be a k-current on $\mathbb{R}^n$ with compact support and $\partial T = 0$ and let $f_0, f_1 : \mathbb{R}^m \to \mathbb{R}^n$ be (smooth) maps that are <u>homotopic</u>, that is, there exists a (smooth) map $F : [a_0, a_1] \times \mathbb{R}^m \to \mathbb{R}^n$ s.t. $F(a_0, \cdot) = f_0(\cdot)$ and $F(a_1, \cdot) = f_1(\cdot)$.

We want to show that the currents

$$T_0 := (f_0)_\# T \quad \text{and} \quad T_1 := (f_1)_\# T$$

are cobordant, that is, $T_1 - T_0 = \partial S$ for some $(k+1)$-current S. (Moreover we want to use S to show that, if T has finite mass, the flat distance between T_1 and T_0 is small when f_1 and f_0 are close.)

If we think in term of sets (surfaces) rather than currents the natural candidate for S is $F([a_0,a_1] \times T)$:



the surface S is obtained by connecting $f_0(x) \in T_0$ and $f_1(x) \in T_1$ by the path $t \mapsto F(t,x)$ $\forall x$

We can easily replicate this construction in the framework of currents, and this results in the so-called homotopy formula.

<u>Homotopy formula (case $\partial T = 0$)</u>

Set $I := [\![[a_0, a_1], e, 1]\!]$ as usual, and let $S := F_\#(I \times T)$.

The proof of the next claims follows closely that of the cone construction (indeed the cone construction is but a particular case of the homotopy formula), and we omit the details.

homotopy formula

<u>Claim 1</u> $\boxed{T_1 - T_0 = \partial S}$ Indeed, since $\partial T = 0$, we have

$$\begin{aligned}
\partial S = \partial(F_\#(I\times T)) &= F_\#(\partial(I\times T)) \\
&= F_\#(\partial I \times T - \cancel{I \times \overset{0}{\overset{\|}{\partial T}}}) \\
&= F_\#(\delta_{a_1}\times T) - F_\#(\delta_{a_0}\times T) \\
&= (f_1)_\# T - (f_0)_\# T = T_1 - T_0 .
\end{aligned}$$

<u>Claim 2</u> If T has finite mass we can take f_0, f_1, F of class $\mathscr{C}^1$, and then, writing $T = \tau\mu$ with $\|\tau\| = 1$ μ-a.e.,

$$\begin{aligned}
\mathbb{F}(T_1 - T_2) \le \mathbb{M}(S) &\le \int_{a_0}^{a_1} \int_{\mathbb{R}^m} |dF(t,x)|^{k+1} \, d\mu(x)\, dt \\
&\le \Big[\sup_{\substack{a_0 \le t \le a_1 \\ x \in \operatorname{supp}(T)}} |dF(t,x)| \Big]^{k+1} (a_1 - a_0) \cdot \mathbb{M}(T)
\end{aligned}$$

<u>Claim 3</u> If T is rectifiable so is S, and if T is rectifiable with integral multiplicity (which means that T is integral, since $\partial T = 0$) then S is integral.

<u>Improving the mass estimate</u>

The estimate in Claim 2 above follows from the usual estimate for the mass of the push-forward, and can be improved thanks to the product structure of $I\times T$.

Let indeed ω be a k-form on $\mathbb{R}^n$. Then, writing $T = \tau\mu$ with $\|\tau\| = 1$ μ-a.e. (as above), we get

$$\langle S ; \omega \rangle = \langle I \times T ; F^\# \omega \rangle$$

computed at (t,x)

computed at x

Recall that $e \in \mathbb{R}$ is identified with $(e,0) \in \mathbb{R} \times \mathbb{R}^m$ and similarly $\tau \in \Lambda_k(\mathbb{R}^m)$ is identified with a $(k+1)$-vector in $\mathbb{R} \times \mathbb{R}^m$

$$= \int_{a_0}^{a_1} \int_{\mathbb{R}^m} \langle F^\# \omega ; e \wedge \tau \rangle \, d\mu(x) \, dt$$

computed at $(F(t,x))$

$$= \int_{a_0}^{a_1} \int_{\mathbb{R}^m} \langle \omega ; (dF)_\# (e \wedge \tau) \rangle \, d\mu(x) \, dt$$

computed at (t,x)

$$= \int_{a_0}^{a_1} \int_{\mathbb{R}^m} \langle \omega ; (d_t F)_\# e \wedge (d_x F)_\# \tau \rangle \, d\mu(x) \, dt$$

differentials of F w.r.t. the t and the x variables (computed at (t,x)).

Since $|e| = \|\tau\| = 1$, $|(d_t F)_\# e| \le |d_t F|$ and $\|(d_x F)_\# \tau\| \le |d_x F|^k$

$$\le \int_{a_0}^{a_1} \int_{\mathbb{R}^m} \|\omega\| \, |d_t F| \, |d_x F|^k \, d\mu(x) \, dt .$$

Taking the supremum over all ω with $\|\omega\|_\infty \le 1$ we get

$$\mathbb{F}(T_1 - T_0) \le \mathbb{M}(S) \le \int_{a_0}^{a_1} \int_{\mathbb{R}^m} |d_t F| \, |d_x F|^k \, d\mu(x) \, dt$$

$$(*) \qquad \le \sup_{\substack{a_0 \le t \le a_1 \\ x \in \mathrm{supp}(T)}} \left(|d_t F| \cdot |d_x F|^k \right) \cdot \delta \cdot \mathbb{M}(T)$$

<u>Remark</u> If F is a linear homotopy between f_0 and f_1, that is

$$F(t,x) := \frac{t}{\delta} f_1(x) + \left(1 - \frac{t}{\delta}\right) f_0(x) \quad \forall\, t \in [0,\delta],\ x \in \mathbb{R}^m$$

then

$$d_t F = \frac{f_1(x) - f_0(x)}{\delta} \, dt \ , \quad d_x F = \frac{t}{\delta} \, df_1(x) + \left(1 - \frac{t}{\delta}\right) df_2(x)$$

and therefore

$$|d_t F| \le \frac{1}{\delta}|f_1(x) - f_0(x)| \ , \ |d_x F| \le |df_0(x)| + |df_1(x)|$$

and estimate (*) becomes

$$\mathbb{F}(T_1 - T_0) \le \mathbb{M}(S) \le \int_{\mathbb{R}^m} |f_1(x) - f_0(x)| \left(|df_0(x)| + |df_1(x)| \right)^k d\mu(x)$$

(**) $$\le \|f_1 - f_0\|_\infty \int_{\mathbb{R}^m} \left(|df_0(x)| + |df_1(x)| \right)^k d\mu(x)$$

$$\le \|f_1 - f_0\|_\infty \cdot \sup_{x \in supp(T)} \left(|df_0(x)| + |df_1(x)| \right)^k \cdot \mathbb{M}(T).$$

(In the following we will need the estimate in the second line.)

<u>An application</u>

Let $M, \tilde{M}$ be k-dimensional, compact, oriented manifolds (or surfaces in some Euclidean space) and let $f_0, f_1 : M \to \tilde{M}$ be (smooth) maps with degree d_0 and d_1, respectively. Using the homotopy formula we can prove one of the fundamental results of degree theory, namely that $d_0 = d_1$ if f_0 and f_1 are homotopic.

Let indeed $T := [M, \underset{\text{orientation of}}{\tau_M}, 1]$.

We have seen in the previous lectures that for $i = 0, 1$ $T_i = (f_i)_\# T = [\tilde{M}, \tau_{\tilde{M}}, d_i]$. Moreover, if f_0 and f_1 are homotopic then T_0 and T_1 are cobordant by some $(k+1)$-current S in $\tilde{M}$ (recall that in this case F takes values in $\tilde{M}$) and since $\tilde{M}$ has dimension k, S must vanish. Hence

$$0 = \partial S = T_1 - T_0 = [\tilde{M}, \tau_{\tilde{M}}, d_1 - d_0] \Rightarrow d_1 = d_0 .$$

We conclude by general version of the homotopy formula, concerning the case $\partial T \neq 0$.

Homotopy formula (general case)

We take everything as in the previous case, except that we no longer assume $\partial T = 0$. Thus

$$\begin{aligned}
\partial F_\#(I\times T) &= F_\#(\partial(I\times T)) \\
&= F_\#(\partial I \times T - I\times \partial T) \\
&= F_\#(\delta_{a_1}\times T) - F_\#(\delta_{a_0}\times T) - F_\#(I\times\partial T) \\
&= \underbrace{(f_1)_\# T}_{T_1} - \underbrace{(f_0)_\# T}_{T_0} - F_\#(I\times\partial T).
\end{aligned}$$

Then the homotopy formula becomes

$$T_1 - T_0 = \partial \underbrace{F_\#(I\times T)}_{\overset{!!}{S}} + \underbrace{F_\#(I\times\partial T)}_{\overset{!!}{R}}$$

Moreover, if T is <u>normal</u> we can take f_0, f_1, F of class $\mathscr{C}^1$, and writing

$$T = \tau\mu \quad \text{with } \|\tau\| = 1 \ \mu\text{-a.e.}$$
$$\partial T = \tau'\mu' \quad \text{with } \|\tau'\| = 1 \ \mu'\text{-a.e.},$$

we get

$$(*) \quad \begin{aligned}
\mathbb{F}(T_1 - T_0) &\le \mathbb{M}(S) + \mathbb{M}(R) \\
&\le \int_{a_0}^{a_1}\int_{\mathbb{R}^m} |d_tF|\,|d_xF|^k \, d\mu(x)\,dt \\
&\quad + \int_{a_0}^{a_1}\int_{\mathbb{R}^m} |d_tF|\,|d_xF|^{k-1} \, d\mu'(x)\,dt \\
&\le \delta\left(L'\cdot \mathbb{M}(T) + L''\cdot \mathbb{M}(\partial T)\right)
\end{aligned}$$

where
$$L' := \sup_{\substack{a_0 \le t \le a_1 \\ x \in \operatorname{supp}(T)}} |d_t F|\,|d_x F|^k \,,$$
$$L'' := \sup_{\cdots\cdots} |d_t F|\,|d_x F|^{k-1} \,.$$

Finally, if F is a linear homotopy, that is,
$$F(t,x) = \frac{t}{\delta} f_1(x) + \left(1 - \frac{t}{\delta}\right) f_0(x) \quad \forall t \in [0,\delta],\, x \in \mathbb{R}^m$$
then estimate $(*)$ yields

$$
\begin{aligned}
\mathbb{F}(T_1 - T_0) &\le \mathbb{M}(S) + \mathbb{M}(R) \\
(**) \qquad &\le \|f_1 - f_0\|_\infty \Big(\int_{\mathbb{R}^n} (|df_0| + |df_1|)^k \, d\mu \\
&\qquad + \int_{\mathbb{R}^n} (|df_0| + |df_1|)^{k-1} \, d\mu' \Big) \\
&\le \|f_1 - f_0\|_\infty \left(L^k \, \mathbb{M}(T) + L^{k-1} \, \mathbb{M}(\partial T) \right)
\end{aligned}
$$

where
$$L := \sup_{x \in \operatorname{supp}(T)} \left(|df_0(x)| + |df_1(x)| \right) .$$

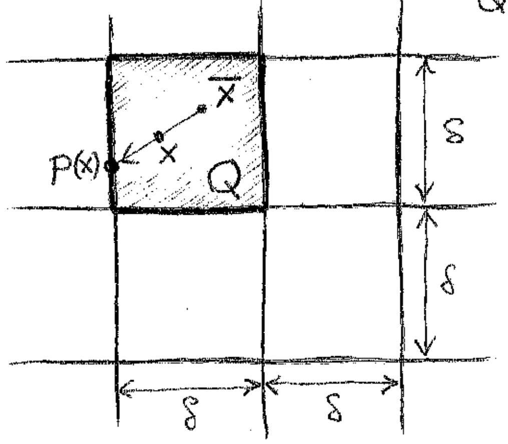
<u>Remark</u> Note that if T is integral than both R and S are integral.

Currents 13/14 — Lecture 17 — 20/5/14

This lecture is devoted to the Polyhedral Deformation Theorem. The purpose, given a k-current T in $\mathbb{R}^n$, is to find a polyhedral current P close to T.

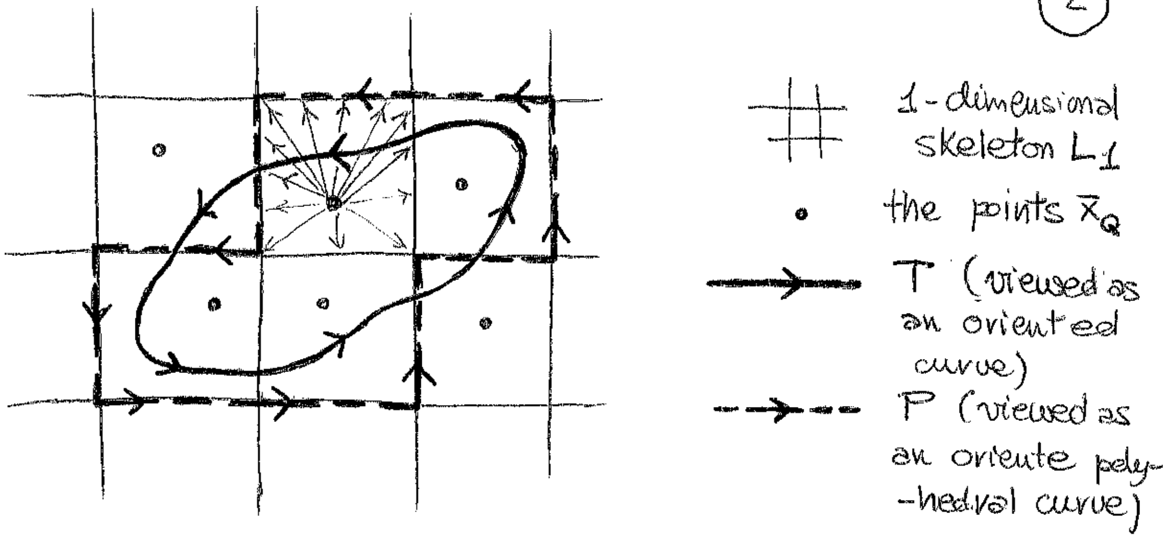
In this lecture we confine ourselves to the case $\partial T = 0$.

To explain the idea we consider the case $n=2, k=1$. We fix $\delta>0$ and choose a δ-grid as in the figure, then for every square Q in the grid we choose a point $\bar{x}=\bar{x}_Q$ in the interior and consider the "radial" retraction $p=p_Q$ of $Q\setminus\{\bar{x}\}$ on ∂Q



Now, let f be the map defined by $f=p_Q$ on every square Q on the grid. Thus f is a retraction of $\mathbb{R}^2$ (minus the points $\bar{x}_Q$...) onto the 1-dimensional skeleton L_1 of the grid.

Finally we take $P := f_\# T$.



Now P is 1-current with $\partial P = 0$ (because $\partial T = 0$ and $\partial P = \partial(f_\# T) = f_\#(\partial T) = 0$) supported on the one dimensional skeleton L_1.

Now the constancy lemma suggests that the restriction of P to every segment I in the grid must be of the form $[I, \tau_I, m]$ with τ_I is a given (constant) orientation of I, and m is a constant multiplicity. Were this correct, P would be a polyhedral current (up to small details).

Moreover, since $|f(x) - x| = O(\delta)$; the various estimates for the homotopy formula suggest that $\mathbb{F}(T-P) = \mathbb{F}(T - f_\# T) = O(\delta) \cdot \mathbb{M}(T)$.

There are, however, some difficulties to overcome.

1| How do we extend the previous construction to arbitrary k and n?

We take a δ-grid in $\mathbb{R}^n$, and construct a retraction f_{n-1} of $\mathbb{R}^n$ (minus a discrete family of points) onto the $(n-1)$-dimensional skeleton L_{n-1} of the grid as we did

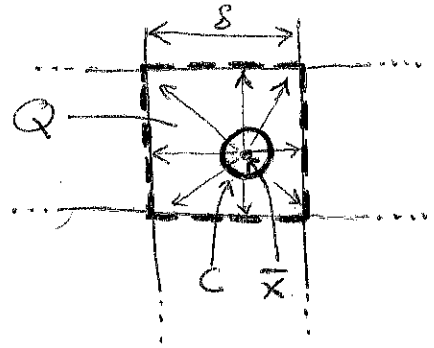
before. Then we let $T_{n-1} := (f_{n-1})_\# T$.
If $k = n-1$, then T_{n-1} is an $(n-1)$-dimensional current without boundary on L_{n-1}, and the argument used before suggests that T_{n-1} is polyhedral, and thus we would take $P := T_{n-1}$.
On the other hand, if $k < n-1$, there is no reason why T_{n-1} should be polyhedral. So we construct a retraction f_{n-2} of L_{n-1} (minus some points) onto L_{n-2} and set $T_{n-2} := (f_{n-2})_\# T_{n-1}$.
And we keep going until we get $T_k := (f_k)_\# T_{k+1}$.
Now T_k is a k-dimensional current without boundary supported on the k-dimensional skeleton L_k, and therefore it should be polyhedral. We would then take $P := T_k$....

Since we expect that $\mathbb{F}(T_{h+1} - T_h) = O(\delta)\,\mathbb{M}(T_{h+1})$ for $h = n-1, \dots, k$ then $\mathbb{F}(T-P) = O(\delta) \cdot \mathbb{M}(T)$.

2| Since the retraction p_Q is singular at the retraction point $\bar{x}_Q$, the map f is far from being of class C^1. Thus it is not clear how to define $f_\# T$.
Of course a definition can be given if we choose the points $\bar{x}_Q$ away from the support of T (but recall that in general $\mathrm{supp}(T)$ may be $\mathbb{R}^n$) but even in that case we have a problem with the estimates for the homotopy formula that should

yield the flat norm estimate $\mathbb{F}(T-P) = O(\delta)\cdot \mathbb{M}(T)$.
Indeed $|dp_Q(x)|$ $(= |df(x)|)$ tends to $+\infty$ as x tends to $\bar{x}_Q$, and therefore, even if $\bar{x}_Q \notin \mathrm{supp}(T)$ we do not have any (useful) bound for $\sup\limits_{x\in \mathrm{supp}(T)} |df(x)|$.

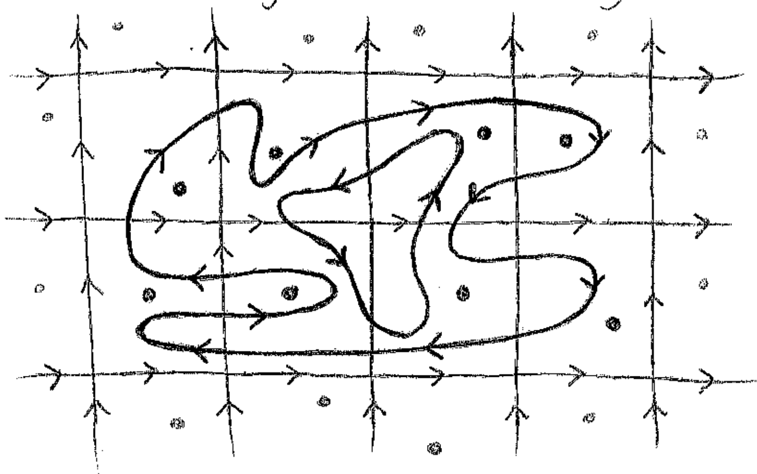
Consider indeed the following situation: $T = [C, \tau_C, m]$ where C is the circle in the picture with radius $\frac{1}{m}$, τ_C is the counter-clockwise orientation of C, and m is a positive number. If we choose as $\bar{x}$ the center of C (as in the picture) then $f_\# T = (p_Q)_\# T = [\partial Q, \tau_Q, m]$ where τ_Q is the counter-clockwise orientation of ∂Q

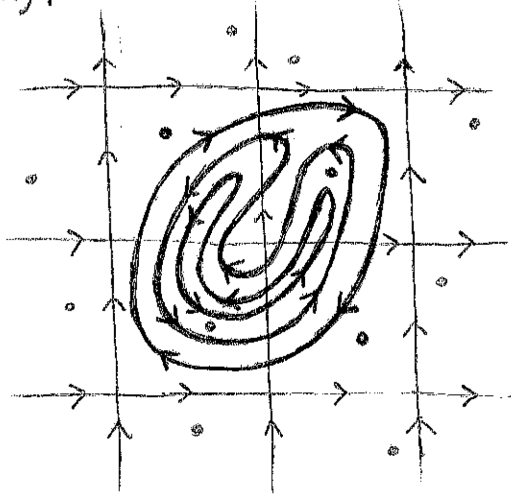


But now an easy computation shows that
$\mathbb{M}(T) = 2\pi$ while $\mathbb{M}(P) = 4m\delta$ and $\mathbb{F}(T-P) \approx m\delta$, and therefore no estimate of type $\mathbb{F}(T-P) = O(\delta)\,\mathbb{M}(T)$ may hold as $m \to +\infty$.
In this case is clear that this is simply due to the "wrong" choice of $\bar{x}$: had we taken $\bar{x}$ outside the circle C, everything would have been fine (compute $\mathbb{M}(P)$ and $\mathbb{M}(T-P)$ to be sure....)
We will actually show that we get the "right" estimate by carefully choosing the points $\bar{x}$.

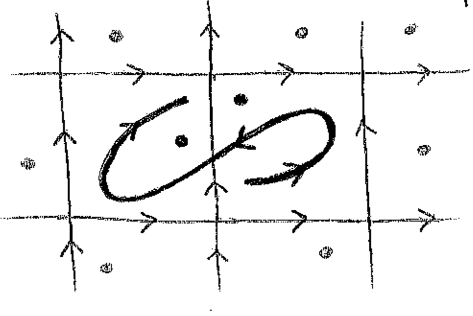
3] We sketched above an argument to prove that $P = f_{\#}T$ is indeed polyhedral. That argument, however, is far from complete, and quite some care is needed to make it work properly.

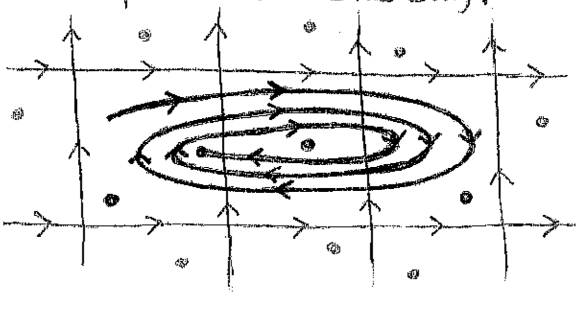
Exercise For each picture below it is given a rectifiable 1-current T (denoted by an oriented support, possibly disconnected, while the multiplicity is assumed to be one), a grid, and a family of projection points. Identify the polyhedral deform. P (defined as above) by computing the multiplicity of each segment in the 1-dimensional skeleton of the grid (each segment is already oriented).





Consider now some cases where $\partial T \neq 0$. Note that if $\partial T \neq 0$ then $P = f_{\#}T$ is not exactly polyhedral, because the multiplicity on each segmen is not necessarily constant (but in these examples it is piecewise constant).





We now begin with the real proof of the polyhedral deformation theorem. The next is (conceptually) the key lemma.

We fix a δ-grid in $\mathbb{R}^n$ with $\delta>0$, and k-current T in $\mathbb{R}^n$ with $0\le k<n$, $\mathbb{M}(T)<+\infty$, $\partial T=0$, $\mathrm{supp}(T)$ compact.

<u>Lemma 1</u>

There exists a k-current $\tilde{T}$ and a $(k+1)$-current S in $\mathbb{R}^n$ such that:

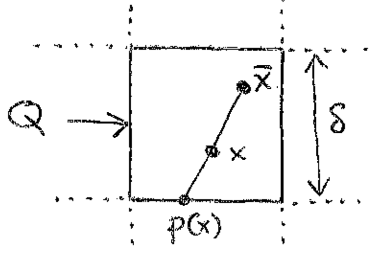
- $\tilde{T}$ is supported on the $(n-1)$-dimensional skeleton of the grid, L_{n-1} (the union of the $(n-1)$-dimensional faces of the cubes in the grid), $\partial\tilde{T}=0$, and $\mathbb{M}(\tilde{T})\le C\,\mathbb{M}(T)$;
- $T-\tilde{T}=\partial S$ and $\mathbb{M}(S)\le C\delta\,\mathbb{M}(T)$; in particular $\mathbb{F}(T-\tilde{T})\le\mathbb{M}(S)\le C\delta\,\mathbb{M}(T)$.

(Here and in the following we use C to denote any constant that depends at most on the dimension n.)

<u>Proof</u>

later we make the choice more precise

For every (closed) cube Q in the grid choose $\bar{x}=\bar{x}_Q$ in the interior of Q and let $p=p_Q$ be the "radial" retraction of $Q\setminus\{\bar{x}\}$ onto ∂Q described in the picture:



We then have

(1) $$|dp(x)| \leq C\frac{\delta}{|x-\bar{x}|}.$$

(The verification is left as an exercise.)

Next we define $f : \mathbb{R}^n \setminus \{\bar{x}_Q\} \longrightarrow L_{n-1} \subset \mathbb{R}^n$ by

$$f(x) := p_Q(x) \quad \forall x \in Q \setminus \{x_Q\} \text{ and } Q \text{ in the grid}$$

(note that f is well-defined and locally Lipschitz), and finally we set

$$\tilde{T} := f_\# T.$$

As pointed out before, $\tilde{T}$ is not really well-defined because f is not of class $\mathcal{C}^1$ on a neighbourhood of $\operatorname{supp}(T)$.

However we now proceed as if it were, and give the "correct" definition of $\tilde{T}$ later.

Estimate for $\mathbb{M}(\tilde{T})$.

Writing $T = \tau\mu$ with $\|\tau\| = 1$ μ-a.e. we get (by the usual formulas for the mass of the push-forward and (1))

(2) $$\mathbb{M}(\tilde{T}) \leq \int_{\mathbb{R}^n} |df(x)|^k d\mu(x) \leq C\int_{\mathbb{R}^n} g^k(x) d\mu(x)$$

where $g(x)$ is defined by

$$g(x) := \frac{\delta}{|x-\bar{x}_Q|} \quad \forall x \in Q \setminus \{x_Q\} \text{ and } Q \text{ in the grid.}$$

Here is the key lemma:

<u>Lemma 2</u> For each Q in the grid we can choose $\bar{x} = \bar{x}_Q$ so that

$$(3) \qquad \int_Q \frac{\delta^k}{|x-\bar{x}|^k}\, d\mu(x) \le C\,\mu(Q)\,,$$

and consequently

$$(3') \qquad \int_{\mathbb{R}^n} g^k(x)\, d\mu(x) \le C\,\mu(\mathbb{R}^n) = C\,\mathbb{M}(T)\,.$$

<u>Proof (of lemma 2)</u>

Estimate (3′) follows immediately from (3) (but be aware that the cubes Q, being closed, may overlap). To prove (3) it suffices to show that the average of the left-hand side over all $\bar{x} \in Q$ is $\le C\mu(Q)$ (then some $\bar{x}$ such that (3) holds must necessarily exist). Indeed

$$\fint_Q \left(\int_Q \frac{\delta^k}{|x-\bar{x}|^k}\, d\mu(x) \right) d\bar{x}$$

<u>Fubini</u> $\longmapsto$ $$= \delta^{k-n} \int_Q \left(\int_Q \frac{1}{|x-\bar{x}|^k}\, d\bar{x} \right) d\mu(x)$$

because $Q \subset B(x, \mathrm{diam}(Q))$ and $\mathrm{diam}(Q) = \sqrt{n}\,\delta$ $\longmapsto$ $$\le \delta^{k-n} \int_Q \left(\int_{B(x,\sqrt{n}\delta)} \frac{1}{|x-\bar{x}|^k}\, d\bar{x} \right) d\mu(x)$$

Compute the inner integral using polar coord. $\longmapsto$ $$= \delta^{k-n} \int_Q \underbrace{\left(\int_0^{\sqrt{n}\delta} \frac{1}{r^k}\, c\, r^{n-1}\, dr \right)}_{= C\delta^{n-k}} d\mu(x)$$

$$= C\,\mu(Q) \qquad \square$$

For the rest of the proof we choose all $\bar{x}_Q$ so that (3′) holds.

In particular estimates (2) and (3′) yield

$$\mathbb{M}(\tilde{T}) \le C\,\mathbb{M}(T)$$

as desired.

Construction of S and estimate for $\mathbb{M}(S)$

To construct S such that

$$\partial S = T - \tilde{T} = T - f_{\#} T$$

we use the homotopy formula with $f_1 :=$ the identity map and $f_0 := f$. We then obtain, using one of the estimates on $\mathbb{M}(S)$ given before,

$$\mathbb{M}(S) \leq \underbrace{\Big(\sup_{x \in \dots} |f(x) - x|\Big)}_{\overset{\wedge}{\operatorname{diam}(Q) = \sqrt{n}\,\delta}} \int_{\mathbb{R}^n} \underbrace{\big(|df(x)| + 1\big)^k}_{\overset{\wedge}{2^k (1 + |df(x)|^k)}} d\mu(x)$$

$$\leq C\,\delta \int_{\mathbb{R}^n} 1 + |df(x)|^k \, d\mu(x)$$

$$\leq C\,\delta \left(\mu(\mathbb{R}^n) + \int_{\mathbb{R}^n} |df(x)|^k d\mu(x) \right)$$

but now $\mu(\mathbb{R}^n) = \mathbb{M}(T)$ while estimates (2) and (3') yield $\int_{\mathbb{R}^n} |df(x)|^k d\mu(x) \leq C\, \mathbb{M}(T)$.
We have thus obtained

$$\mathbb{M}(S) \leq C\,\delta\, \mathbb{M}(T).$$

The "correct" construction of $\tilde{T}$ and S.

We construct $\tilde{T}$ and S as limits as $\varepsilon \to 0$ of $\tilde{T}_\varepsilon$ and S_ε defined as above with f_ε in place of f where f_ε is given by

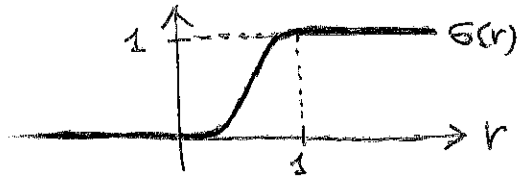
$$f_\varepsilon(x) = P_{Q,\varepsilon}(x) \quad \text{if } x \in Q$$

and $P_\varepsilon = P_{Q,\varepsilon}$ is a smooth map from Q to Q

obtained by "smoothing" p in an ε-neighbourhood of the singularity $\bar{x}$, e.g. by setting $p_\varepsilon(\bar{x}) = \bar{x}$ and

$$p_\varepsilon(x) := \bar{x} + (p(x) - \bar{x}) \cdot \sigma\left(\frac{|x-\bar{x}|}{\varepsilon}\right) \quad \forall x \in Q \setminus \{\bar{x}\},$$

where $\sigma : \mathbb{R} \to [0,1]$ is a smooth map as in the figure



Then p_ε is a smooth, proper map and

(4) $$p_\varepsilon(x) = p(x) \quad \text{if } |x - \bar{x}| \geq \varepsilon$$

Thus f_ε is smooth and $\tilde{T}_\varepsilon := (f_\varepsilon)_\# T$ is well-defined, and $\partial \tilde{T}_\varepsilon = 0$ because $\partial T = 0$. Moreover one can show that

(5) $$|dp_\varepsilon(x)| \leq C \frac{\delta}{|x-\bar{x}|}$$

(it is important that C does not depend on ε). Therefore we can estimate $\mathbb{M}(T_\varepsilon)$ as we estimated $\mathbb{M}(\tilde{T})$ and obtain $\mathbb{M}(T_\varepsilon) \leq C\, \mathbb{M}(T)$.

But we want to prove more, namely that $\tilde{T}_\varepsilon$ converge in the mass norm as $\varepsilon \to 0$, so that we can finally define

$$\tilde{T} := \lim_{\varepsilon \to 0} \tilde{T}_\varepsilon$$

(then automatically $\partial \tilde{T} = 0$ and $\mathbb{M}(\tilde{T}) \leq C\, \mathbb{M}(T)$).

Take indeed $\varepsilon > \varepsilon' > 0$. Then $p_{\varepsilon'}(x) = p_{\varepsilon}(x)$ $(= p(x))$ if $|x - \bar{x}| \geq \varepsilon$, and therefore, letting

$$A_\varepsilon := \{x : |x - \bar{x}_Q| < \varepsilon \text{ for some } Q \text{ in the grid}\}$$

we have

$$f_\varepsilon(x) = f_{\varepsilon'}(x) \quad \text{if } x \notin A_\varepsilon .$$

Then

these are the same because $f_\varepsilon = f_{\varepsilon'}$ on $\mathbb{R}^n \setminus A_\varepsilon$

$$\begin{aligned} T_\varepsilon - T_{\varepsilon'} &= (f_\varepsilon)_\# T - (f_{\varepsilon'})_\# T \\ &= (f_\varepsilon)_\#(1_{A_\varepsilon} \cdot T) + \overbrace{(f_\varepsilon)_\#(1_{\mathbb{R}^n \setminus A_\varepsilon} \cdot T)} \\ &\qquad - (f_{\varepsilon'})_\#(1_{A_\varepsilon} \cdot T) - \overbrace{(f_{\varepsilon'})_\#(1_{\mathbb{R}^n \setminus A_\varepsilon} \cdot T)} \\ &= (f_\varepsilon)_\#(1_{A_\varepsilon} \cdot T) - (f_{\varepsilon'})_\#(1_{A_\varepsilon} \cdot T) \end{aligned}$$

and then

$$\mathbb{M}(T_\varepsilon - T_{\varepsilon'}) \leq \mathbb{M}((f_\varepsilon)_\#(1_{A_\varepsilon} T)) + \mathbb{M}((f_{\varepsilon'})_\#(1_{A_\varepsilon} T)) .$$

Now, using (5) we get that $|df_\varepsilon(x)| \leq C g(x)$ and then

$$\mathbb{M}((f_\varepsilon)_\#(1_{A_\varepsilon} \cdot T)) \leq \int_{A_\varepsilon} |df_\varepsilon(x)|^k d\mu(x) \leq C \int_{A_\varepsilon} g^k(x)\, d\mu(x)$$

and the same estimate holds for $(f_{\varepsilon'})_\#(1_{A_\varepsilon} T)$.

Thus

$$\text{(6)} \qquad \mathbb{M}(T_\varepsilon - T_{\varepsilon'}) \leq C \int_{A_\varepsilon} g^k(x)\, d\mu(x) .$$

Now estimate (3') in Lemma 2 implies that $g^k \in L^1(\mu)$, and on the other hand $\mu(\bigcap_{\varepsilon > 0} A_\varepsilon) = 0$ (explain why) and therefore

$$\int_{A_\varepsilon} g^k(x)\, d\mu(x) \xrightarrow[\varepsilon \to 0]{} 0 ,$$

which, together with 6, implies that $\varepsilon \mapsto \tilde{T}_\varepsilon$ is a Cauchy "sequence", that is, $\forall\, \delta>0\ \exists\, \bar{\varepsilon}$ s.t. $\forall\, 0<\varepsilon'<\varepsilon\leq\bar{\varepsilon}$ there holds $\mathbb{M}(\tilde{T}_\varepsilon-\tilde{T}_{\varepsilon'})\leq\delta$.
Since the norm $\mathbb{M}$ is complete, this means that $\tilde{T}_\varepsilon$ converge w.r.t. $\mathbb{M}$ as $\varepsilon\to0$.

In a similar way one checks that S_ε, constructed using the homotopy formula with T and f_ε, converge to some S in the mass norm as $\varepsilon\to0$, and that $\mathbb{M}(S_\varepsilon)\leq C\delta\,\mathbb{M}(T)$ for all $\varepsilon>0$. It follows immediately that $\partial S=\lim_{\varepsilon\to0}\partial S_\varepsilon=\lim_{\varepsilon\to0}(T-\tilde{T}_\varepsilon)=T-\tilde{T}$, and that $\mathbb{M}(S)\leq C\delta\,\mathbb{M}(T)$.
This concludes the proof. □

<u>Remarks</u> • A careful examination of the last part of the proof above shows that some further corrections are needed.

1) The maps f_ε, as defined now, are not smooth on L_{n-1} but only (locally) Lipschitz. To solve this we can further modify $P_{Q,\varepsilon}$ so that it agrees with the identity on a ε-neighbourhood of ∂Q, and not just on Q.

2) The construction of $\tilde{T}_\varepsilon$ makes sense and gives the desired result if ε is so small that $\overline{B(\bar{x}_Q,\varepsilon)}$ is contained in the interior of Q for all Q.
Such an ε can be found if only finitely many Q's

are "relevant" to the construction, that is, if T has compact support. If T does not have compact support we need a slightly more refined construction.

∘ It is important to notice (for further use) that if T is rectifiable with integral multiplicity so are $\tilde{T}$ and S. This would be immediate if we had defined $\tilde{T}$ and S as we first did, but if we define them as limits of $\tilde{T}_\varepsilon$ and S_ε (which are rectifiable with integral multiplicity) we must use the fact that the limit of a sequence of rectifiable currents with integral multiplicity <u>in the mass norm</u> is also rectifiable with integral multiplicity. This fact is not difficult to prove, but still requires a proof (note moreover that this result does not hold for any weaker limit, for example in the flat norm...)

Lemma 1 admits the following "generalization" from which the polyhedral deformation theorem will be derived straight away.

We do not prove this generalization in detail, but only indicate which modifications should be taken in the proof of Lemma 2.

We fix a k-current T with compact support such that $\partial T = 0$ and $\mathbb{M}(T) < +\infty$, and a δ-grid in $\mathbb{R}^n$.

Lemma 3

Assume that T is supported on the h-dimensional skeleton L_h of the grid (the union of the h-dimensional faces of the cubes in the grid) and that $k < h \leq n$.
Then there exists a k-current $\tilde{T}$ and a $(k+1)$-current S such that:

- $\tilde{T}$ is supported on L_{h-1}, $\partial \tilde{T} = 0$, $\mathbb{M}(\tilde{T}) \leq C\, \mathbb{M}(T)$;
- S is supported on L_h, $T - \tilde{T} = \partial S$, $\mathbb{M}(S) \leq C\delta\, \mathbb{M}(T)$, and in particular $\mathbb{F}(T - \tilde{T}) \leq \mathbb{M}(S) \leq C\delta\, \mathbb{M}(T)$.

Moreover, if T is rectifiable so are $\tilde{T}$ and S, and if T is rectifiable with integral multiplicity (that is, integral — recall that $\partial T = 0$) so are $\tilde{T}$ and S.

Remark If $h = n$ then Lemma 3 is exactly Lemma 1. For $h < n$ Lemma 3 can be proved essentially in the same way as Lemma 1: for every h-dimensional face Q of the grid we choose a point $\bar{x}_Q$ in the interior of Q, and consider the retraction p_Q of $Q \setminus \{\bar{x}_Q\}$ onto ∂Q; then we define the map $f : L_h \setminus \{\bar{x}_Q : Q \in \ldots\} \to L_{h-1}$ by $f = p_Q$ on $Q \setminus \{\bar{x}_Q\}$ for every Q, and set

$$\tilde{T} := f_\# T$$

and then construct S using the homotopy formula (with the linear homotopy between f and the identity map).

As for the proof of Lemma 1, the first key step is to show that the points $\bar{x}_Q$ can be chosen in such a way that the estimates $\mathbb{M}(\tilde{T}) \leq C\,\mathbb{M}(T)$ and $\mathbb{M}(S) \leq C\delta\,\mathbb{M}(T)$ hold. The second key step is to give a correct definition of $\tilde{T}$ and S as limits of $\tilde{T}_\varepsilon$ and S_ε constructed as above with f replaced by suitable regularizations f_ε. Note that to apply the definition of push-forward and the homotopy formula the map f_ε must be defined (and of class $\mathscr{C}^1$) on $\mathbb{R}^n$.... this requires a construction for f_ε which is slight more complicated than that in the proof of Lemma 1.

<u>Polyhedral Deformation Theorem</u> (case $\partial T = 0$)

Take the k-current T and the δ-grid as above. Then there exists a polyhedral k-current P of the form

$$(*) \qquad P = \sum_{Q\ldots} [Q; \tau_Q, m_Q] \quad \longleftarrow \; m_Q = 0 \text{ for all } Q \text{ except finitely many}$$

(where the sum is taken over all k-dimensional faces Q of the grid, τ_Q is a (given) constant orientation of Q, and m_Q is a suitable constant multiplicity) and a $(k+1)$-current S such that

- $\partial P = 0$ and $\mathbb{M}(P) \leq C\,\mathbb{M}(T)$;
- $T - P = \partial S$ and $\mathbb{M}(S) \leq C\delta\,\mathbb{M}(T)$, and in particular $\mathbb{F}(T-P) \leq \mathbb{M}(S) \leq C\delta\,\mathbb{M}(T)$.

Moreover, if T is rectifiable so are P and S, and if T is rectifiable with integral multiplicity (i.e., integral) so are P and S.

<u>Proof</u>

By applying Lemma 3 iteratively we can construct a sequence of currents $\tilde{T}_h$ and S_h for $h = n-1, n-2, \dots, k$ so that, setting $\tilde{T}_n := T$, for every h there holds

- $\tilde{T}_h$ is supported on L_h, $\partial \tilde{T}_h = 0$, $\mathbb{M}(\tilde{T}_h) \le C\,\mathbb{M}(\tilde{T}_{h+1})$
- S_h is supported on L_{h+1}, $\tilde{T}_{h+1} - \tilde{T}_h = \partial S_h$, and $\mathbb{F}(\tilde{T}_{h+1} - \tilde{T}_h) \le \mathbb{M}(S_h) \le C\delta\,\mathbb{M}(\tilde{T}_{h+1})$.

Then $\mathbb{M}(\tilde{T}_h) \le C\,\mathbb{M}(\tilde{T}_{h+1}) \le \dots \le C\,\mathbb{M}(\tilde{T}_n = T)$, and in particular

$$\mathbb{M}(\tilde{T}_k) \le C\,\mathbb{M}(T). \tag{1}$$

Moreover

$$
\begin{aligned}
T - \tilde{T}_k &= (\overset{\overset{T}{\shortparallel}}{\tilde{T}_n} - \tilde{T}_{n-1}) + (\tilde{T}_{n-1} - \tilde{T}_{n-2}) + \dots + (\tilde{T}_{k+1} - \tilde{T}_k) \\
&= \partial S_{n-1} + \partial S_{n-2} + \dots + \partial S_k \qquad (2) \\
&= \partial(\underbrace{S_{n-1} + \dots + S_k}_{=:S}),
\end{aligned}
$$

and since $\mathbb{M}(S_h) \le C\delta\,\mathbb{M}(\tilde{T}_{h+1}) \le C\delta\,\mathbb{M}(T)$ for every h,

$$\mathbb{M}(S) \le C\delta\,\mathbb{M}(T). \tag{3}$$

We therefore set $P := \tilde{T}_k$ and take S as above, and then (1)–(3) show that the statement of the Polyhedral Deformation Theorem holds provided that we show that $P := \tilde{T}_k$ is of the form $(*)$.

To this end we denote by A_Q the interior of the k-dimensional face Q of the grid and note that L_k is the disjoint union of all A_Q and of L_{k-1}.
We can thus write $\tilde{T}_k$ as

$$\tilde{T}_k = \Big[\sum_Q (1_{A_Q}\cdot \tilde{T}_k)\Big] + (1_{L_{k-1}}\cdot \tilde{T}_k)$$

First we observe that $1_{L_{k-1}}\cdot T = 0$ because $\mathcal{H}^k(L_{k-1})=0$ and we proved in one of the previous lectures that given a normal current T (and $\tilde{T}_k$ is normal) of the form $T = \tau\mu$ with $\tau \neq 0$ μ-a.e., then μ is absolutely continuous w.r.t. $\mathcal{H}^k$.

Next we prove that $1_{A_Q}\cdot \tilde{T}_k$ is of the form $[Q, \tau_Q, m_Q]$ with m_Q constant.
To this end we choose U open set in $\mathbb{R}^n$ such that $L_k \cap U = A_Q$ (we can, because A_Q is open in L_k) and note that the restriction of $\tilde{T}_k$ to U is a k-current without boundary supported on the smooth k-surface A_Q, and since A_Q is connected, one of the variants of the constancy lemma ensures that the restriction of $\tilde{T}_k$ to U is of the form $[A_Q, \tau_Q, m_Q] = [Q, \tau_Q, m_Q]$, and this implies the desired representation for $1_{A_Q}\cdot \tilde{T}_k$.

□

As a corollary of the Polyhedral Deformation Theorem we obtain the following approximation result:

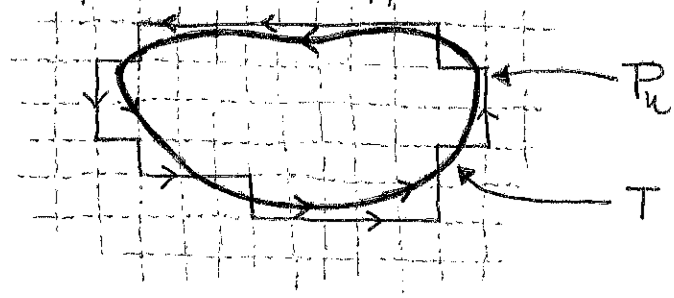
Corollary

Let T be a k-dimensional current in $\mathbb{R}^n$ with $0 \le k < n$ such that $\partial T = 0$, $\mathbb{M}(T) < +\infty$, supp(T) is compact. Then there exists a sequence of polyhedral k-currents P_n such that $\partial P_n = 0$, $\mathbb{M}(P_n) \le C\, \mathbb{M}(P) < +\infty$, $\mathbb{F}(T - P_n) \to 0$.

Moreover, if T is integral then also the P_n are integral.

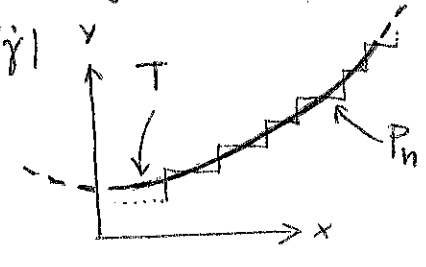
Note that this approximation result is far from optimal, as we could not claim that $\mathbb{M}(P_n) \to \mathbb{M}(T)$, that is, we do not have approximation "in mass".

A typical example of such approximation is the following:



and it is clear that approximation "in mass" cannot be achieved unless the tangent to T is parallel to the axis of the grid (a.e.).

More precisely, if T is the 1-current associated to a closed path γ in $\mathbb{R}^2$ and the grid is parallel to the x and y axes, then $\mathbb{M}(T) = \int |\dot\gamma|$ while $\mathbb{M}(P_n)$ can at most converge to $\int |\dot\gamma_x| + |\dot\gamma_y|$ which in general is strictly larger than $\int |\dot\gamma|$



In the next lecture we prove an approximation in mass.

In this lecture we prove the Polyhedral Deformation Theorem for currents with boundary (while in the previous lecture we only considered currents without boundary).

Therefore, in the following we fix a k-dimensional normal current T in $\mathbb{R}^n$ with compact support, and a δ-grid in $\mathbb{R}^n$.

We proceed step by step, as in the previou lecture...

Lemma 1

If $k<n$ then there exists a $(k+1)$-current S and k-currents $\tilde{T}$ and R such that

- $\tilde{T}$ is supported on the $(n-1)$-dimensional skeleton L_{n-1} of the grid, $\tilde{T}$ is normal, and $\mathbb{M}(\tilde{T}) \leq C\,\mathbb{M}(T)$, $\mathbb{M}(\partial\tilde{T}) \leq C\,\mathbb{M}(\partial T)$;
- $T-\tilde{T} = R+\partial S$ and $\mathbb{M}(R) \leq C\delta\,\mathbb{M}(\partial T)$, $\mathbb{M}(S) \leq C\delta\,\mathbb{M}(T)$; In particular $\mathbb{F}(T-\tilde{T}) \leq \mathbb{M}(R)+\mathbb{M}(S) \leq C\delta(\mathbb{M}(T)+\mathbb{M}(\partial T))$.

Remarks • From these statements we further obtain that

- $\partial T - \partial\tilde{T} = \partial R$ and $\mathbb{F}(\partial T-\partial\tilde{T}) \leq \mathbb{M}(R) \leq C\delta\,\mathbb{M}(\partial T)$
- R and S are normal, and $\mathbb{M}(\partial R) \leq C\,\mathbb{M}(\partial T)$, $\mathbb{M}(\partial S) \leq C(\mathbb{M}(T)+\delta\,\mathbb{M}(\partial T))$

Indeed $T-\tilde{T} = R+\partial S \Rightarrow \partial T - \partial\tilde{T} = \partial R$, and this yield both the estimate on $\mathbb{F}(\partial T-\partial\tilde{T})$ and that on $\mathbb{M}(\partial R)$; moreover $T-\tilde{T} = R+\partial S \Rightarrow \partial S = T-\tilde{T}-R$ and this yields the estimate on $\mathbb{M}(\partial S)$.

∘ The proof will show furthermore that if T is an integral current so will be R and S.

<u>Proof</u> We proceed as in the proof of Lemma 1 in the previous lecture (case $\partial T=0$) and only indicate the main differences.

We thus construct the retraction f from $\mathbb{R}^n \setminus \{\bar{x}_Q\}$ on L_{n-1} and set $\tilde{T} := f_\# T$. Thus $\partial\tilde{T} = f_\#(\partial T)$

$F(t,x) := tx + (1-t)f(x)$

Next we consider the linear homotopy F between f and the identity map, thus

$I := [[0,1], e, 1]$

$$T - \tilde{T} = T - f_\# T = \underbrace{\partial F_\#(I \times T)}_{\overset{\shortparallel}{S}} + \underbrace{F_\#(I \times \partial T)}_{\overset{\shortparallel}{R}}$$

Now, setting $g(x) := \frac{\delta}{x - \bar{x}_Q}$ for every $x \in Q \setminus \{\bar{x}_Q\}$ and every Q, and writing $T = \tau\mu$ with $\|\tau\| = 1$ μ-a.e, $\partial T = \tau'\mu'$ with $\|\tau'\| = 1$ μ'-a.e, we get the following estimates:

$$\mathbb{M}(\tilde{T}) \leq C \int_{\mathbb{R}^n} g^k(x)\, d\mu(x),$$

$$\mathbb{M}(\partial\tilde{T}) \leq C \int_{\mathbb{R}^n} g^{k-1}(x)\, d\mu'(x),$$

$$\mathbb{M}(S) \leq C \int_{\mathbb{R}^n} (1 + g^k(x))\, d\mu(x),$$

$$\mathbb{M}(R) \leq C \int_{\mathbb{R}^n} (1 + g^{k-1}(x))\, d\mu'(x).$$

To obtain the desired estimates, we need a variant of lemma 2 in previous lecture, stating that we can choose the points $\bar{x}_Q \in Q$ in such a way that

$$\int_{\mathbb{R}^n} g^k(x)\, d\mu(x) \leq C \|\mu\| = C\, \mathbb{M}(T),$$

and

$$\int_{\mathbb{R}^n} g^{k-1}(x)\, d\mu'(x) \leq C \|\mu'\| = C\, \mathbb{M}(\partial T).$$

The next part of the proof consists in giving the "correct" definition of $\tilde{T}, R, S$ as limits (as $\varepsilon \to 0$) of $\tilde{T}_\varepsilon, R_\varepsilon, S_\varepsilon$ constructed as above with f replaced by a suitably "smoothed" map f_ε .....

$\square$

Next we generalize Lemma 1 (cf. Lemma 3 in the previous lecture):

<u>Lemma 2</u>

If T is supported on the h-dimensional skeleton L_h of the grid, then we can take $\tilde{T}, R, S$ which satisfy all the statements in lemma 1 plus the following: $\tilde{T}$ is supported on L_{h-1}, R and S are supported on L_h.

And finally we have

<u>Polyhedral Deformation Theorem</u> (<u>general case</u>)

Let T be a normal k-current in $\mathbb{R}^n$ with compact support, then there exists a polyhedral k-current P, a k-current R and a $(k+1)$-current S such that

- $P = \sum_Q [Q, \tau_Q, m_Q]$ where Q ranges among

the k-dimensional faces of the grid, τ_Q is a (given) constant orientation of Q, m_Q is a constant multiplicity, and $m_Q = 0$ for all Q except finitely many (thus the sum is finite and P is polyhedral),
$\mathbb{M}(P) \le C(\mathbb{M}(T) + \delta\, \mathbb{M}(\partial T))$, $\mathbb{M}(\partial P) \le C\, \mathbb{M}(\partial T)$;

- $T - P = R + \partial S$, and $\mathbb{M}(R) \le C\delta\, \mathbb{M}(\partial T)$, $\mathbb{M}(S) \le C\delta\, \mathbb{M}(T)$; in particular $\mathbb{F}(T-P) \le \mathbb{M}(R) + \mathbb{M}(S) \le C\delta(\mathbb{M}(T) + \mathbb{M}(\partial T))$.

Furthermore

- $\partial T - \partial P = \partial R$ and $\mathbb{F}(\partial T - \partial P) \le \mathbb{M}(R) \le C\delta\, \mathbb{M}(\partial T)$, and $\mathbb{M}(\partial S) \le C(\mathbb{M}(T) + \delta\, \mathbb{M}(\partial T))$; in particular R and S are normal;
- if in addition T is integral then R and S are integral.

<u>Proof</u>

The proof of this theorem follows closely that of the previous version of the Polyhedral Deformation Theorem in the previous lecture, except of one additional step at the very end.

We thus use lemma 2 to construct $\tilde{T}_h, R_h, S_h$ for $h = n-1, n-2, \dots, k$ so that, setting $\tilde{T}_n := T$,

- $\tilde{T}_h$ is supported on L_h,
- S_h and R_h are supported on L_{h+1} and $\tilde{T}_{h+1} - \tilde{T}_h = R_h + \partial S_h$.

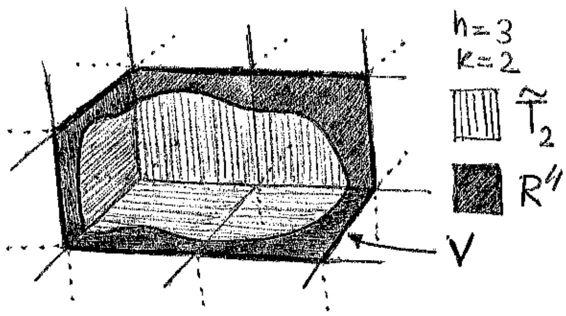
(I omit the various estimates....)

Then
$$T-\tilde T_k = \tilde T_h - \tilde T_k = (\tilde T_h - \tilde T_{h-1}) + (\tilde T_{h-1} - \tilde T_{h-2}) + \dots + (\tilde T_{k+1} - \tilde T_k)$$
$$= (R_{h-1} + \partial S_{h-1}) + (R_{h-2} + \partial S_{h-2}) + \dots + (R_k + \partial S_k)$$
$$= \underbrace{(R_{h-1} + \dots + R_k)}_{=: R'} + \partial \underbrace{(S_{h-1} + \dots + S_k)}_{=: S}$$

However, since $\tilde T_k$ may have boundary, the fact that it is supported on L_k is not enough to deduce that $\tilde T_k$ is polyhedral.

We thus need an additional step.

Applying lemma 3 from the previous lecture to the current $\partial \tilde T_k$ (note that $\partial(\partial \tilde T_k) = 0$) we can find a $(k-1)$-current V supported on L_{k-1} and a k-current R'' supported on L_k such that $\partial \tilde T_k - V = \partial R''$ (plus the usual estimates: $\mathbb{M}(V) \le C\, \mathbb{M}(\partial \tilde T_k) \le C\, \mathbb{M}(\partial T)$, $\mathbb{M}(R') \le$ $\le C\delta\, \mathbb{M}(\partial \tilde T_k) \le C\delta\, \mathbb{M}(\partial T)$).



We finally set
$$P := \tilde T_k - R''$$

Then
$$T - P = T - \tilde T_k + R'' = \underbrace{R' + R''}_{=: R} + \partial S$$

Moreover $\partial P = \partial \tilde T_k - \partial R'' = V$ is supported on L_{k-1}.

Thus P is a normal k-current supported on L_k with boundary supported on L_{k-1} and we can proceed as in the proof in the previous lecture to show that P is indeed a polyhedral current of the desired form.

$\square$

From the Polyhedral Def. Theor. we obtain the following polyhedral approximation:

Corollary

Let T be a normal k-current with compact support in $\mathbb{R}^n$. Then there exists a sequence of polyhedral currents P_n such that $\mathbb{F}(T-P_n) \to 0$ and $\mathbb{M}(P_n) \leq C\,\mathbb{M}(T)$, $\mathbb{M}(\partial P_n) \leq C\,\mathbb{M}(\partial T)$. And if T is integral so is each P_n.

This result can be improved as follows:

Theorem (Strong Polyhedral Approximation)

Let T be as above. Then there exists a sequence of polyhedral currents P_n such that

$$\mathbb{F}(T-P_n) \to 0, \quad \mathbb{M}(P_n) \to \mathbb{M}(T), \quad \mathbb{M}(\partial P_n) \to \mathbb{M}(\partial T).$$

Moreover, if T is integral so is each P_n; and if $\partial T = 0$ then $\partial P_n = 0$ for every n.

Sketch of proof for the case $\partial T = 0$

Fix $\varepsilon > 0$. We want to find a polyhedral current P such that $\partial P = 0$, $\mathbb{M}(P) \leq \mathbb{M}(T) + C\varepsilon$, $T - P = \partial S$ with $\mathbb{M}(S) \leq C\varepsilon$. We split the construction of P in three steps.

Step 1 We can find finitely many, closed discs D_i with dimension k endowed with constant orientations τ_i, and constant multiplicities m_i such that, setting $T' := \sum [D_i, \tau_i, m_i]$, there holds $\mathbb{M}(T') \leq \mathbb{M}(T)$ and $\mathbb{F}(T-T') \leq C\varepsilon$.

The idea is the following: write $T = \tau\mu$ with $\|\tau\| = 1$ μ-a.e.
Now, μ-a.e. $\bar{x}$ is a point of L^1-approximate continuity of τ, which implies that $\int_{B(\bar{x},r)} \|\tau(x) - \tau(\bar{x})\| \, d\mu(x) \le \varepsilon \, \mu(B(\bar{x},r))$ for r sufficiently small.

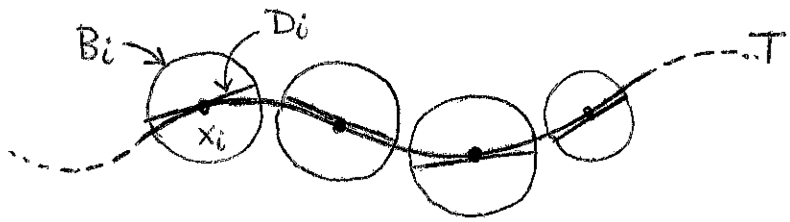
Therefore we can use a corollary of Besicovitch covering theorem (mentioned now for the first time in this course) to find finitely many disjoint closed balls $B_i = \overline{B(x_i, r_i)}$ such that

(1) $\quad r_i \le \varepsilon$; $\int_{\overline{B}_i} \|\tau(x) - \tau(x_i)\| \, d\mu(x) \le \varepsilon \, \mu(B_i)$; $\mu(\mathbb{R}^n \setminus \bigcup_i B_i) \le \varepsilon$.

Assume now that $\tau(x_i)$ is a <u>simple</u> k-vector for every x_i, and let V_i be the k-dimensional plane spanned by $\tau(x_i)$.
We then set

(2) $\quad D_i := B_i \cap (x_i + V_i)$; $\tau_i := \tau(x_i)$; $m_i := \dfrac{\mu(B_i)}{\mathcal{H}^k(D_i)}$.

Thus m_i is chosen in such a way that $\mathbb{M}([D_i, \tau_i, m_i]) = \mu(B_i)$.



Using (1) and (2) it is easy to show that, having set $T' := \sum_i [D_i, \tau_i, m_i]$ then $\mathbb{M}(T') \le \mathbb{M}(T)$.

To prove that $\mathbb{F}(T - T') \le C\varepsilon$ we use the following facts: letting $T_i := 1_{B_i} \cdot \tau \cdot \mu$, then

(i) $\quad \mathbb{M}(T - \sum_i T_i) \le \varepsilon$;

(ii) $\quad \mathbb{M}(T_i - 1_{B_i} \tau_i \mu) \le \varepsilon \, \mu(B_i)$;

(iii) $\mathbb{F}(1_{B_i} \cdot \tau_i \cdot \mu - c_i \tau_i \delta_{x_i}) \le C\varepsilon$ where $c_i := \mu(B_i)$
(here we use that $r_i \le \varepsilon$, and some work...);

(iv) $\mathbb{F}(c_i \tau_i \delta_{x_i} - [D_i, \tau_i, m_i]) \le C\varepsilon$.

Now, what should we do if the k-vectors $\tau_i := \tau(x_i)$ are not all simple?
In this case we should remember the definition of the mass norm of k-vectors (here is the only point where we use it in the entire course) and write τ_i as a convex combination of unitary simple k-vectors τ_{ij}, that is

$$\tau_i = \sum_j \alpha_{ij} \tau_{ij}$$

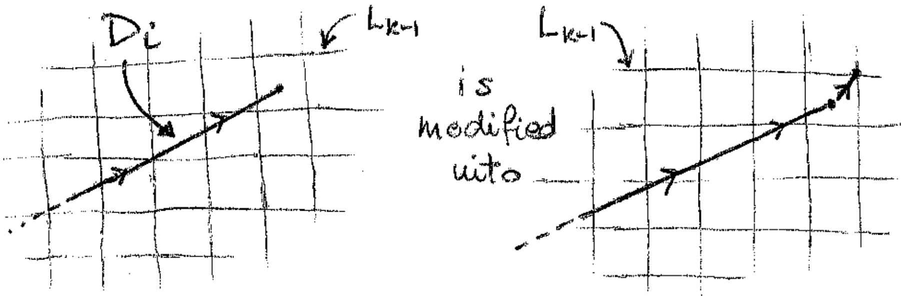
with $\alpha_{ij} \ge 0$, $\sum_j a_{ij} = 1$, $\|\tau_{ij}\| = 1$ (not possible with the Euclid. norm).

Then for each j we let V_{ij} be the span of τ_{ij}, set $D_{ij} := B_i \cap (x_i + V_{ij})$, $m_{ij} := \alpha_{ij} \dfrac{\mu(B_i)}{\mathcal{H}^k(D_{ij})}$, and finally we set

$$T' := \sum_{ij} [D_{ij}, \tau_{ij}, m_{ij}]$$

Step 2 We can find a polyhedral current P' such that $\mathbb{M}(P') \le \mathbb{M}(T)$ and $\mathbb{F}(T - P') \le C\varepsilon$, and a δ-grid such that $\partial P'$ is supported on the $(k-1)$-skeleton L_{k-1} of the grid.

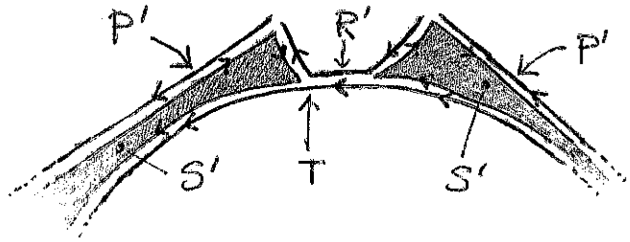
The idea is to choose a δ-grid with δ much smaller than the radius of each disc D_i in Step 1, and "modify" each disc D_i close to the boundary so to obtain a polyhedral current with boundary supported on L_{k-1}.
The details are omitted...



Note that P' would be a good approximation if only $\partial P'=0$. The problem is that we have no control whatsoever on $\partial P'$. This is solved in the next step.

Step 3 We can modify P' so to obtain a polyhedral current P with $\partial P=0$ such that $\mathbb{M}(P) \le \mathbb{M}(T)+C\varepsilon$ and $T-P=\partial S$ with $\mathbb{M}(S)\le C\varepsilon$, which implies $\mathbb{F}(T-P)\le C\varepsilon$. (This would conclude the proof.)

Since $\mathbb{F}(T-P')\le C\varepsilon$, there exist R' and S' s.t. $T=P'+R'+\partial S'$ and $\mathbb{M}(R')$, $\mathbb{M}(S')\le C\varepsilon$.



Note that $\partial T=0 \Rightarrow \partial P'+\partial R'=0$. Thus $P'+R'$ would be an approximation of T without boundary (as desired) but unfortunately R' is not polyhedral. The idea is then to deform R' into a polyhedral current P'' keeping the same boundary.
More precisely we obtain P'' by applying the Polyhedral Deform. Thm. to R' and to the grid in Step 2. Thus $R'=P''+R''+\partial S''$ with $\mathbb{M}(P'')\le$ $\le C\,\mathbb{M}(R')\le C\varepsilon$, $\mathbb{M}(R'')+\mathbb{M}(S'')\le C\delta(\mathbb{M}(R')+\mathbb{M}(\partial R'))$.

Now we observe that

depends on Step2!

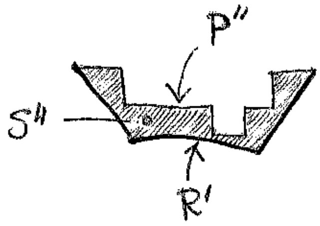
- $\partial R' = -\partial P'$ and then $\mathbb{M}(\partial R') = \mathbb{M}(\partial P') \le C\, \mathbb{M}(\partial T')$ and therefore, having chosen δ small enough in Step 2, we get $\delta\, \mathbb{M}(\partial T') \le \varepsilon$ (check that this can indeed be done) and therefore the previous estimate on $\mathbb{M}(R'') + \mathbb{M}(S'')$ becomes

$$\mathbb{M}(R'') + \mathbb{M}(S'') \le C\varepsilon$$

- Since $\partial R' = -\partial P'$ is supported on the $(k-1)$-skeleton of the grid, a careful examination of the proof of the Polyhedral Deform. Thm. shows that $\partial P'' = \partial R'$ and $R'' = 0$ (all the retractions involved in the proof keep the points of L_{k-1} fixed ....). Thus

$$R' = P'' + \partial S''$$

and $\mathbb{M}(P'') \le C\varepsilon$, $\mathbb{M}(S'') \le C\varepsilon$.



Finally we set

$$P := P' + P''$$

Thus P is polyhedral, $\partial P = \partial P' + \partial P'' = \partial P' + \partial R' = 0$, and $\mathbb{M}(P) \le \mathbb{M}(P') + \mathbb{M}(P'') \le \mathbb{M}(T) + C\varepsilon$. Moreover

$$T = P' + R' + \partial S' = P' + (P'' + \partial S'') + \partial S'$$
$$= P + \partial(\underbrace{S' + S''}_{\overset{!!}{S}})$$

and $\mathbb{M}(S) \le \mathbb{M}(S') + \mathbb{M}(S'') \le C\varepsilon$.

$\square$

In this section we give a few application of the polyhedral deformation theorem.

Isoperimetric Theorem (first version).
Let T be an integral k-current with compact support in $\mathbb{R}^n$ s.t. $0<k<n$ and $\partial T=0$.
Then there exists S integral current with compact support such that $\partial S = T$ and $\mathbb{M}(S) \leq C\, \mathbb{M}(T)^{1+\frac{1}{k}}$

Constant depending only on n (and k)

Proof
Let us fix for the time being $\delta>0$, to be chosen properly later, and use the Polyhedral Deformation Theorem to construct the deformation P of T on a δ-grid. Thus $T-P=\partial S$ and P, S satisfy

$$\mathbb{M}(P) \leq \underbrace{C\, \mathbb{M}(T)}_{= L}, \quad \mathbb{M}(S) \leq C\delta\, \mathbb{M}(T).$$

Now, let δ be chosen so that $L<\delta^k \leq 2L$.
Then $P=0$.
Indeed P is an integral polyhedral current supported on a δ-grid, and therefore its mass is an integral multiple of δ^k. But $\mathbb{M}(P) \leq L < \delta^k$, which implies $\mathbb{M}(P)=0$, and then $T=\partial S$. Moreover

$$\mathbb{M}(S) \leq C\delta\, \mathbb{M}(T) \leq C(2L)^{\frac{1}{k}}\, \mathbb{M}(T) \leq C\left(\mathbb{M}(T)\right)^{1+\frac{1}{k}}.$$

$\square$

Remarks.

- It is clear why in the proof we need $k>0$, but is it really needed? The answer is yes: if T is the 0-current associated to the Dirac mass δ_{x_0}, then T is not the boundary of any 1-current S with finite mass (although it is the boundary of a 1-current S with locally finite mass, e.g. the current associated to a half-line starting from x_0). Moreover, if $T = \delta_{x_1} - \delta_{x_0}$, then T is the boundary of an (integral) 1-current S, but the mass of S is larger that $|x_1 - x_2|$, and therefore cannot be controlled by any universal constant.

- Why do we need that T is integral? What happens if we only assume that T has finite mass?
  Under this more general assumption the Theorem fails.
  Assume by contradiction that it holds, take $T \neq 0$, and apply the theorem to $T_\lambda := \lambda T$ with $\lambda > 0$.
  Then there exists S_λ s.t. $T_\lambda = \partial S_\lambda$ and $\mathbb{M}(S_\lambda) \leq C\, \mathbb{M}(T_\lambda)^{1+\frac{1}{k}}$.
  Thus $T = \lambda^{-1} T_\lambda = \partial(\underbrace{\lambda^{-1} S_\lambda})$ and $\mathbb{M}(\tilde{S}_\lambda) \leq \lambda^{-1}\, \mathbb{M}(T_\lambda)^{1+\frac{1}{k}} \sim \lambda^{1/k}$.
  Hence $\tilde{S}_\lambda \to 0$ as $\lambda \to 0$. $\tilde{S}_\lambda$
  Hence $T = \partial 0 = 0$, which is a contradiction since $T \neq 0$.

Among the many possible generalizations of the Isoperimetric Theorem we quote the following:

Isoperimetric Theorem (second version)

Let $0<k<n$ and let T be a k-integral current contained in M, n-dimensional compact smooth manifold, such that $T=\partial\tilde{S}$ where $\tilde{S}$ is a $(k+1)$-integral current in M.

Then there exists an integral current S in M such that $T=\partial S$ and

$$\mathbb{M}(S)\leq C\,\mathbb{M}(T)\ ;\quad \mathbb{M}(S)\leq C\,\mathbb{M}(T)^{1+\frac{1}{k}}$$

where the constant C depends only on M.

Variant The ambient manifold M can be replaced by an open set $U\subset\mathbb{R}^n$ with smooth boundary, and it is assumed that $T, S, \tilde{S}$ have compact support in U.

Sketch of the proof

We can assume that M is a submanifold of class C^2 of some $\mathbb{R}^N$, and choose r such that the projection on M is a retraction of the r-neighbourhood $\mathcal{I}_r M$ of M onto M.

We fix for the time being $\delta>0$ s.t. $\sqrt{N}\delta<r$ and a δ-grid of $\mathbb{R}^N$ (δ will be properly chosen later; the ineq. $\sqrt{N}\delta<r$ ensures that every cube in the grid intersecting M is contained in $\mathcal{I}_r M$).

Step 1

Let $\tilde{S}'$ be the deformation of $\tilde{S}$ on the δ-grid according to the polyhedral deformation theorem, and let $T':=\partial\tilde{S}'$. Then $T-T'=\partial\tilde{S}-\partial\tilde{S}'=\partial R$ where R is an integral current, and there holds $\mathbb{M}(T'=\partial\tilde{S}')\leq C\,\mathbb{M}(T=\partial\tilde{S})$;
$\mathbb{M}(R)\leq C\delta\,\mathbb{M}(T=\partial\tilde{S})$ (C depends only on N,k).

Step 2

Now we want to find S' polyhedral current such that $\partial S' = \partial \tilde{S}' = T'$ and $\mathbb{M}(S') \leq C\, \mathbb{M}(T')$ where the constant C depends only on N and δ.

We observe that the cubes Q_i in the grid that intersect the submanifold M are only finitely many, and by construction T' and $\tilde{S}'$ are integral polyhedral currents given by sums of the k and $(k+1)$-dimensional faces of these cubes only.

Now, let V^h be the real vector space of all real polyhed. currents given by sums of the h-faces of these cubes, and let W^k be the subset of the currents in V^k tha are integral and are boundaries of integral currents in V^{k+1}. Thus V^k is finite dimensional, and W^k is a discrete subgroup of V^k. By a not-so-well-known theorem, we can find a family of generators $\{T_i\}$ of W^k (generators in the sense of (abelian) groups) that are also linearly independent (as elements of V^k). Now, each T_i is of the form $T_i = \partial S_i$ with $S_i \in V^{k+1}$ and integral. Thus there exists a (unique) linear operator $\phi : \mathrm{Span}(W^k) \to V^{k+1}$ s.t. $\phi : T_i \to S_i\ \forall i$, and clearly $T = \partial(\phi(T))$ for every $T \in \mathrm{Span}(W^k)$ and $\phi(T)$ is integral if $T \in W^k$. Moreover, since linear operator between finite dimensional normed spaces are <u>always</u> bounded, $\mathbb{M}(\phi(T)) \leq C\, \mathbb{M}(T)$. We finally set $S' := \phi(T')$ where T' is taken as in the previous step.

Step 3

Let p be the projection of $U_r M$ onto M. We set

$$S := p_\#(S'+R).$$

T is supp. on M

Since $T = T'+R = \partial(S'+R)$, then $T = p_\# T = \partial p_\#(S'+R) = \partial S$.

Moreover S is integral because S' and R are integral, and

$$\mathbb{M}(S) \le C(\mathbb{M}(S')+\mathbb{M}(R)) \le C(\mathbb{M}(T')+\delta\,\mathbb{M}(T)) \le C\,\mathbb{M}(T).$$

see Steps 1&2 — see Step 1

Step 4

It remains to prove that $\mathbb{M}(S) \le C\,\mathbb{M}(T)^{1+1/k}$. When $\mathbb{M}(T)$ is "large" this inequality follows from $\mathbb{M}(S) \le C\,\mathbb{M}(T)$. When $\mathbb{M}(T)$ is "small" we preceed as in the proof of the first version of the Isoper. Th: we let T' be the def. of T on the δ-grid, so that $T-T' = \partial S'$, and choose δ so that $\mathbb{M}(T') \le C\,\mathbb{M}(T) \le \delta^k$ and $\mathbb{M}(T) \sim \delta^k$. Then $T = \partial S'$ and $\mathbb{M}(S') \sim \delta^{k+1} \sim \mathbb{M}(T)^{1+1/k}$ and finally se set $S := p_\# S'$. $\square$

Homology groups defined via currents

Let M be a smooth compact n-manifold with $\partial M = \emptyset$, and $0<k<n$. In the class of "polyhedral" integral k-currents on M we consider the subclass X_k^P on the currents without boundary, and within X_k^P we consider the subclass Y_k^P of the currents that are boundaries (of integral polyhedral $(k+1)$-currents on M). Then the k-th (singular) homology group of M with coefficients in $\mathbb{Z}$ is usually defined as the quotient $H_k(M) := \dfrac{X_k^P}{Y_k^P}$.

(Note that I didn't specify exactly what a polyhedral current on M is...)

Now one can replicate this "classical" construction in the framework of integral currents by removing the constraint that the currents are polyhedral, and define in a similar way X_k, Y_k, and X_k/Y_k.

Proposition 1

$$\frac{X_k}{Y_k} = \frac{X_k^P}{Y_k^P}.$$

More precisely, this means that:

(i) for every $T \in X_k$ there exists $P \in X_k^P$ cobordant to T in M, that is $T - P \in Y_k^P$; this shows that the inclusion $X_k^P \xrightarrow{i} X_k/Y_k$ is surjective, and so is $X_k^P/Y_k^P \xrightarrow{\tilde{i}} X_k/Y_k$.

(ii) if $P, P' \in X_k^P$ are cobordant, that is, $P - P' \in Y_k$, then we also have that $P - P' \in Y_k^P$; this show that the inclusion $\tilde{i}$ considered before is injective.

Without entering into details, let's say that M is a submanifold of $\mathbb{R}^N$ as in the last proof, and that we call "polydral" current in M the projection (onto M) of the currents in V^k (which is not quite correct but can be made correct with some effort). Then the previous statement follows easily from the Polyhedral Deformation Theorem.

For our purposes the fundamental resuet is the following:

Theorem 2 Take X_k and Y_k as above. Then every equivalence class $[T] \in X_k/Y_k$ is a closed subset of X_k (closed under convergence in the sense of currents with uniformly bounded masses).

Proof

A sequence of currents T_n in $[T]$ can be represented as $T_n = T + \partial\tilde{S}_n$ with $\tilde{S}_n$ integral. By the Isoperimetric Theorem (2nd version) we can find S_n s.t. $\partial S_n = \partial\tilde{S}_n$ and $\mathbb{M}(S_n) \leq C\, \mathbb{M}(\partial\tilde{S}_n) = \mathbb{M}(T_n - T) \leq \mathbb{M}(T_n) + \mathbb{M}(T)$. Now, if $T_n \to T_\infty$ and $\mathbb{M}(T_n) \leq c < +\infty$, then $\mathbb{M}(S_n) \leq C < +\infty$, and up to subsequence we can assume that S_n converges to some S_∞ which is integral (we use F&F compactness theorem). Hence $T_\infty = T + \partial S_\infty$ and then $T_\infty \in [T]$.

$\square$

An immediate corollary of this result (and of F&F Compactness theorem) is the following:

Corollary 3

Let M be as above, and $0<k<n$. Then every homology class $[T] \in X_k/Y_k$ contains a current which minimizes the mass (within the class).

We have thus proved existence for the homological Plateau problem.

We conclude this lecture with a result for normal currents akin to the second version of the Isoper. Th.

Proposition 4

Let M, n, k as in the statement of the Isoper. Th. (2nd v.). Let T be a k-current with finite mass on M s.t. $T = \partial\tilde{S}$, where $\tilde{S}$ is a $(k+1)$-current with finite mass on M. Then there exists a $(k+1)$-current S in M s.t. $T = \partial S$ and $\mathbb{M}(S) \le C\,\mathbb{M}(T)$, where C depends only on M.

The proof is very similar to that of the Isoper. Th. (2nd v.) and in fact simpler. We leave it as an exercise.

Proceeding as before, one can define for every k with $0 < k < n$ the class $\tilde{X}_k$ of all k-currents on M with finite mass an no boundary, and the subclass $\tilde{Y}_k$ of all currents that are boundaries of $(k+1)$-normal currents on M. It can be shown that the k-th (singular) homology space with coefficients in $\mathbb{R}$ of M, usually defined in terms of simplicial chains with coefficients in $\mathbb{R}$ (or polyhedral currents with real multiplicities), is equivalent to the quotient $\tilde{X}_k / \tilde{Y}_k$.
Using Proposition 4 one can prove that

Corollary 5 Every homology class $[T] \in \tilde{X}_k / \tilde{Y}_k$ is closed (with respect to the convergence in the sense of currents with uniformly bounded masses).

Slicing of currents

Let M and M' be surfaces in $\mathbb{R}^n$ of dimension k and $n-h$ respectively. If M and M' are in general position (or, more precisely, transversal) the intersection $M \cap M'$ is a surface of dimension $k-h$ (or its empty).
Of course this is not true if M and M' are not in general position, but one expects that when M' is chosen from a sufficiently large family of surfaces, then general position will be achived for "most" choices.
For example M and $M'+v$ (the translation of M' by $v \in \mathbb{R}^n$) are in general position for a.e. v.

A particularly interesting case occurs when M' is chosen among the level sets of a sufficiently smooth map $f: \mathbb{R}^n \to \mathbb{R}^h$. We begin with a result in this smooth setting which we then extend to the case where M is replaced by a rectifiable current.

Proposition 1

Let $0 < h \leq k \leq n$, and let M be a smooth surface (with or without boundary) of dimension k in $\mathbb{R}^n$, and $f: \mathbb{R}^n \to \mathbb{R}^h$ a smooth map.
Then for ($\mathcal{L}^h$-) a.e. $y \in f$, the intersection $M_y := M \cap S_y$ where $S_y := f^{-1}(y)$ is a smooth surface of dimension $k-h$ (or is empty).

This statement is an immediate Corollary of Sard Theorem (applied to the restriction of f to M).
We call the surfaces M_y "slices" of M according to f.

Now, if we denote by ∇_τ the tangential gradient (tangential to M) then the simple h-vector

$$\eta(x) := \nabla_\tau f_1(x) \wedge \dots \wedge \nabla_\tau f_h(x)$$

(where f_i are the components of) is non trivial and spans the normal space to M_y at the point x (within $\mathrm{Tan}(M,x)$) for every $x \in M_y$ and a.e. $y \in \mathbb{R}^h$ (this is again Sard th.)

Thus if M is oriented by τ, for a.e. y we can endow M_y by the "canonical" orientation $\tilde{\tau}$ defined by

$$\frac{\eta(x)}{|\eta(x)|} \wedge \tilde{\tau}(x) = \tau(x) \qquad \forall x \in M_y .$$

We can extend such construction to the case where M is replaced by a rectifiable current.

To this purpose we need the so-called coarea formula (the proof of which we omit).

<u>Coarea Formula</u> (first version)

as before, $0 < h \leq k$

Let $f : \mathbb{R}^k \to \mathbb{R}^h$ be a lipschitz map, for every $y \in \mathbb{R}^h$ let $S_y := f^{-1}(y)$, and for $\mathcal{L}^k$-a.e. $x \in \mathbb{R}^n$ let

$$Jf(x) := \left| \nabla f_1(x) \wedge \dots \wedge \nabla f_h(x) \right| . \longleftarrow \text{Jacobian of } f$$

Then for every (Borel) set F in $\mathbb{R}^k$ there holds

$$\int_{y \in \mathbb{R}^h} \mathcal{H}^{k-h}(F \cap S_y) \, d\mathcal{L}^h(y) = \int_F Jf \, d\mathcal{L}^k$$

More generally, for every Borel function $g : \mathbb{R}^k \to [0,+\infty]$ there holds

$$\int_{y\in\mathbb{R}^h} \Big(\int_{S_y} g \, d\mathcal{H}^{k-h} \Big) \, d\mathcal{L}^h(y) = \int_{\mathbb{R}^k} g \; Jf \, d\mathcal{L}^k$$

As one may expect, this formula can be extended to the case $\mathbb{R}^k$ is replaced by a k-rectifiable set.

<u>Coarea formula</u> (second version)
Let E be a k-rectifiable set in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^h$ a lipschitz map, and for every $y \in \mathbb{R}^h$ let $S_y := f^{-1}(y)$ (as before) and for $\mathcal{H}^k$-a.e. $x \in E$ let $\nabla_\tau f(x)$ be the tangential gradient of f at x (defined some lectures ago...) and

$$J_\tau f(x) := \big| \overbrace{\nabla_\tau f_1(x) \wedge \dots \wedge \nabla_\tau f_h(x)}^{\eta(x)} \big| . \leftarrow \text{(tangential) Jacobian of } f$$

Then for every Borel function $g : E \to [0,+\infty]$ there holds

$$\int_{y\in\mathbb{R}^h} \Big(\int_{S_y \cap E} g \, d\mathcal{H}^{k-h} \Big) \, d\mathcal{L}^h(y) = \int_E h \; J_\tau f \, d\mathcal{L}^k .$$

We can now state and prove the equivalent of Proposition 1 for rectifiable currents

<u>Proposition 2</u> (<u>Slicing of rectifiable currents</u>)
Let $0 < h \leq k \leq n$, and let $T = [E, \tau, m]$ be a rectifiable k-current in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^h$ a lipschitz map.
Let then $\tilde{E}$ be the set of all $x \in E$ where f is tangentially differentiable and $\nabla_\tau f(x)$ has rank h

(that is, maximal rank) or equivalently the simple h-vector $\eta(x) := \nabla_C f_1(x) \wedge \dots \wedge \nabla_C f_h(x)$ is non zero. Finally let $E_y := S_y \cap E$ for every $y \in \mathbb{R}^h$. Then

(i) $\mathcal{H}^{k-h}(E_y \setminus \tilde{E}) = 0$ for $(\mathcal{L}^h\text{-})$a.e. y;

(ii) E_y is $(k-h)$-rectifiable for a.e. y;

(iii) for a.e. y, $\eta(x)$ spans the orthogonal complement of $\operatorname{Tan}(E_y, x)$ in $\operatorname{Tan}(E, x)$ for $\mathcal{H}^{k-h}$-a.e. $x \in E_y$ and then we can orient E_y by $\tilde{\tau}$ defined by the following identity:

$$\frac{\eta(x)}{|\eta(x)|} \wedge \tilde{\tau}(x) = \tau(x) \quad \text{for } \mathcal{H}^{k-h}\text{-a.e. } x \in E_y$$

(iv) there holds

$$\int_{y \in \mathbb{R}^h} \left(\int_{E_y} |m| \, d\mathcal{H}^{k-h} \right) d\mathcal{L}^h(y) = \int_E |m| \, J_C f \, d\mathcal{H}^k$$
$$\leq (\operatorname{Lip}(f))^h \, \mathbb{M}(T) < +\infty;$$

in particular

$$\int_{E_y} |m| \, d\mathcal{H}^{k-h} < +\infty \quad \text{for a.e. } y.$$

(v) The $(k-h)$-rectifiable current $T_y := [E_y, \tilde{\tau}, m]$ is well-defined for a.e. y, and

$$\int_{y \in \mathbb{R}^h} \mathbb{M}(T_y) \, d\mathcal{L}^h(y) = \int_E |m| \, J_C f \, d\mathcal{H}^k \leq (\operatorname{Lip}(f))^h \, \mathbb{M}(T)$$

We call the currents T_y slices of T according to f.

Proof

(i) By applying the coarea formula (2nd v.) with $g := 1_{E \setminus \tilde{E}}$ we obtain

$$\int \mathcal{H}^{k-l}(E_y \setminus \tilde{E})\, d\mathcal{L}^l(y) = \int_{E\setminus\tilde{E}} J_\tau f \, d\mathcal{H}^k \underset{\uparrow}{=} 0$$

because $\mathcal{H}^k(E \setminus \tilde{E}) = 0$

Hence $\mathcal{H}^{k-l}(E_y \setminus \tilde{E}) = 0$ for a.e. y.

(ii) Let's assume first that f is of class $\mathcal{C}^1$, and let $E = \bigcup_{i=0}^{\infty} E_i$ with $\mathcal{H}^k(E_0) = 0$ and E_i contained (for $i>0$) in S_i, surface of class $\mathcal{C}^1$ and dimension k in $\mathbb{R}^n$; we can also assume (w.n.l.g) that $\mathrm{Tan}(E_i, x) = \mathrm{Tan}(S_i, x)$ for every $x \in E_i$. Then for every $x \in \tilde{E} \cap E_i$ the tang. gradient of f at x has maximal rank, and therefore $\tilde{E} \cap E_i \cap S_y$ is contained in a $\mathcal{C}^1$ surface of dimension $k-l$ for every y and every $i>0$.
Since $E_y := E \cap S_y$ is covered by the sets $\tilde{E} \cap E_i \cap S_y$, $i>0$, and by $E_y \cap E_0$, $E_y \setminus \tilde{E}$, we conclude that E_y is rectifiable if $\mathcal{H}^{k-l}(E_y \cap E_0) = 0$ and $\mathcal{H}^{k-l}(E_y \setminus \tilde{E}) = 0$.
The second condition holds for a.e. y by statement (i).
Concerning the first condition, we can proceed as in the proof of statement (i) to show that it holds for a.e. y.
The case where f is Lipschitz (and not $\mathcal{C}^1$) can be reduced to the previous one using the Lusin property of Lipschitz functions with $\mathcal{C}^1$ functions (on E...).

(iii) This statement follows essentially from the same argument used to prove statement (ii). (Recall that $\eta(x) := \nabla_C f_1(x) \wedge \ldots \wedge \nabla_C f_k(x)$).

(iv) It suffices to apply the area formula to the function m.

(v) This stament is a straightforward consequence of statements (ii), (iii) and (iv). $\square$

Next I derive two properties of the slicing of rectifiable currents, one of this will be used to define the slicing in a more general setting.

Proposition 3

Let $T, f, \{T_y\}$ be as in the previous statement, f of class $\mathcal{C}^1$. Then for every $(k-h)$-form $\omega \in \mathcal{D}^{k-h}(\mathbb{R}^n)$ there holds

$$(*) \qquad \int_{y \in \mathbb{R}^h} \langle T_y ; \omega \rangle \, d\mathcal{L}^h(y) = \langle T ; df_1 \wedge \ldots \wedge df_h \wedge \omega \rangle .$$

To prove this result we need the following lemmas:

Lemma 4

Let $v, \tilde{v}$ be simple h-, $\tilde{h}$-vectors in $\mathbb{R}^n$ s.t. $v \wedge \tilde{v} \neq 0$, let $w = w_1 \wedge \ldots \wedge w_h$ be a simple h-covector s.t. w_i is null on span $\tilde{v}$ for every i, and finally let $\tilde{w}$ be an $\tilde{h}$-covector.

Then

(1) $$\langle w\wedge\tilde w\,;\,v\wedge\tilde v\rangle = \langle w\,;\,v\rangle\cdot\langle\tilde w\,;\,\tilde v\rangle .$$

<u>Proof</u> We choose $e_1,\dots,e_n$ base of $\mathbb{R}^n$ so that $v=e_1\wedge\dots\wedge e_h$ and $\tilde v=e_{h+1}\wedge\dots\wedge e_{h+\tilde h}$.
Then we can write each $w_i=\sum_{j=1}^{n}w_{ij}\,e_j^*$ where $w_{ij}=0$ for $h<j\le h+\tilde h$.
We also write $\tilde w=\sum\tilde w_{\underline j}\,e^*_{\underline j}$.
Since the identity (1) is linear in $\tilde w$, we can assume that $\tilde w=e^*_{\underline j}$ for some $\tilde h$-index $\underline j$.
Next we write $\underline i=(1,\dots,h+\tilde h)$; $\underline i'=(1,\dots,h)$; $\underline i''=(h+1,\dots,h+\tilde h)$.
Then

$$\langle w\wedge\tilde w\,;\,v\wedge\tilde v\rangle=\langle w\wedge\overset{e^*_{\underline j}}{\overset{\|}{\tilde w}}\,;\,e_{\underline i}\rangle=(w\wedge e^*_{\underline j})_{\underline i}=\begin{cases}(w)_{\underline i'} & \text{if } \underline j=\underline i''\,;\\ 0 & \text{if } \underline j\neq\underline i''.\end{cases}$$

this is the key point, and follows from the fact that writing w in terms of the basis, no e_i appears with $h<i\le h+\tilde h$.

On the other hand

$$\langle w\,;\,v\rangle=\langle w\,;\,e_{\underline i'}\rangle=(w)_{\underline i'}$$

and

$$\langle\tilde w\,;\,\tilde v\rangle=\langle e^*_{\underline j}\,;\,e_{\underline i''}\rangle=\begin{cases}1 & \text{if } \underline j=\underline i''\,,\\ 0 & \text{if } \underline j\neq\underline i''.\end{cases}$$

$\square$

<u>Lemma 5</u>

Let $v_1,\dots,v_h\in W$ space with scalar product, and for $v\in V$ let v° be the covector associated to v by the scalar product, that is $\langle v^\circ\,;\,w\rangle:=v\cdot w$. Then

(2) $$\langle v_1^\circ\wedge\dots\wedge v_h^\circ\,;\,v_1\wedge\dots\wedge v_h\rangle=|v_1\wedge\dots\wedge v_h|^2.$$

<u>Proof</u> By choosing an orthonormal basis on V we can assume that $W=\mathbb{R}^n$. Let V be the matrix with

columns $v_1, \dots, v_k$. Then we know that

$$|v_1 \wedge \dots \wedge v_k|^2 = \det(V^t V) ;$$

$$\langle v_1^\circ \wedge \dots \wedge v_k^\circ ; v_1 \wedge \dots \wedge v_k \rangle = \det A \text{ where } A_{ij} = \langle v_i^\circ ; v_j \rangle ;$$

$$(V^t V)_{ij} = v_i \cdot v_j = \langle v_i^\circ ; v_j \rangle = A_{ij} \Rightarrow A = V^t V .$$

Putting these formulas together we get (2).

$\square$

<u>Proof of Proposition 3</u>

Take $x \in E$ s.t. $\eta = \eta(x) = \nabla_\tau f_1(x) \wedge \dots \wedge \nabla_\tau f_k(x)$ is well defined and $\neq 0$.

At such point we have by definition that $\frac{\eta}{|\eta|} \wedge \tilde{\tau} = \tau$.

Hence

$$\begin{aligned} &\langle df_1 \wedge \dots \wedge df_k \wedge \omega ; \tau \rangle \\ &= \langle df_1 \wedge \dots \wedge df_k \wedge \omega ; \tfrac{\eta}{|\eta|} \wedge \tilde{\tau} \rangle \\ \text{by Lemma 4} \longrightarrow &= \tfrac{1}{|\eta|} \langle df_1 \wedge \dots \wedge df_k ; \eta \rangle \langle \omega ; \tilde{\tau} \rangle \end{aligned}$$

On the other hand

$$\begin{aligned} \langle df_1 \wedge \dots \wedge df_k ; \eta \rangle &= \langle df_1 \wedge \dots \wedge df_k ; \nabla_\tau f_1 \wedge \dots \wedge \nabla_\tau f_k \rangle \\ &= \langle d_\tau f_1 \wedge \dots \wedge d_\tau f_k ; \nabla_\tau f_1 \wedge \dots \wedge \nabla_\tau f_k \rangle \\ \text{by Lemma 5} \longrightarrow \quad &= |\nabla_\tau f_1 \wedge \dots \wedge \nabla_\tau f_k|^2 \\ &= |\eta|^2 \end{aligned}$$

We have thus proved that

$$(3) \qquad \langle df_1 \wedge \dots \wedge df_k \wedge \omega ; \tau \rangle = \underset{\substack{\parallel \\ Jf}}{|\eta|} \langle \omega ; \tilde{\tau} \rangle .$$

Therefore

$$\langle T \,;\, df_1 \wedge \dots \wedge df_\ell \wedge \omega \rangle = \int_E \langle df_1 \wedge \dots \wedge df_\ell \wedge \omega \,;\, \tau \rangle \, m \, d\mathcal{H}^k$$

by (3) $\longrightarrow$ $$= \int_E \langle \omega ; \tilde{\tau} \rangle \, m \, Jf \, d\mathcal{H}^k$$

by the coarea formula $\longrightarrow$ $$= \int_{y \in \mathbb{R}^\ell} \left(\int_{E_y} \langle \omega ; \tilde{\tau} \rangle \, m \, d\mathcal{H}^{k-\ell} \right) d\mathcal{L}^\ell(y)$$

$$= \int_{y \in \mathbb{R}^\ell} \langle T_y \,;\, \omega \rangle \, d\mathcal{L}^\ell(y).$$

and (*) is proved. □

Remark

The slices $\{T_y : y \in \mathbb{R}^\ell\}$ are "essentially" determined (up to a negligible subset of y) by the fact that each T_y is supported on $S_y := f^{-1}(y)$ and that formula (*) holds, and this can be used to define the slicing of an arbitrary current T with finite mass w.r.t. a map $f : \mathbb{R}^n \to \mathbb{R}^\ell$ of class C^1.

More precisely, one can prove the following

Proposition 6

Let $T = \tau\mu$ a k-current with finite mass in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^\ell$ a map of class C^1. Then there exists a measure λ on $\mathbb{R}^\ell$ and a family of $(k-\ell)$-currents with finite mass T_y, $y \in \mathbb{R}^\ell$ s.t.

(i) T_y is supported in $S_y := f^{-1}(y)$ for (λ-almost) every $y \in \mathbb{R}^h$;

(ii) $\int_{y\in\mathbb{R}^h} \mathbb{M}(T_y)\, d\lambda(y) < +\infty$;

(iii) for every $\omega \in \mathcal{D}^{k-h}(\mathbb{R}^n)$ there holds

$$\langle T; df_1 \wedge \dots \wedge df_h \wedge \omega \rangle = \int_{y\in\mathbb{R}^h} \langle T_y; \omega \rangle \, d\lambda(y).$$

Moreover the family $\{T_y\}$ is "unique" in the sense that given $\{\tilde{T}_y\}$ such that (i)-(iii) hold, then $\tilde{T}_y = T_y$ for λ-a.e. y. We might say that T_y are the slices of T according to f and λ....

<u>Remarks</u> ∘ We must actually assume a certain Borel regularity of the map $y \mapsto T_y$....

∘ Contrary to the case of rectifiable currents, the measure λ cannot always be taken equal to $\mathcal{L}^k$.

∘ In the following we give a different definition of slicing of a normal current, which is based on a different property of the slicing of rectifiable currents.

∘ The proof of Proposition 6 relies on the theorem of disintegration of measures (w.r.t. a map) and is actually rather simple, but it is not essential to this course, and we omit it.

We now proceed with another key property of the slicing of rectifiable currents.

Proposition 7

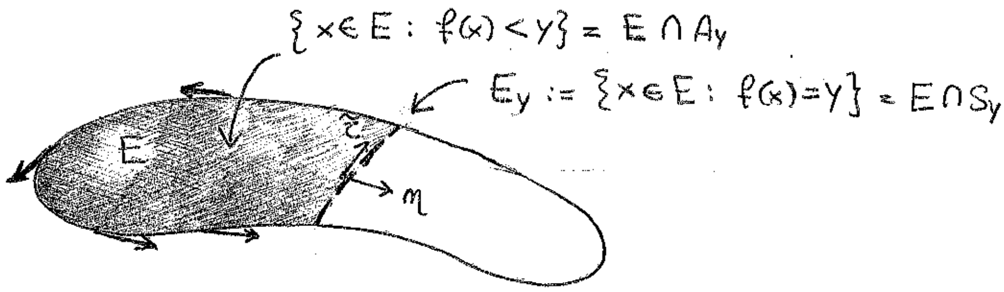
Let T be a k-rectifiable current in $\mathbb{R}^n$ with boundary of finite mass, $f:\mathbb{R}^n\to\mathbb{R}$ a Lipschitz and C^1 function and let $\{T_y\}$ be the slicing of T according to f, as defined in Proposition 2.

Then for $(\mathcal{L}^1)$ a.e. $y\in\mathbb{R}$ there holds

$$(**)\qquad T_y=\partial(1_{A_y}\cdot T)-1_{A_y}\cdot\partial T \quad\text{where } A_y:=f^{-1}((-\infty,y]).$$

Proof

The geometric idea is quite simple: if T is the current associated to a smooth surface E,

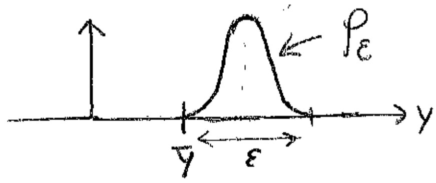


then it is clear (see the picture) that

$\partial(E\cap A_y)=(\partial E)\cap A_y\cup E_y$ that is, $E_y=\partial(E\cap A_y)\setminus(\partial E)\cap A_y$.

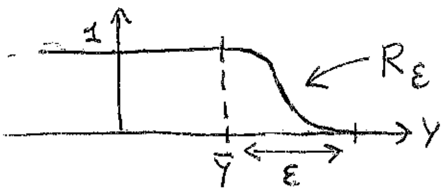
To prove $(**)$ rigorously, let us fix a smooth function $\rho:\mathbb{R}\to\mathbb{R}$ positive, compactly supported in $[0,1]$, with integral $=1$, a point $\bar{y}\in\mathbb{R}$, and for every $\varepsilon>0$

$$\rho_\varepsilon(y):=\frac{1}{\varepsilon}\rho\left(\frac{y-\bar{y}}{\varepsilon}\right)$$



Moreover let $R_\varepsilon:\mathbb{R}\to\mathbb{R}$ s.t.

$$R_\varepsilon'=-\rho_\varepsilon,\quad R_\varepsilon(+\infty)=0.$$



Then, for every $\omega \in \mathcal{D}^{k-1}(\mathbb{R}^n)$,

$$\int_{y\in\mathbb{R}^n} \langle T_y ; \omega\rangle \, \rho_\varepsilon(y)\, dy =$$

short for $d\mathcal{L}^1(y)$

$$= \int \langle T_y ; (\rho_\varepsilon \circ f)\cdot \omega\rangle \, dy$$

by Proposition 3 $\longmapsto$ $= \langle T ; (\rho_\varepsilon \circ f)\, df \wedge \omega\rangle$

$d(R_\varepsilon \circ f) = -(\rho_\varepsilon \circ f)\, df$ $\longmapsto$ $= -\langle T ; \, d(R_\varepsilon \circ f)\wedge \omega\rangle$

$d((R_\varepsilon\circ f)\omega) = d(R_\varepsilon \circ f)\wedge\omega + (R_\varepsilon\circ f)\, d\omega$ $\longmapsto$ $= \langle T ; (R_\varepsilon\circ f) d\omega\rangle - \langle T ; d((R_\varepsilon\circ f)\omega)\rangle$

$$= \langle (R_\varepsilon \circ f) T ; d\omega\rangle - \langle \partial T ; (R_\varepsilon\circ f)\omega\rangle$$

$$= \langle (R_\varepsilon \circ f) T ; d\omega\rangle - \langle (R_\varepsilon\circ f)\, \partial T ; \omega\rangle .$$

Now, the functions $R_\varepsilon \circ f$ are uniformly bounded and, as $\varepsilon \to 0$, converge pointwise <u>everywhere</u> to the characteristic function $1_{A_{\bar y}}$. Hence $(R_\varepsilon \circ f) T$ and $(R_\varepsilon\circ f)\,\partial T$ converge, in the sense of measures, to $1_{A_{\bar y}} T$ and $1_{A_{\bar y}} \partial T$, and
$\langle (R_\varepsilon\circ f) T ; d\omega\rangle - \langle (R_\varepsilon \circ f)\partial T ; \omega\rangle \longrightarrow \langle 1_{A_{\bar y}} T ; d\omega\rangle - \langle 1_{A_{\bar y}} \partial T ; \omega\rangle$.
Let us assume in addition that

(H) the functions $y \longmapsto \langle T_y ; \omega\rangle$ are L^1-approximately continuous at $\bar y$ for a dense family of ω (dense in the space $\mathcal{C}^1_0(\mathbb{R}^n, \Lambda^{k-1}\mathbb{R}^n)$ of continuous $(k-1)$-forms vanishing at infinity).

Then for every such ω there holds

$$\int \langle T_y ; \omega\rangle \, \rho_\varepsilon(y)\, dy \xrightarrow[\varepsilon\to 0]{} \langle T_{\bar y} ; \omega\rangle$$

and therefore, putting all pieces together,

$$\langle T_{\bar y};\omega\rangle = \langle 1_{A_{\bar y}}T;d\omega\rangle - \langle 1_{A_{\bar y}}\partial T;\omega\rangle .$$

Since this identity holds for a dense family of ω, it also holds for all ω (in $\mathcal{E}'_0(\mathbb{R}^n,\Lambda^{k-1}\mathbb{R}^n)$) which implies (**) for $\bar y$, that is,

$$T_{\bar y} = \partial(1_{A_{\bar y}}T) - 1_{A_{\bar y}}\partial T .$$

It remains to show that assumption (H) is satisfied by ($\mathcal{L}^1$-)a.e. $\bar y$. Note that the function $y\mapsto\langle T_y,\omega\rangle$ belongs to L^1 for every ω (in $\mathcal{E}_0(\mathbb{R}^n,\Lambda^{k-1}\mathbb{R}^n)$) because

$$\int |\langle T_y,\omega\rangle|\,dy \le \|\omega\|_\infty \int \mathbb{M}(T_y)\,dy$$
$$\le (\mathrm{Lip}(f))^k\,\|\omega\|_\infty\,\mathbb{M}(T) < +\infty,$$

and then $y\mapsto\langle T_y,\omega\rangle$ is L^1-approx. continuous at a.e. $\bar y\in\mathbb{R}$. Hence, for every countable (dense) family of ω the functions $y\mapsto\langle T_y,\omega\rangle$ are <u>simultaneously</u> L^1-approx. continuous at $\mathcal{L}^1$-a.e. y.

$\square$

Proposition 7 paves the way for a definition of slicing of currents that are not rectifiable.

<u>Slicing of normal currents ($h=1$)</u>

Let T be a normal k-current in $\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}$ a Lipschitz and $\mathcal{E}^1$ function. We define the slices of T according to f as

$$(**)\qquad T_y := \partial(1_{A_y}T) - 1_{A_y}\partial T$$

where $A_y := f^{-1}((-\infty, y])$ as before, and $y \in \mathbb{R}$.

Now, the idea would be to define the slicing for maps $f : \mathbb{R}^n \to \mathbb{R}^h$ iteratively in $h = 1, 2, 3 \dots$, that is, the slices of T according to $f = (f_1, f_2)$ are the slices according to f_2 of the slices according to f_1, and so on. To do so, we must first show that the codimension-1 slices defined above are normal currents for a.e. y (so that we can further slice them).

Proposition 8

Let T be a normal k-current in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz & C^1, and let T_y be defined in $(**)$ above. Then T_y has finite mass for a.e. y, and more precisely

(4) $$\int_{y \in \mathbb{R}} \mathbb{M}(T_y)\, dy \le \operatorname{Lip}(f) \cdot \mathbb{M}(T) < +\infty .$$

To prove this result we need the following lemma.

Lemma 9

Let λ_n be a sequence of (possibly vector-valued) measures that converge to λ in the sense of measures on $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel function. For every $y \in \mathbb{R}$ let $A_y := f^{-1}((-\infty, y])$ as above. Then for all y except countably many, $1_{A_y} \lambda_n \to 1_{A_y} \lambda$ in the sense of measures.

~~Proof~~

Up to subseq. we can assume that the positive measures $|\lambda_n|$ converge to some μ, and it is known that $\mu \geq |\lambda|$, and that $\int g \, d\lambda_n \to \int g \, d\lambda$ for every bounded function $g : \mathbb{R}^n \to \mathbb{R}$ s.t. $g \to 0$ at infinity and the set of discontinuity points of g is μ-null.

Hence, if $\mu(f^{-1}(y)) = 0$ then for every $\varphi \in \mathcal{E}_0(\mathbb{R}^n)$ the function $1_{A_y}\varphi$ satisfies the assumptions above and therefore $\int_{A_y} g \, d\lambda_n \to \int_{A_y} g \, d\lambda$. Then $1_{A_y}\lambda_n \to 1_{A_y}\lambda$. Thus the thesis holds if $\mu(f^{-1}(y)) = 0$, and this holds for all y except countably many because the sets $\{f^{-1}(y)\}$ are pairwise disjoint and μ is finite. $\square$

Proof of proposition 8

We already know that (6) holds if T is rectifiable. We prove it for a general normal current T by approximation. Let then T_n be a sequence of normal rectifiable currents s.t. $T_n \to T$ and $\partial T_n \to \partial T$ in the sense of measures (such as those provided by the polyhedral deformation theorem).

By Lemma 9, $1_{A_y} T_n \to 1_{A_y} T$ and $1_{A_y} \partial T_n \to 1_{A_y} \partial T$ for a.e. y and then $(T_n)_y \to T_y$ in the sense of distrib.

Then
$$\int \mathbb{M}(T_y)\, dy \leq \int \liminf_{n\to\infty} \mathbb{M}((T_n)_y)\, dy$$

by Fatou's lemma $\longmapsto$
$$\leq \liminf_{n\to\infty} \int \mathbb{M}((T_n)_y)\, dy$$
$$\leq \mathrm{Lip}(f) \liminf_{n\to\infty} \mathbb{M}(T_n) < +\infty.$$

This suffices to show that $\mathbb{M}(T_y) < +\infty$ for a.e. y.
To get (4) we must choose T_n so that $\mathbb{M}(T_n) \to \mathbb{M}(T)$, which is possible by the Strong Polyhedral Approx. Theorem. $\square$

Proposition 10

Let T, f, T_y as in Proposition 8. Then, for every $y \in \mathbb{R}$,

$$(5) \qquad \partial(T_y) = -\underbrace{(\partial T)_y}_{\text{slice of } \partial T} .$$

In particular $\partial(T_y)$ has finite mass for a.e. y and

$$(6) \qquad \int \mathbb{M}(\partial(T_y))\, dy \leq \mathrm{lip}(f)\ \mathbb{M}(\partial T) .$$

Proof

For every $\omega \in \mathcal{D}^{k-2}(\mathbb{R}^n)$ we have, according to (**)

$$\langle (\partial T)_y ; \omega \rangle = \langle 1_{A_y} \partial T ; d\omega \rangle - \langle 1_{A_y} \underbrace{\partial^2 T}_{=0} ; \omega \rangle ,$$

$$\langle \partial(T_y) ; \omega \rangle = \langle T_y ; d\omega \rangle = \langle 1_{A_y} T ; \underbrace{d^2\omega}_{=0} \rangle - \langle 1_{A_y} \partial T , d\omega \rangle ,$$

and (5) is proved; (6) follows from (5) and (4) (applied to ∂T instead of T). $\square$

Corollary 11

Let T, f, T_y be as in Proposition 8. Then T_y is normal for a.e. y, and more precisely

$$\int_{y \in \mathbb{R}} \mathbb{M}(T_y) + \mathbb{M}(\partial T_y)\, dy \leq \mathrm{lip}(f) \left(\mathbb{M}(T) + \mathbb{M}(\partial T) \right) .$$

Proof Just apply Propositions 8 and 10. $\square$

Slicing of normal currents (every h)

As mentioned above, we define the slicing of a normal k-current T on $\mathbb{R}^n$ according to a Lipschitz & $\mathcal{C}^1$ $f : \mathbb{R}^n \to \mathbb{R}^h$ by recursion on h.

For $h=1$ the definition is given by (**).

For $h>1$ and $y=(y_1,\dots,y_h)\in\mathbb{R}^h$ we set

$$T_y := (T_{\tilde{y}})_{y_h}$$

where $\tilde{y}=(y_1,\dots,y_{h-1})$, $T_{\tilde{y}}$ is the slice of T according to $\tilde{f}:=(f_1,\dots,f_{h-1})$ and $(T_{\tilde{y}})_{y_h}$ is the slice of $T_{\tilde{y}}$ according to f_h.

One can prove the following (we omit the details):

Proposition 12

The slice T_y is well-defined and is a normal current for $\mathcal{L}^h$-a.e. $y\in\mathbb{R}^h$, and more precisely

$$\int_{y\in\mathbb{R}^h} \mathbb{M}(T_y)+\mathbb{M}(\partial T_y)\,dy \le (\mathrm{Lip}(f))^h\,(\mathbb{M}(T)+\mathbb{M}(\partial T)).$$

Moreover, for a.e. y,

$$\partial(T_y) = (-1)^h (\partial T)_y .$$

Remark The slicing of normal currents can also be defined when f is Lipschitz (and not also of class $\mathcal{C}^1$), and Proposition 12 still holds. We will not need this in the following.

We begin with a property of slicing that is essential for the rest of this lecture.

Theorem 1

Let T be a normal k-current in $\mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^h$ $(0 < h \leq k)$ a Lipschitz (and C^1) map, $\{T_y : y \in \mathbb{R}^h\}$ the slices of T according to f, and X the space of $(k-h)$-currents with finite flat norm, endowed with the flat norm.

Then the map

$$y \in \mathbb{R}^h \longmapsto T_y \in X$$

is BV.

Remark BV maps from (an open subset of) $\mathbb{R}^m$ to a normed space Y can be defined in different ways.

We choose the following definition: $g: \mathbb{R}^m \to Y$ is BV if there exists $C < +\infty$ s.t.

$$\| \tau_v g - g \|_1 \leq C |v| \quad \forall v \in \mathbb{R}^m$$

where $\tau_v g$ is the translation $\tau_v g(x) := g(x-v)$ and $\| \tau_v g - g \|_1 = \int | g(x-v) - g(x) |_Y \, dx$ (as usual...).

When $m=1$ there is also the classical (and not perfectly equivalent) definition of BV function: $g: \mathbb{R} \to Y$ has bounded variation if there exists $C < +\infty$ s.t.

$$\sum_{i=1}^{N} | g(t_i) - g(t_{i-1}) |_Y < C$$

for every N and every $t_0 < t_1 < \dots < t_N$.
Note that Theorem 1 holds with both definition.

Proof of Theorem 1

We begin by proving that for $h=1$, $y \mapsto T_y$ is BV in the classical sense. Let $T = \tau\mu$, $\partial T = \tau'\mu'$.
unitary
Take $y_0 < y_1 < \dots < y_N$. Then

$$\begin{aligned} T_{y_i} - T_{y_{i-1}} &= \partial\left(1_{\{\rho \le y_i\}} T\right) - 1_{\{\rho \le y_i\}} \partial T \\ &\quad - \partial\left(1_{\{\rho \le y_{i-1}\}} T\right) + 1_{\{\rho \le y_{i-1}\}} \partial T \\ &= \partial\left(1_{\{y_{i-1} < \rho \le y_i\}} T\right) \\ &\quad - 1_{\{y_{i-1} < \rho \le y_i\}} \partial T \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{F}(T_{y_i} - T_{y_{i-1}}) &\le \mathbb{M}\left(1_{\{y_{i-1} < \rho \le y_i\}} T\right) + \mathbb{M}\left(1_{\{\dots\}} \partial T\right) \\ &= (\mu + \mu')\left(\{y_{i-1} < \rho \le y_i\}\right), \end{aligned}$$

and then

$$\begin{aligned} \sum_{i=1}^{N} \mathbb{F}(T_{y_i} - T_{y_{i-1}}) &\le (\mu + \mu')\left(\bigcup_{i=1}^{N} \{y_{i-1} < \rho \le y_i\}\right) \\ &= (\mu + \mu')\left(\{y_0 < \rho \le y_N\}\right) \\ &\le (\mu + \mu')(\mathbb{R}^n) = \mathbb{M}(T) + \mathbb{M}(\partial T). \end{aligned}$$

Now we prove that $y \mapsto T_y$ is BV in the "real" sense. We begin with the case $h=1$, where the proof is close to the previous one. Fix $v>0$. Then, as seen above,

$$\mathbb{F}(T_y - T_{y-v}) \leq (\mu+\mu')(\{y-v<f\leq y\})$$

and then

$$\int_{y\in\mathbb{R}} \mathbb{F}(T_y - T_{y-v})\, dy \leq \int_{y\in\mathbb{R}} (\mu+\mu')(\{y-v<f\leq y\})\, \overset{d\mathcal{L}^1(y)}{\overset{\shortparallel}{dy}}$$

$$= \int_{y\in\mathbb{R}} \left(\int_{x\in\mathbb{R}^n} 1_{\{y-v<f(x)\leq y\}}\, d(\mu+\mu')(x)\right) dy$$

Fubini $\longmapsto$ $$= \int_{x\in\mathbb{R}^n} \left(\int_{y\in\mathbb{R}} 1_{\{f(x)\leq y<f(x)+v\}}\, dy \right) d(\mu+\mu')(x)$$

$$= \int_{\mathbb{R}^n} v\, d(\mu+\mu') = v\,(\mathbb{M}(T)+\mathbb{M}(\partial T)),$$

and the proof that $y \mapsto T_y$ is BV is complete.

We just give an idea of the proof for higher h by looking at the case $h=2$.

By definition for $y=(y_1,y_2)$ we have $T_y := (T_{y_1})_{y_2}$ (the "inner" slicing being according to f_1, the "outer" one according to f_2). Fix again $v>0$. By the previous computation we have

$$\int \mathbb{F}(T_{(y_1,y_2)} - T_{(y_1,y_2-v)})\, dy_2 \leq v\,(\mathbb{M}(T_{y_1}) + \mathbb{M}(\partial T_{y_1}))$$

and the

$$\int \mathbb{F}(T_{(y_1,y_2)} - T_{(y_1,y_2-v)})\, dy_1 dy_2 \leq v \int \mathbb{M}(T_{y_1}) + \mathbb{M}(\partial T_{y_1})\, dy_1$$

see the previous lect. $\longmapsto$ $$\leq v\, \mathrm{lip}(f)(\mathbb{M}(T)+\mathbb{M}(\partial T))$$

This proves that

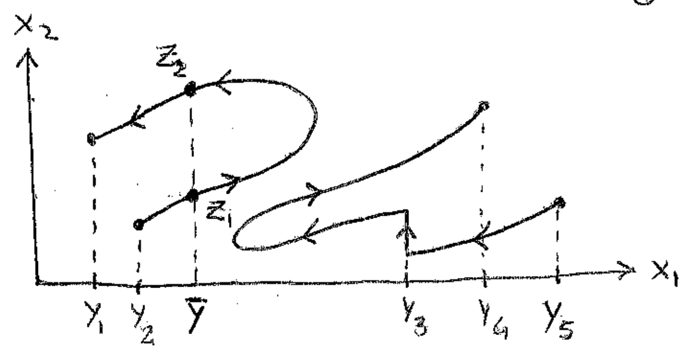
$$\int \mathbb{F}(T_y - T_{y-w})\, dy \leq C|w| \quad \text{with } C<+\infty$$

for w of the form $w=(0,v)$, and then also for $w=(v,0)$.

From these two special case we (easily) obtain the inequality for all $\omega \in \mathbb{R}^2$ (possibly doubling the constant C....).

Example

Let $T = [E, \tau, 1]$ where E is the union of (oriented) curves in the plane described in the picture below, and let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) := x_1$.



Check that $T_{\bar{y}} = \delta_{z_1} - \delta_{z_2}$ using both the definition of slicing for rectifiable currents and that for normal currents.

Check that the map $y \mapsto T_y$ is discontinuous at the points $y_1, \dots, y_5$ (and only at these points).

We can now state the main result of this lecture, which is a criterion for rectifiability of normal currents based on the rectifiability of slices.

This criterion is due to B. White. The proof we sketch is due to L. Ambrosio and B. Kirchheim, who exploited the fact that the slicing map $y \mapsto T_y$ is BV

(Theorem 1, due to R. Jerrard).

Let P be the set of all (orthogonal) projections f of $\mathbb{R}^n$ on a k-dimensional coordinate plane, that is, $f(x) := (x_{i_1}, \dots, x_{i_k})$ for some multiindex $\underline{i} = (i_1, \dots, i_k) \in I(n,k)$. Now, if T is a rectifiable current, then, for every such f, the slices T_y of T according to f are rectifiable 0-currents for ($\mathcal{L}^k$-) a.e. $y \in \mathbb{R}^k$. The following converse holds

<u>Theorem 2</u> (<u>Rectifiability-by-slicing criterion</u>)
Let T be a normal k-current in $\mathbb{R}^n$, and assume that for <u>every</u> $f \in P$, the slices T_y of T according to f are <u>integral</u> 0-currents for ($\mathcal{L}^k$-) a.e. y.
Then T is a rectifiable current with integral multiplicity.

This result can be stated in stronger forms, but we are not going for it now...
We neither give a complete proof, but rather a (hopefully informative) sketch of the main steps of the proof.

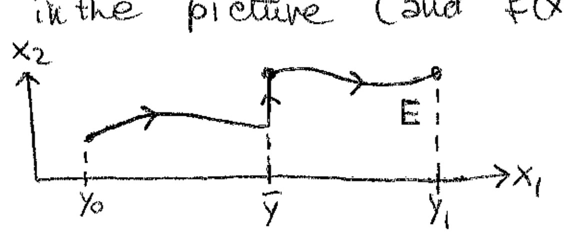
<u>Sketch of proof</u>
The proof is divided in three steps. The first step consists in showing that the current T is supported on a k-rectifiable set E.

The second step consists in showing that a normal k-current T supported on a k-rectifiable set E must be rectifiable. Since we already proved such a statement in the case E is a k-surface of class $\mathscr{C}^1$, we omit the proof of this generalisation.
In the third step we notice that T has integral multiplicity because the slices T_y have integral multiplicit. (Then if $\partial T = 0$ this suffices to show that T is integral; the case $\partial T \neq 0$ requires some additional work.)
Below we only focus on the first step, which is the key one.

Let f_i be fixed. Since T_y is an integral 0-current for a.e. y, it can be written in the form

$$T_y = \sum_i m_i \delta_{x_i}$$

The idea is that T is supported on the set E given by the union of the supports $\{x_i\}$ of T_y over all y.
Now, this is not quite correct, and indeed T is supported on the union of all such sets E as f varies in T.
The point is explained in the example below, where $T = [E, \tau, 1]$ and E is the oriented curve in the plane given in the picture (and $f(x_1, x_2) = x_1$)



Indeed T_y is a single Dirac mass for all $y \in (y_0, y_1]$, including $\bar{y}$. Thus the union of the supports of these Dirac masses is E minus the vertical segment in the middle. To recover this segment we must consider also the slices of T according to the other projection $f(x_1, x_2) := x_2$.

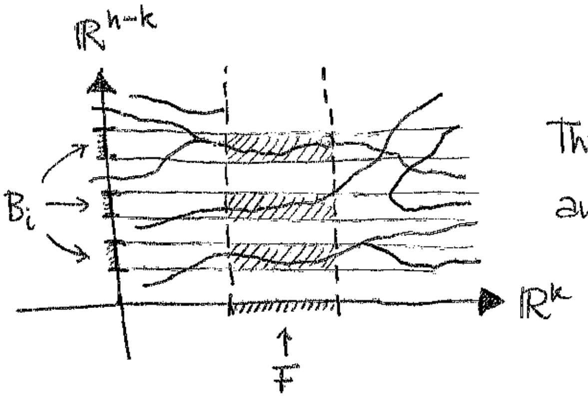
The second problem with such definition of E, is that in order to have that E is rectifiable, we might need in the end to throw away a negligible subset of y (this is somehow hidden in the sketch of proof below).

Modulo this remark, we now prove that the set E defined above (the union of the supports of all T_y) is k-rectifiable. We assume that $f(x) = (x_1, \dots, x_k)$.

We actually consider the following subset of E: Fix N disjoint closed balls $B_1, \dots, B_N$ in $\mathbb{R}^{n-k}$ and integers $m_1, \dots, m_N$, and let F be the set of all $y \in \mathbb{R}^k$ such that $T_y = \sum_{i=1}^{N} m_i \delta_{x_i}$ with $x_i = (y_i, z_i)$ and $z_i \in B_i$. Let then E_F be the union of the supports of all T_y with $y \in F$.

We are going to prove that E_F is rectifiable.

The rectifiability of E follows by the fact that it can be written as union of countably many such E_F.



The set E (union of the curves) and the sets F, $E_F = E \cap (F \times \mathbb{R}^{n-k})$.

Let now X be the space of 0-currents endowed with the flat distance, and let $\tilde{X}$ be the set of all currents in X of the form $\mu = \sum_{i=1}^{N} m_i \delta_{x_i}$ with $x_i \in \mathbb{R}^k \times B_i$, $x_i = (y_i, z_i)$.
For every i let $L_i : \tilde{X} \to \mathbb{R}^{n-k}$ be the map that to every μ as above associates the second coordinate z_i of x_i.
It can be verified that L_i is Lipschitz (on $\tilde{X}$ we consider again the flat distance). Hence it can be extended to a Lipschitz map from X to $\mathbb{R}^{n-k}$ (we apply Mc Shane extension lemma to each component of L_i).

Then the map $g_i : y \mapsto L_i(T_y)$ is BV (from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$), and for every $y \in F$, the support of T_y is the set $\{ (y, g_i(y)) : i = 1, \dots, N \}$.
Hence E_F is contained in the union of the graphs of the maps g_i.

To conclude the proof we recall that BV maps have the Lusin property with C^1 maps, and therefore the graphs of BV maps are rectifiable (provided a suitable negligible subset of the domain is discarded).

Hence the graphs of all g_i are (essentially) k-rectifiable, and so is E_F.

$\square$

Remark

Theorem 2 allow us to deduce the k-rectifiability of a normal current T by checking the 0-rectifiability of the "intersections" of T with $(n-k)$-planes (parallel to the coordinate $(n-k)$-planes).
Note that there is no analogous statement for sets.

We conclude this lecture by sketching the proofs of the Boundary Rectifiability Theorem and the Closure Theorem (that is, Federer - Fleming compactness theorem).

Boundary Rectifiability Theorem

Let T be a rectifiable k-current with integral multiplicity whose boundary has finite mass. Then T is an integral current (that is, the boundary is rectifiable with integral multiplicity).

Sketch of proof

Step 1: the statement is true if $k=1$ (we omit the proof, which is actually not complicated — for instance it can be obtained from the polyhedral approximation theorem).

Step 2 ($k>1$)

We want to prove that ∂T is a rectifiable current with integral multiplicity using Theorem 2.

Let indeed $f: \mathbb{R}^n \to \mathbb{R}^{k-1}$ be a projection (as in Th. 2) and let T_y, $(\partial T)_y$ be the slices of T and ∂T according to f.

Then we know that T_y is normal for a.e. y (because T is) and rectifiable with integral multiplicity (because T is). Hence $\partial(T_y)$ is a rectifiable 0-current with integral multiplicity by Step 1, and so is $(\partial T)_y$ because $\partial(T_y) = (-1)^{h-1}(\partial T)_y$.

Thus ∂T satisfies the assumption of Th. 2, and therefore is rectifiable with integral multiplicity.

$\square$

Closure Theorem

Let T_n be a sequence of integral k-currents such that $\mathbb{M}(T_n) + \mathbb{M}(\partial T_n) \leq C < +\infty \quad \forall n$, and T_n converge to some normal current T as $n \to \infty$. Then T is integral.

Sketch of proof

Step 1: the statement is true if $k=0$ (almost trivial).

Step 2 ($k>0$)

By the Boundary Rectifiability Theorem it suffices to prove that T is rectifiable with integral multiplicity, which we obtain by applying Theorem 2.

Let then $f : \mathbb{R}^n \to \mathbb{R}^k$ be a projection as in Thm. 2. (11)
We need to show that the slices T_y (of T according to f) are integral 0-currents for a.e. y.

Now, since $T_h \to T$ and $\mathbb{M}(T_h) + \mathbb{M}(\partial T_h) < +\infty$ we have that for a.e. $y \in \mathbb{R}^k$ there holds

$$(T_h)_y \longrightarrow T_y$$

and

$$\mathbb{M}(T_y) + \mathbb{M}(\partial(T_y)) \leq$$
$$\leq \liminf_{h \to +\infty} \left[\mathbb{M}((T_h)_y) + \mathbb{M}(\partial((T_h)_y)) \right] < +\infty .$$

The proof of this claim is contained in the proof of Proposition 8 in the previous lecture.

Moreover each $(T_h)_y$ is an integral 0-current (because the currents T_h are integral) and therefore by Step 1 we have that T_y is integral.

$\square$